



Instruments for Radioastronomy

Interferometers

We have introduced define the **complex visibility** V from the two independent (real) correlator outputs R_c and R_s as:

$$V = R_c - iR_s = |V| e^{i\phi}$$

R_c cosine response (even)

R_s sine response (odd)

where $|V| = \sqrt{R_c^2 + R_s^2}$ is the **visibility amplitude**

$\phi = \tan^{-1}(R_s/R_c)$ is the **visibility phase**

We have seen the relation between the source brightness and the response of an interferometer:

$$V = \int_{\Omega_{\text{source}}} I_\nu(\hat{s}) e^{-2\pi i \nu \vec{b} \cdot \hat{s} / c} d\Omega$$

$$\tau_g = \frac{\vec{b} \cdot \hat{s}}{c}$$

geometric delay

which is a 2D Fourier transform, giving as a well established way to recover $I_\nu(\hat{s})$ from V .

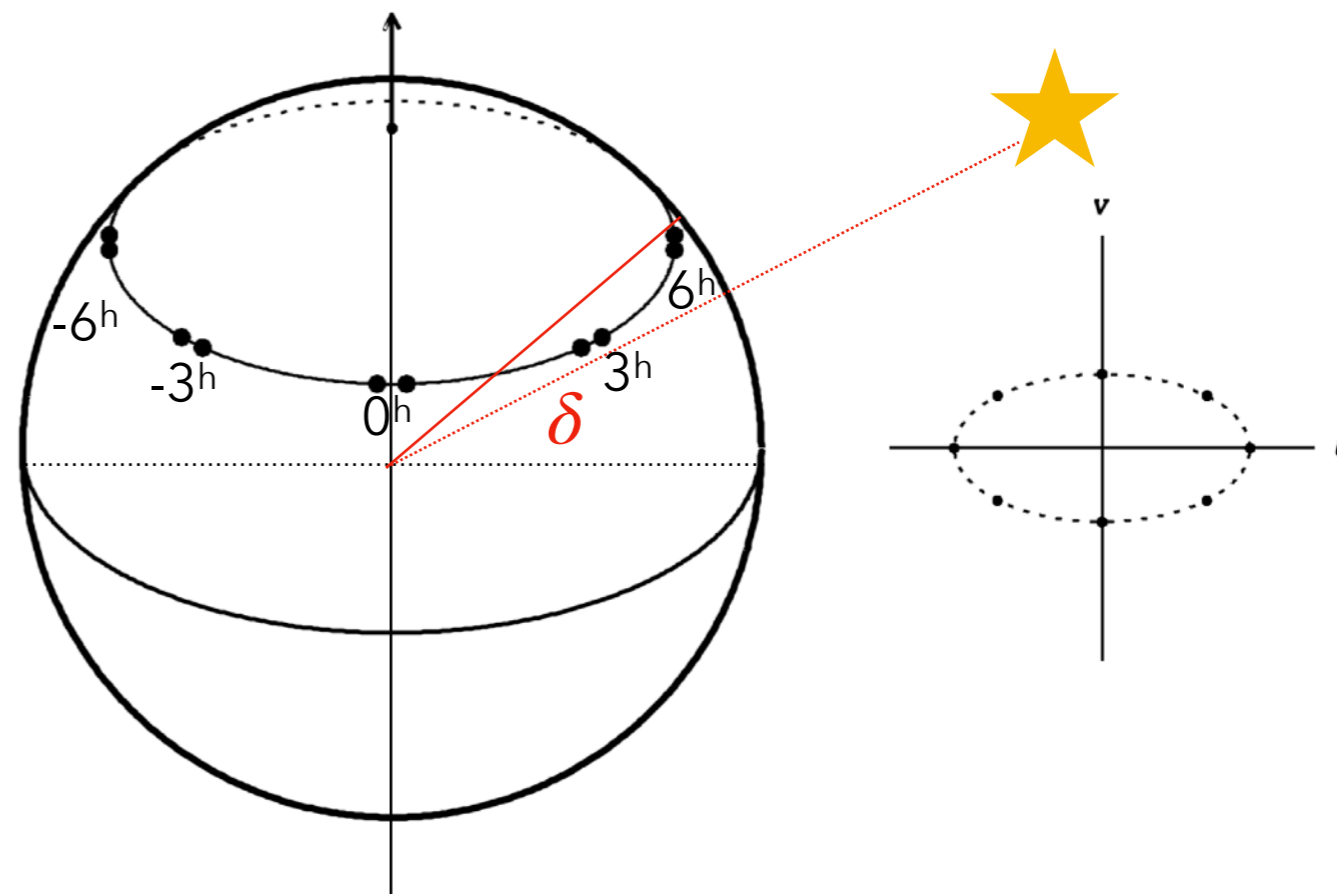
This relation is valid in the ideal case of an interferometer with isotropic response and in the quasi-monochromatic approximation.

Earth rotation aperture synthesis

Most astronomical sources are stationary, that is their brightness does not change on the timescale of the observation. Earth rotation and moving antennas can be exploited to increase the number of effective "antenna pairs".

Consider an east-west two-element interferometer at latitude $+40^\circ$ as seen by a source at declination $\delta = +30^\circ$.

The projected east-west component (in wavelength units) is u and the north-south projected component is v .



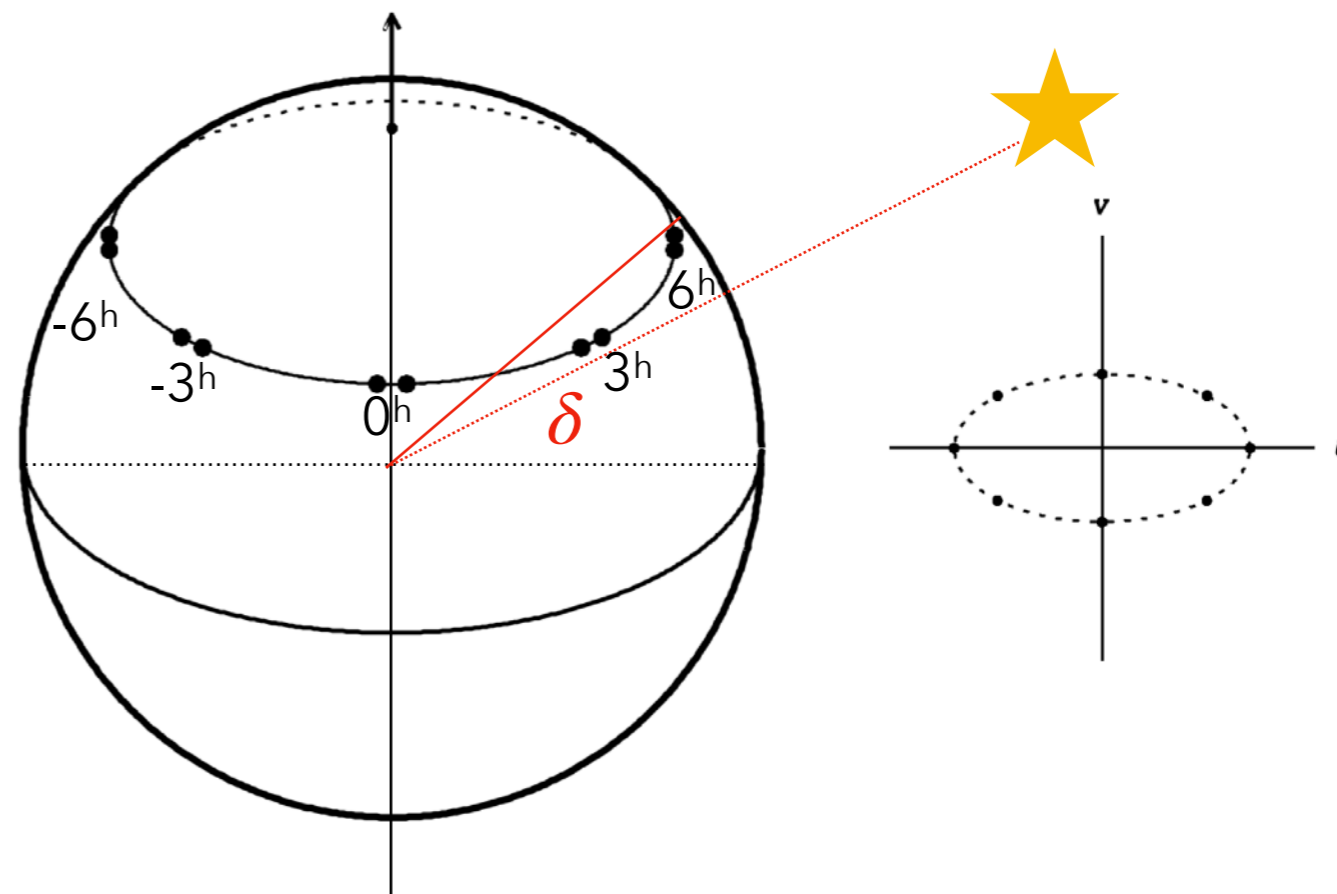
During a 12-hour period, the interferometer traces out a complete ellipse in the (u,v) plane. The maximum value of u equals the actual antenna separation in wavelengths, and the maximum value of v is smaller by the projection factor $\sin\delta$, where δ is the source declination

If the interferometer has $N > 2$ antennas, or if the spacing of the two antennas is changed daily, the (u,v) coverage will become a number of concentric ellipses having the same shape. Thus the synthesized beam obtained can approach an elliptical Gaussian.

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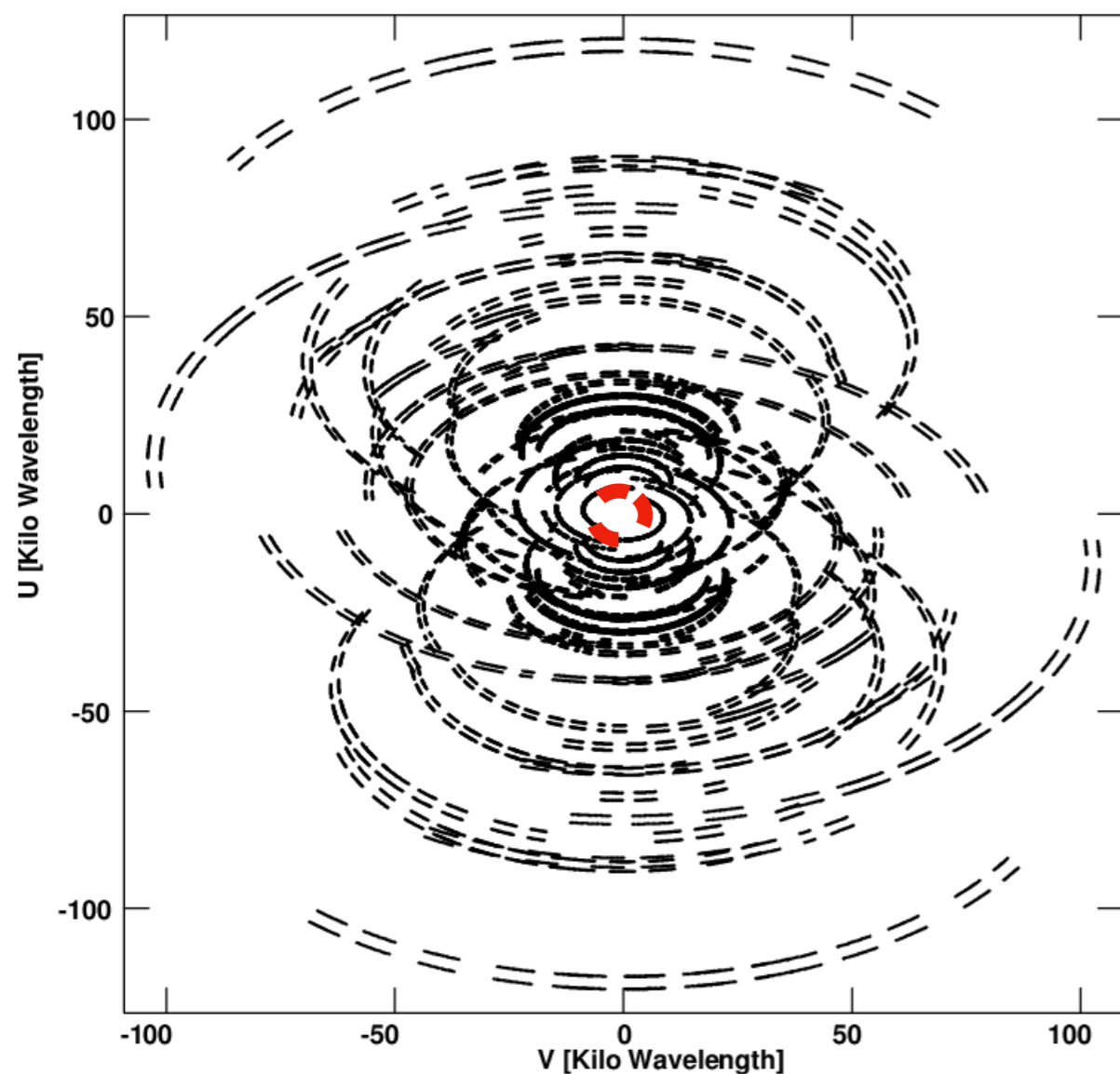
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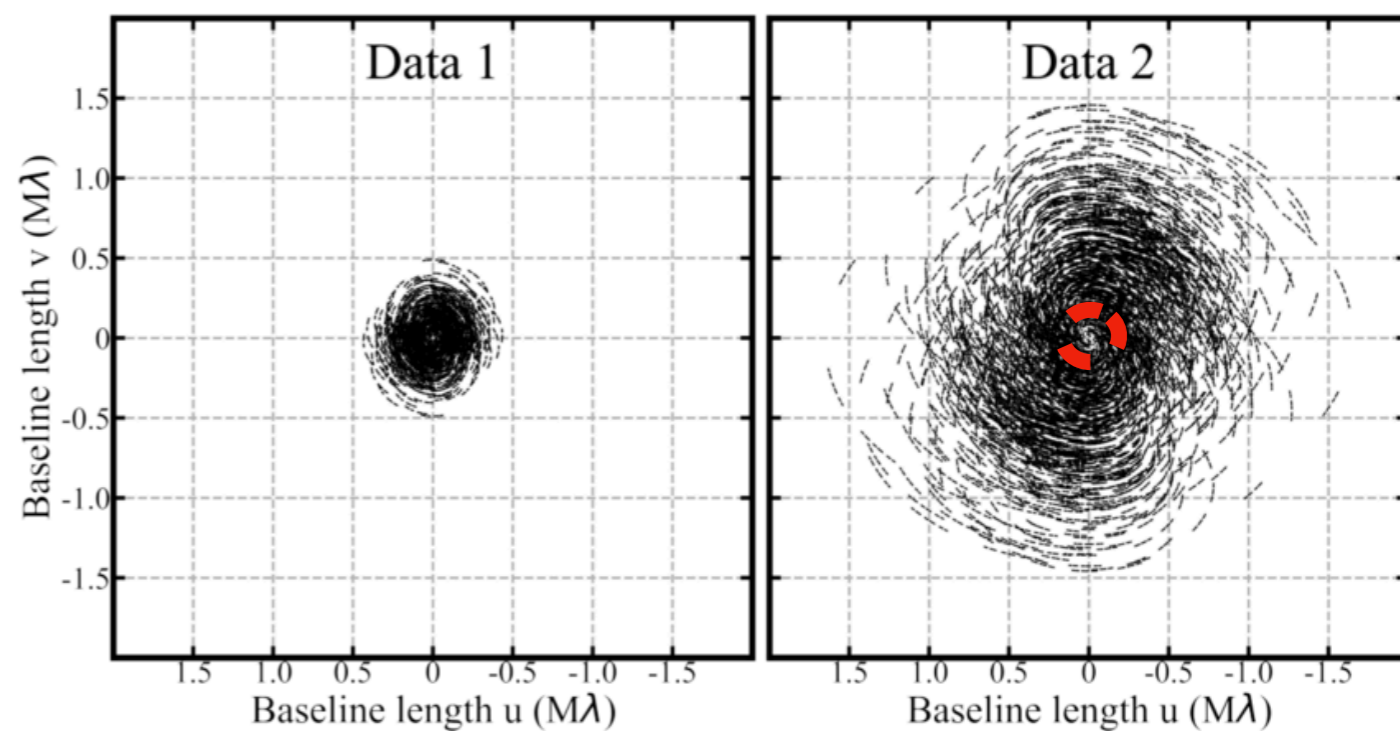
The synthesized beamwidth is $\sim u^{-1}$ rad east-west and $\sim u^{-1}/\sin\delta$ rad in the north-south direction.

The synthesized beam is almost circular for a source near the celestial pole, but the north-south beamwidth is very large for a source near the celestial equator.



IRAM PdBI (now NOEMA)
4 to 6 antennas in 3 configurations
On source time ~27 hours

De Breuck et al. 2003



Two ALMA configurations
compact (left) and extended (right)
On source time ~1 (left) and ~2(right) hours

Yamaguchi et al. 2020

Visibility-brightness relation for a realistic antenna response

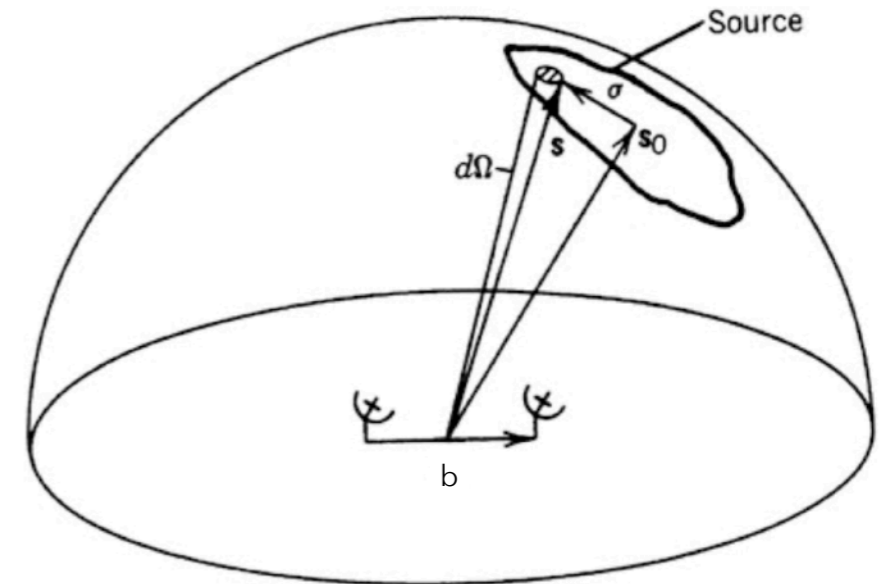
We now consider the relation between source brightness and complex visibility (for an extended source) in the case of a realistic (direction dependent) antenna response, in the quasi-monochromatic approximation. We have:

$$V = \int_{\Omega_{\text{source}}} A(\hat{s}) I_{\nu}(\hat{s}) e^{-2\pi i \nu \vec{b} \cdot \hat{s} / c} d\Omega$$

where $A(\hat{s})$ is the effective area of each (identical) antenna.

Typically, source positions are expressed as with respect to the phase-center: $\hat{s} = \hat{s}_0 + \hat{\sigma}$

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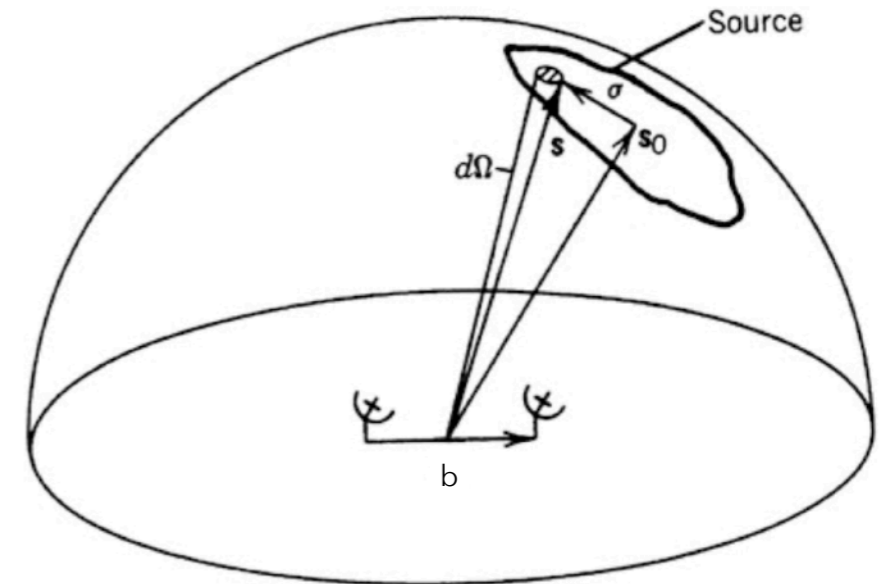
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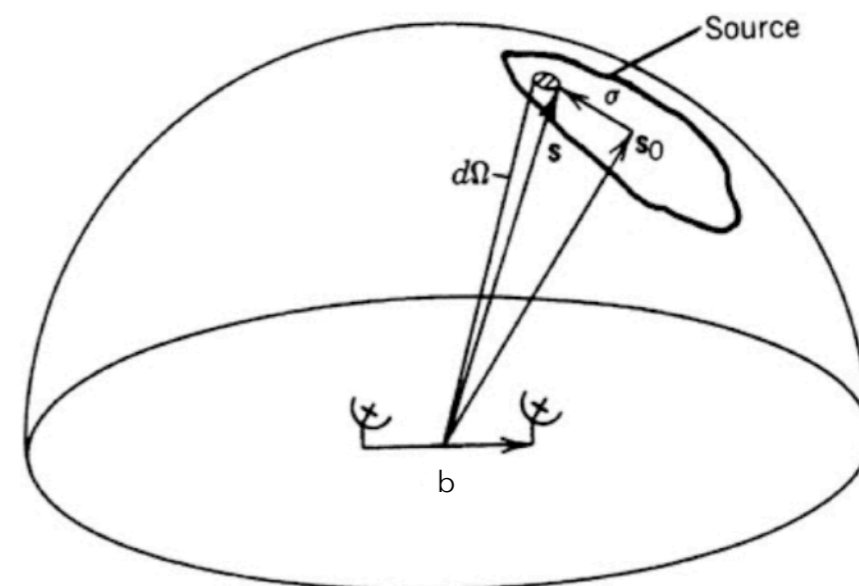
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Let's now express the correlator response as a function of the visibility....

We have seen that the correlator response in the case of a realistic (direction dependent) antenna response, in the quasi-monochromatic approximation, is:

$$\begin{aligned}
 R &= \int_{\Omega_{\text{source}}} A(\hat{s}) I_{\nu}(\hat{s}) \cos(2\pi\nu \vec{b} \cdot \hat{s}/c) d\Omega \\
 &= \cos\left(2\pi\nu \frac{\vec{b} \cdot \hat{s}_0}{c}\right) \int_{\Omega_{\text{source}}} A(\hat{\sigma}) I_{\nu}(\hat{\sigma}) \cos(2\pi\nu \vec{b} \cdot \hat{\sigma}/c) d\Omega \\
 &\quad - \sin\left(2\pi\nu \frac{\vec{b} \cdot \hat{s}_0}{c}\right) \int_{\Omega_{\text{source}}} A(\hat{\sigma}) I_{\nu}(\hat{\sigma}) \sin(2\pi\nu \vec{b} \cdot \hat{\sigma}/c) d\Omega = \\
 &= \cos\left(2\pi\nu \frac{\vec{b} \cdot \hat{s}_0}{c}\right) |V| \cos\phi - \sin\left(2\pi\nu \frac{\vec{b} \cdot \hat{s}_0}{c}\right) |V| \sin\phi = \\
 &= |V| \cos(2\pi\nu\tau_g - \phi)
 \end{aligned}$$



Note:

We do not measure $r = FT(I)$

We measure $R =$ something related to V , which resembles the $FT(I)$



Complex visibility for a finite bandwidth interferometer

The definition of visibility for quasi-monochromatic interferometers may be generalized to interferometers with finite bandwidths and integration times, which are necessary for high sensitivity.

$$V = \int_{\Omega_{\text{source}}} A(\hat{s}) I_{\nu}(\hat{s}) e^{-2\pi i \nu \tau_g} d\Omega$$
 quasi-monochromatic

$$V = \int_{\Omega_{\text{source}}} \left(\int_{\nu_0 - \Delta\nu/2}^{\nu_0 + \Delta\nu/2} A(\hat{s}) I_{\nu}(\hat{s}) e^{-2\pi i \nu \tau_g} d\nu \right) d\Omega$$
 finite bandwidth $\Delta\nu$ centered on frequency ν_0

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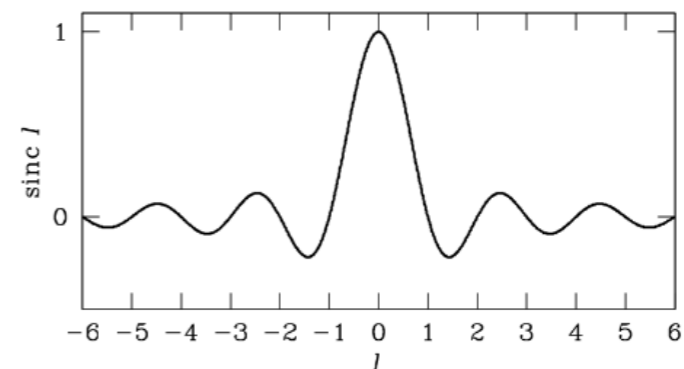
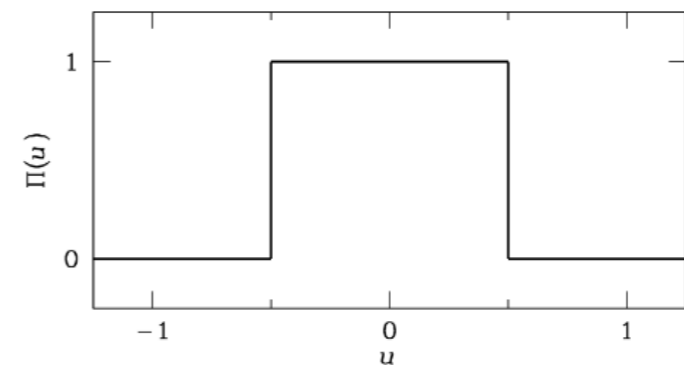
$$V = \int_{\Omega_{\text{source}}} \left(\int_{\nu_0 - \Delta\nu/2}^{\nu_0 + \Delta\nu/2} A(\hat{s}) I_{\nu}(\hat{s}) e^{-2\pi i \nu \tau_g} d\nu \right) d\Omega \quad \text{finite bandwidth } \Delta\nu \text{ centered on frequency } \nu_0$$

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If the source brightness and response of the interferometer are \sim constant over $\Delta\nu$, the integral over frequency is just the FT of a rectangle function:

$$V \simeq \int A(\hat{s}) I_{\nu}(\hat{s}) \text{sinc}(\Delta\nu \tau_g) e^{-2\pi i \nu \tau_g} d\Omega$$

For a finite bandwidth $\Delta\nu$ and delay τ_g , the fringe amplitude is attenuated by a factor $\text{sinc}(\Delta\nu \tau_g)$. This effect is called **bandwidth smearing**.





Complex visibility for a finite bandwidth interferometer

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$$R = |V| \cos(2\pi \nu_0 \tau_g - \phi) \frac{\text{sinc}(\pi \Delta\nu \tau_g)}{\pi \Delta\nu \tau_g}$$

Correlator response

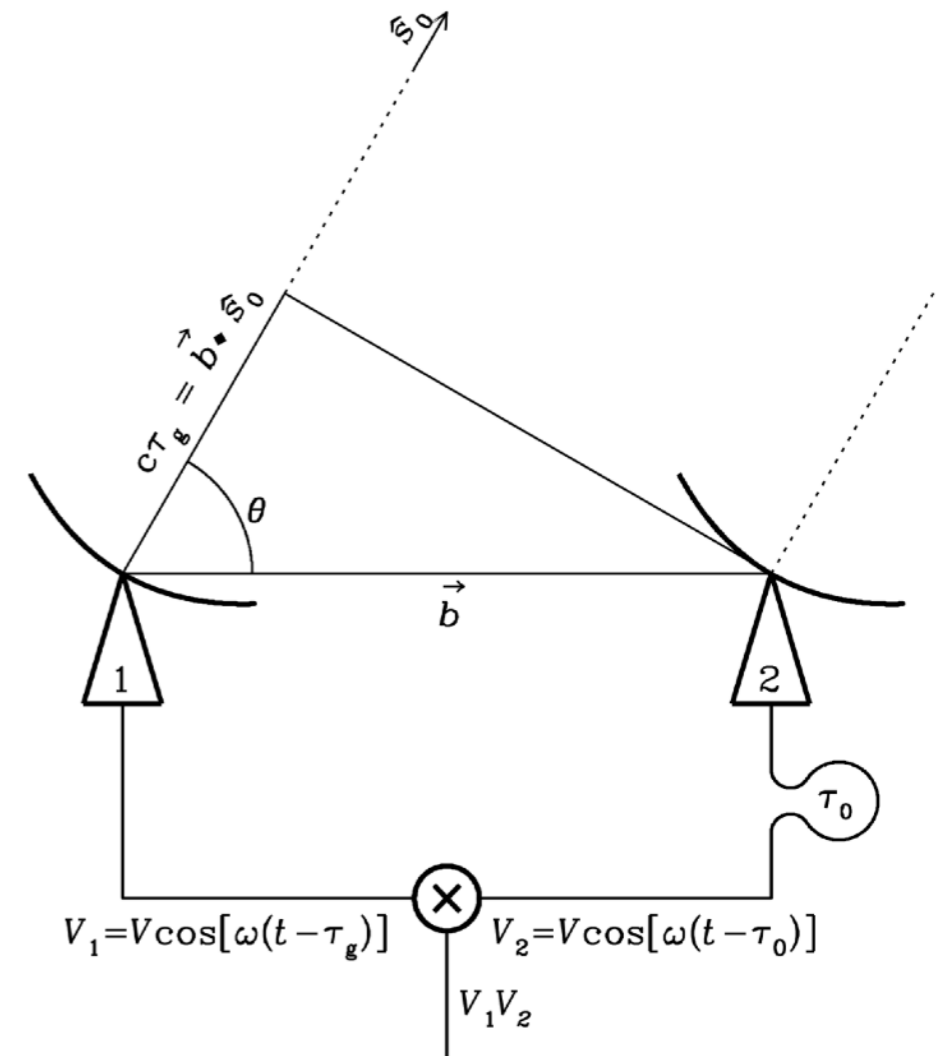


Complex visibility for a finite bandwidth interferometer

This attenuation can be eliminated in a given direction \hat{s}_0 (called the **phase reference position** or **phase center**) by introducing a compensating delay $\tau_0 \sim \tau_g$ in the signal path of the "leading" antenna.

This is usually done with digital electronics by introducing an instrumental delay in the correlator. As Earth rotates, τ_0 must be continuously adjusted so that $|\tau_0 - \tau_g| \ll \Delta\nu^{-1}$. If the delay is compensated, one can measure $R = |V| \cos\phi$

However: τ_g varies with direction, so **delay compensation can be exact in only one direction**.

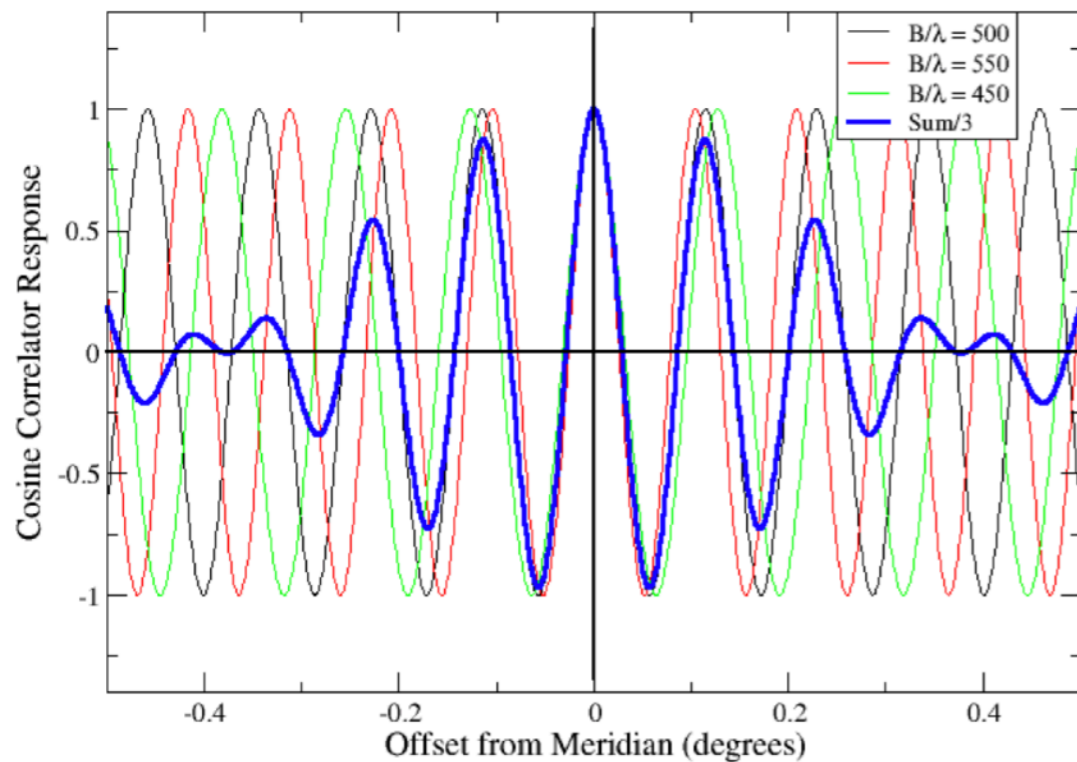
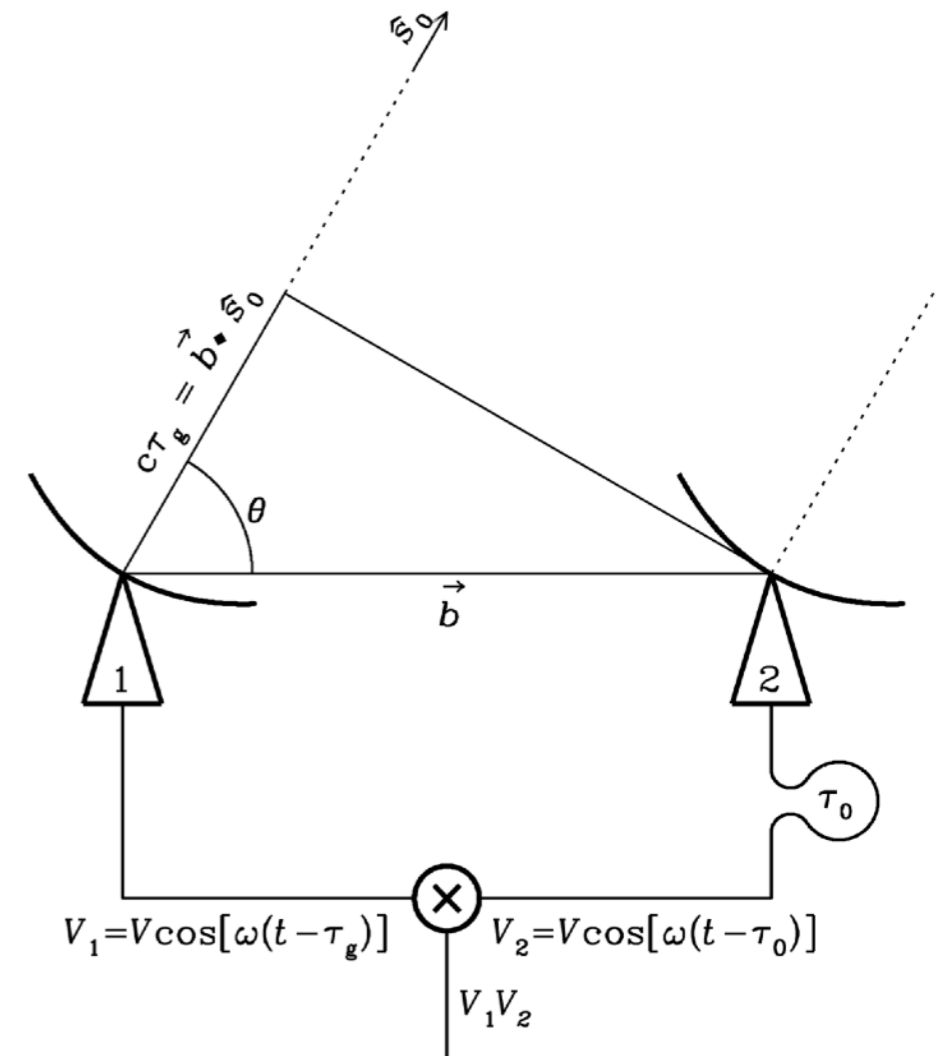


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Example: three wavelength components for the same physical baseline. For each λ , the separation between the fringes is $\sim \lambda/b$. They get increasingly out of step at increasing offset from the phase center



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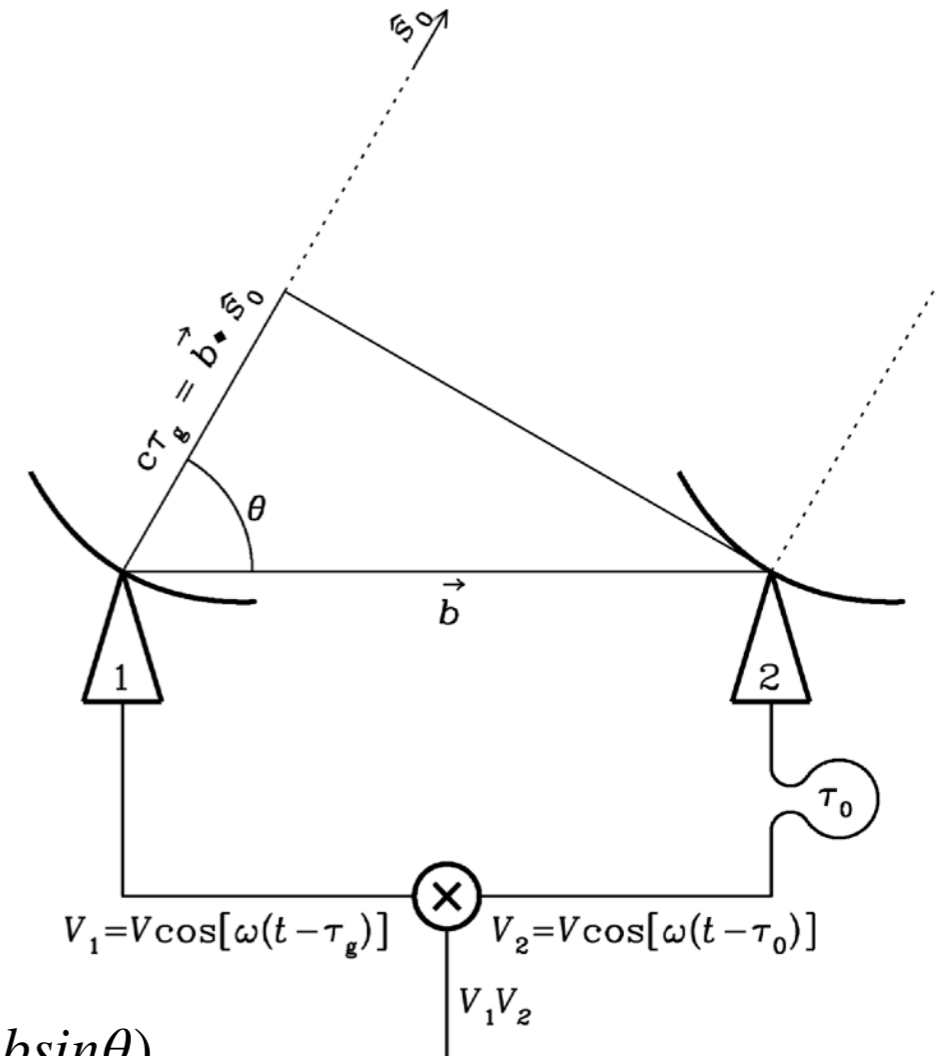
The angular radius $\Delta\theta$ of the "usable" field within which the bandwidth smearing is small can be derived by requiring that

$$\Delta\nu\Delta\tau_g \ll 1$$

which implies

$$\frac{\Delta\theta}{\theta_s} \ll \frac{\nu}{\Delta\nu} *$$

using $|c\Delta\tau_g| = b\sin\theta\Delta\theta$ and the synthesized beamwidth $\theta_s \sim \lambda/(b\sin\theta)$



At larger angular offsets from the phase reference position, **bandwidth smearing** will broaden the **synthesized beam** by convolving it with a rectangle function of angular width $\Delta\theta\Delta\nu/\nu$.

This is why wide field images (large $\Delta\theta$) can be made only by dividing the bandwidth into a number of narrower frequency channels, each satisfying *



Complex visibility for a finite bandwidth

How small does our bandwidth need to be, in order to ensure little bandwidth smearing within a given offset $\Delta\theta$?

$$\frac{\Delta\theta}{\theta_s} \ll \ll \frac{\nu}{\Delta\nu}$$

For example, the synthesized beam of the VLA "B" configuration (maximum $b \sim 10$ km) at $\lambda \sim 20$ cm is $\theta_s \sim 0.2/10^4$ rad ~ 4 arcsec.

We want to image an angular radius $\Delta\theta = 15$ arcmin = 900 arcsec, corresponding to $\sim \theta_p/2$ (primary beam) of the VLA.

This imply a bandwidth $\Delta\nu \ll \ll \frac{\nu\theta_s}{\Delta\theta} = \frac{1.5 \times 10^9 \text{ Hz} \cdot 4 \text{ arcsec}}{900 \text{ arcsec}} \sim 7 \text{ MHz}$





Complex visibility for a finite bandwidth and averaging times

How small does our bandwidth need to be, in order to ensure little bandwidth smearing within a given offset $\Delta\theta$?

$$\frac{\Delta\theta}{\theta_s} \ll \ll \frac{\nu}{\Delta\nu}$$

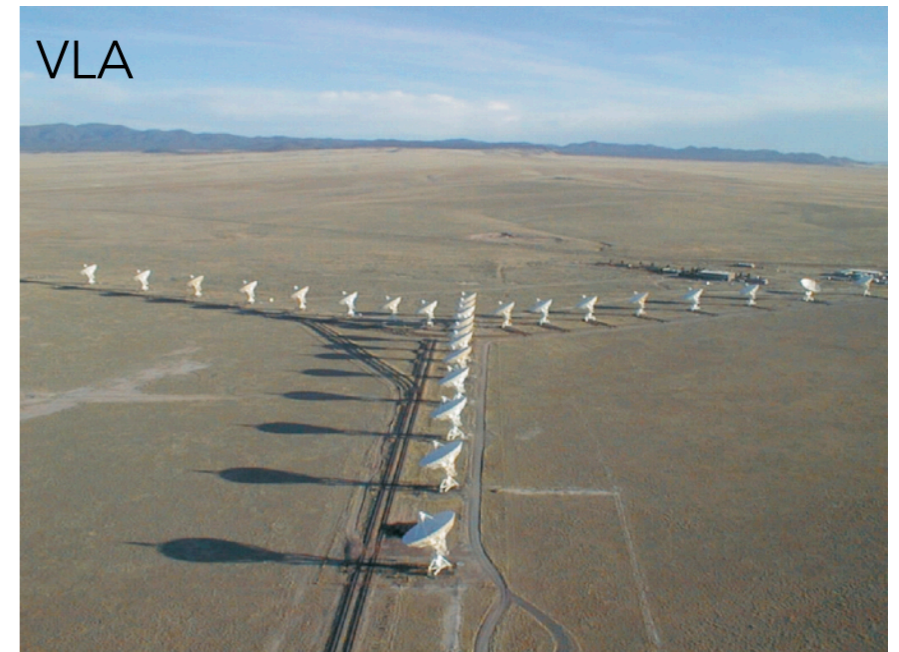
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Additionally, the synthesized beam might be broadened by [time smearing](#), due to excessively large correlator averaging times. To avoid this, the correlator averaging time Δt must be short enough that Earth's rotation will not significantly change the source position in the frame of the interferometer.

If the phase center is set to track the celestial pole, a source $\Delta\theta$ away from the celestial pole will move with an angular rate $2\pi\Delta\theta/P$ [rad/s], where P is the Earth's sidereal rotation period.





Complex visibility for a finite bandwidth and averaging times

How small does our averaging time need to be, in order to ensure little time smearing in an image of angular radius $\Delta\theta$?

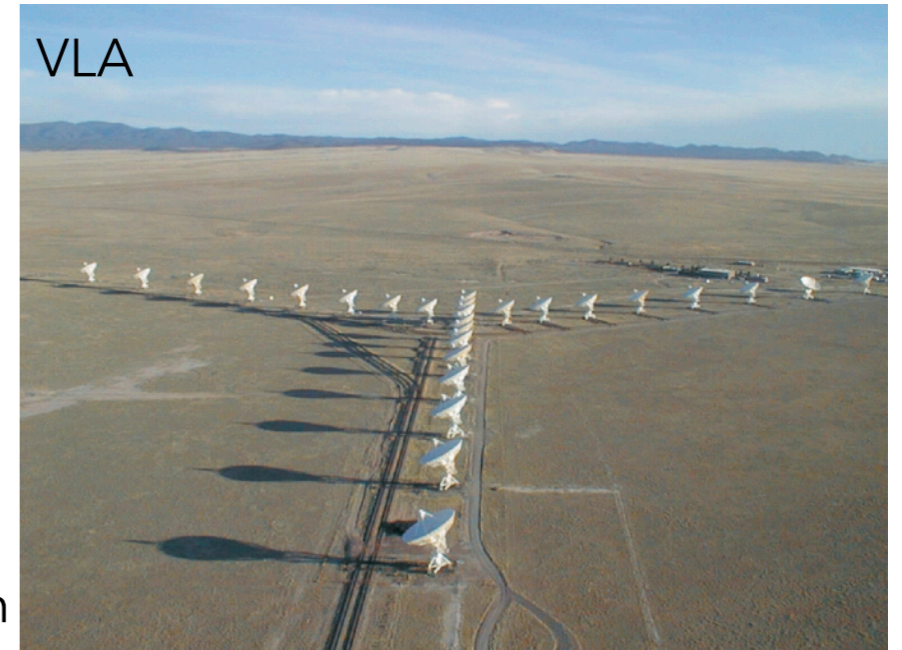
$$\frac{2\pi\Delta t}{P} \ll \frac{\theta_s}{\Delta\theta} \quad P \sim 10^4 \text{ s}$$

$$\frac{\Delta t}{1.4 \times 10^4 \text{ s}} \ll \frac{\theta_s}{\Delta\theta}$$

Again, for the VLA "B" configuration (maximum $b \sim 10$ km) at $\lambda \sim 20$ cm we have $\theta_s \sim 0.2/10^4$ rad ~ 4 arcsec

We want to image an angular radius $\Delta\theta = 15$ arcmin = 900 arcsec, corresponding to $\sim \theta_p/2$ (primary beam) of the VLA.

$$\Delta t \ll \frac{\theta_s}{\Delta\theta} \cdot 1.4 \times 10^4 \text{ s} = \frac{4 \text{ arcsec}}{900 \text{ arcsec}} \cdot 1.4 \times 10^4 \text{ s} \sim 60 \text{ s}$$





Interferometers in three dimensions

In general a baseline \vec{b} is described by a set of three coordinates (u, v, w) . The w -axis is in the reference direction \hat{s}_0 usually chosen to point the source.

The u and v axes point east and north in the (u, v) plane normal to the w -axis. u, v, w are the components of \vec{b} / λ in wavelength units.

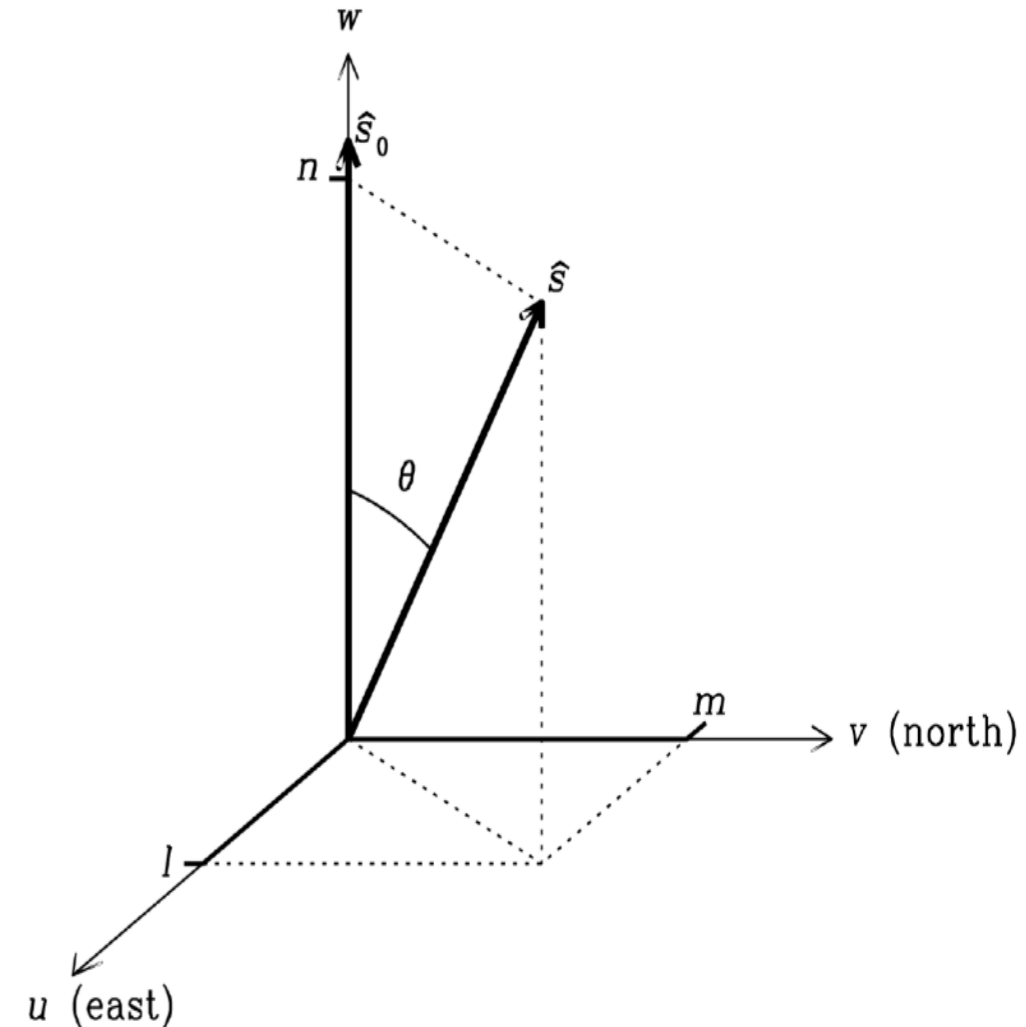
An arbitrary unitary vector \hat{s} has components (l, m, n) , where $n = \cos\theta = \sqrt{1 - m^2 - l^2}$

$$\text{Because } d\Omega = \frac{dl dm}{\sqrt{1 - m^2 - l^2}}$$

the three-dimensional generalization of the complex visibility definition $V = \int_{\Omega_{\text{source}}} A(\hat{s}) I_\nu(\hat{s}) e^{-2\pi i \nu \tau_g} d\Omega$ is

$$V(u, v, w) = \iint \frac{I_\nu(l, m)}{\sqrt{1 - m^2 - l^2}} \exp[-i2\pi(u l + v m + w n)] dl dm$$

which is NOT a FT unless $w = 0$



$$V = \int_{\Omega_{\text{source}}} A(\hat{s}) I_\nu(\hat{s}) e^{-2\pi i \nu \tau_g} d\Omega$$



Interferometers in three dimensions

$$V(u, v, w) = \iint \frac{I_\nu(l, m)}{\sqrt{1 - m^2 - l^2}} \exp[-i2\pi(ul + vm + wn)] dldm$$

For any interferometer, if we consider only directions close to \hat{s}_o we have $n = \cos\theta \sim 1 - \theta^2/2$ and

$$V(u, v, w) \sim \exp(-i2\pi w) \iint \frac{I_\nu(l, m)}{\sqrt{1 - m^2 - l^2}} \exp[-i2\pi(ul + vm - \underline{w\theta^2/2})] dldm$$

The factor $\exp[-i2\pi w\theta^2/2]$ can be kept close to unity if $w\theta^2 \ll 1$, that is by imaging a field of view whose radius $\theta \ll w^{-1/2} \sim (\lambda/b)^{1/2}$

For example, a $\theta \ll 0.01$ rad is sufficiently small for a baseline of $10^4\lambda$. Then

$$V(u, v) = \iint \frac{I_\nu(l, m)}{\sqrt{1 - m^2 - l^2}} \exp[-i2\pi(ul + vm)] dldm$$

Approximation ok in (sub)mm domain, problems start at wavelengths $> \text{cm}$.

A wider field image can be imaged with two-dimensional Fourier transforms by breaking it up into smaller facets, much like a fly's eye, and merging the facets to make the final image.



Interferometer field of view

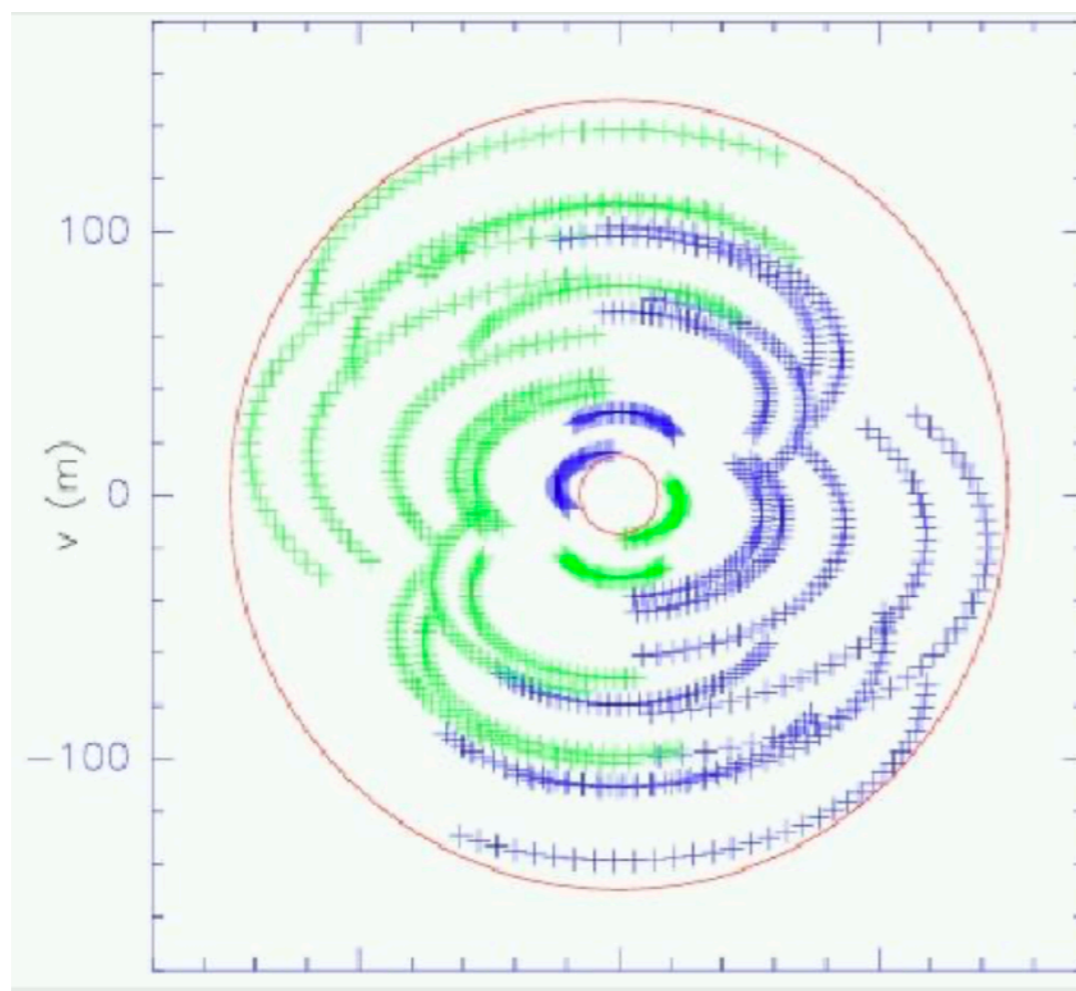
- Field of view is limited by
 - the **antenna primary beam**: the interferometer measures $A \times I$
 - the **2D visibility approximation**
 - the frequency averaging (bandwidth)
 - the time averaging (integration)

Values for Plateau de Bure

θ_s	ν (GHz)	2-D Field	0.5 GHz Bandwidth	1 Min Averaging	Primary Beam
5''	80	5'	80''	2'	60''
2''	80	3.5'	30''	45''	60''
2''	230	3.5'	1.5'	45''	24''
0.5''	230	1.7'	22''	12''	24''

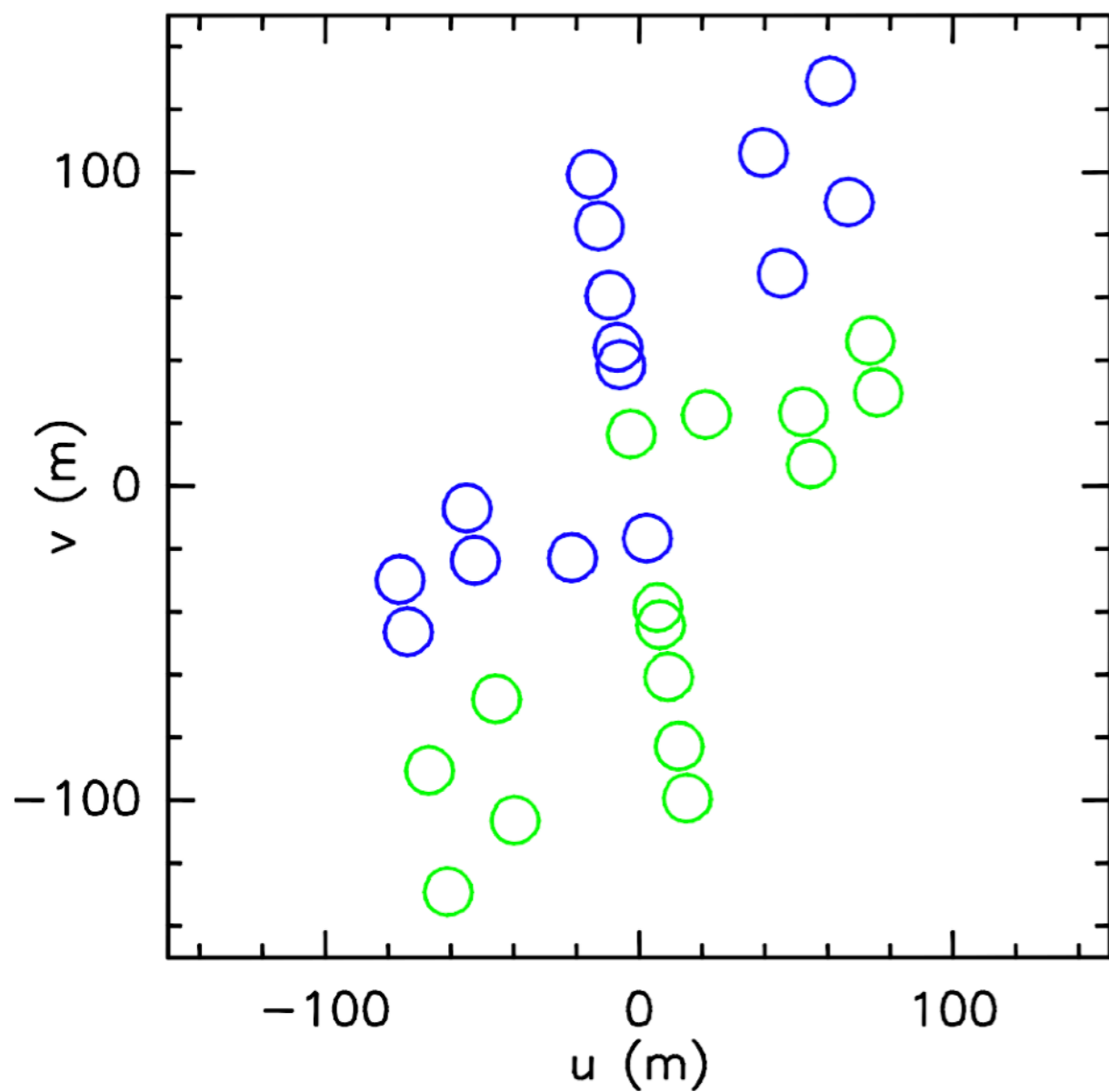
Measurements = uv plane sampling \times visibilities
After FT: dirty map = dirty beam $*$ (prim. beam \times sky)

The FT of the uv plane coverage gives the dirty beam = the PSF of the observations

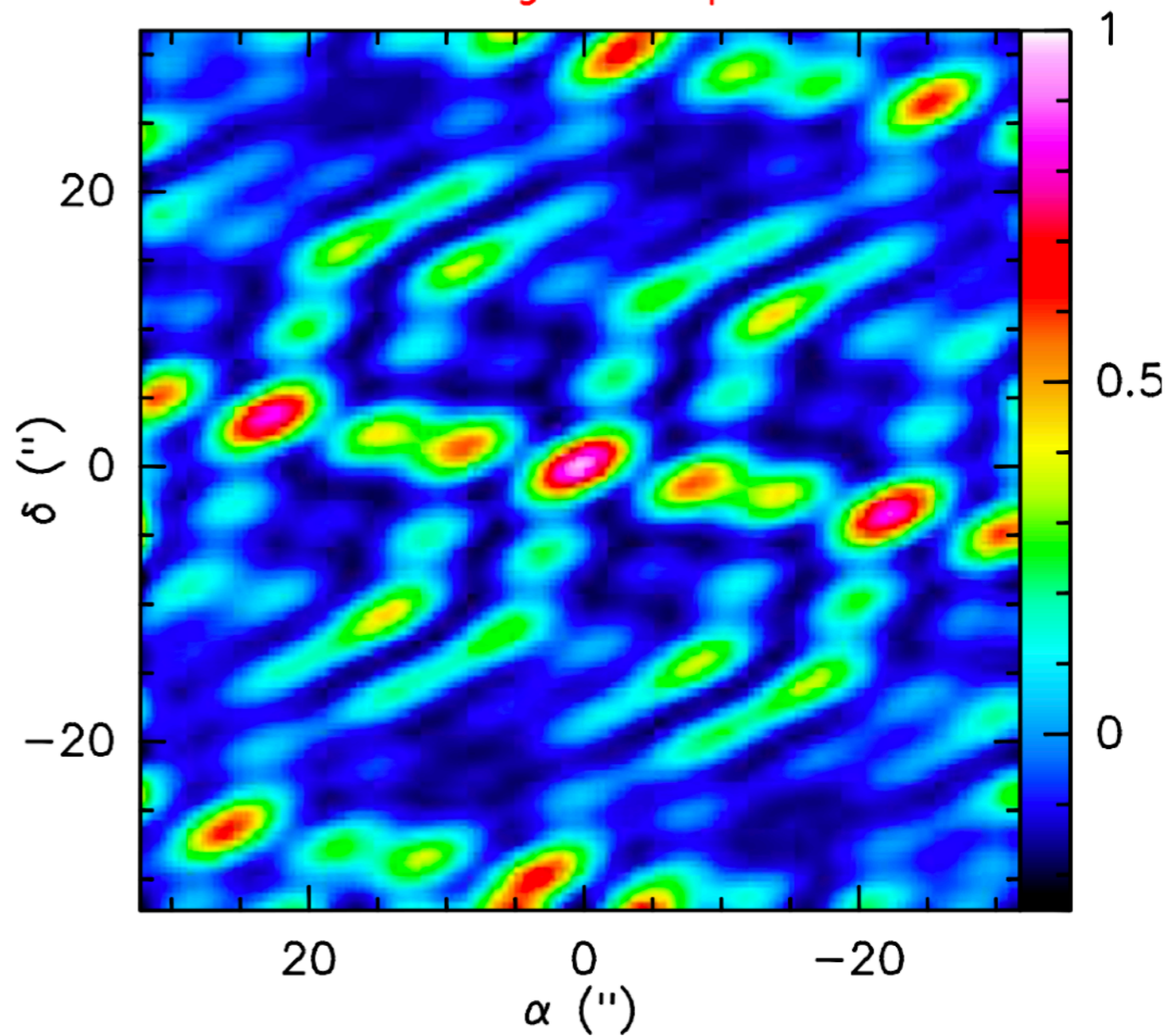


**Max. base-
line gives
the angular
resolution**

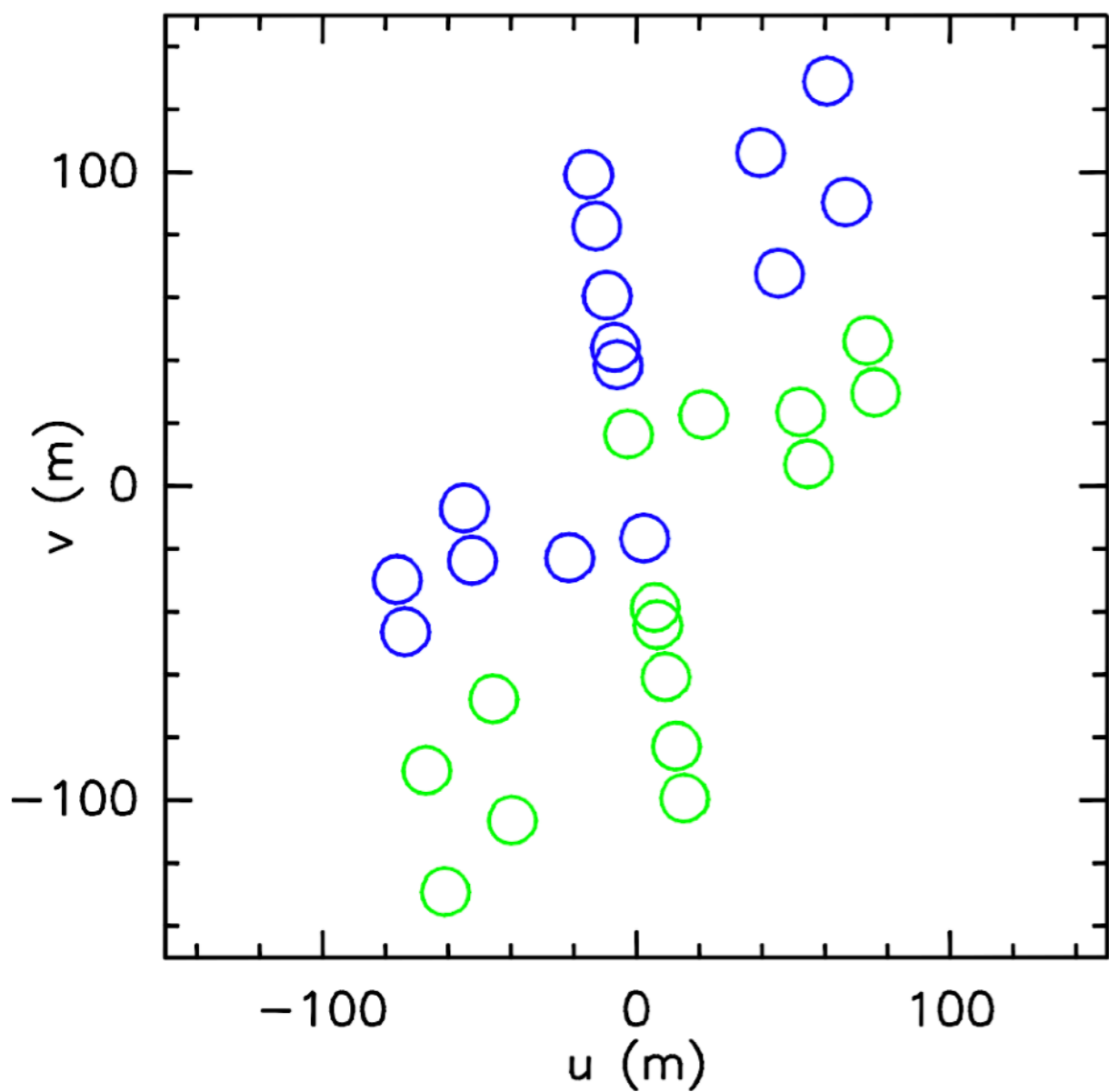
Sampling in the uv plane



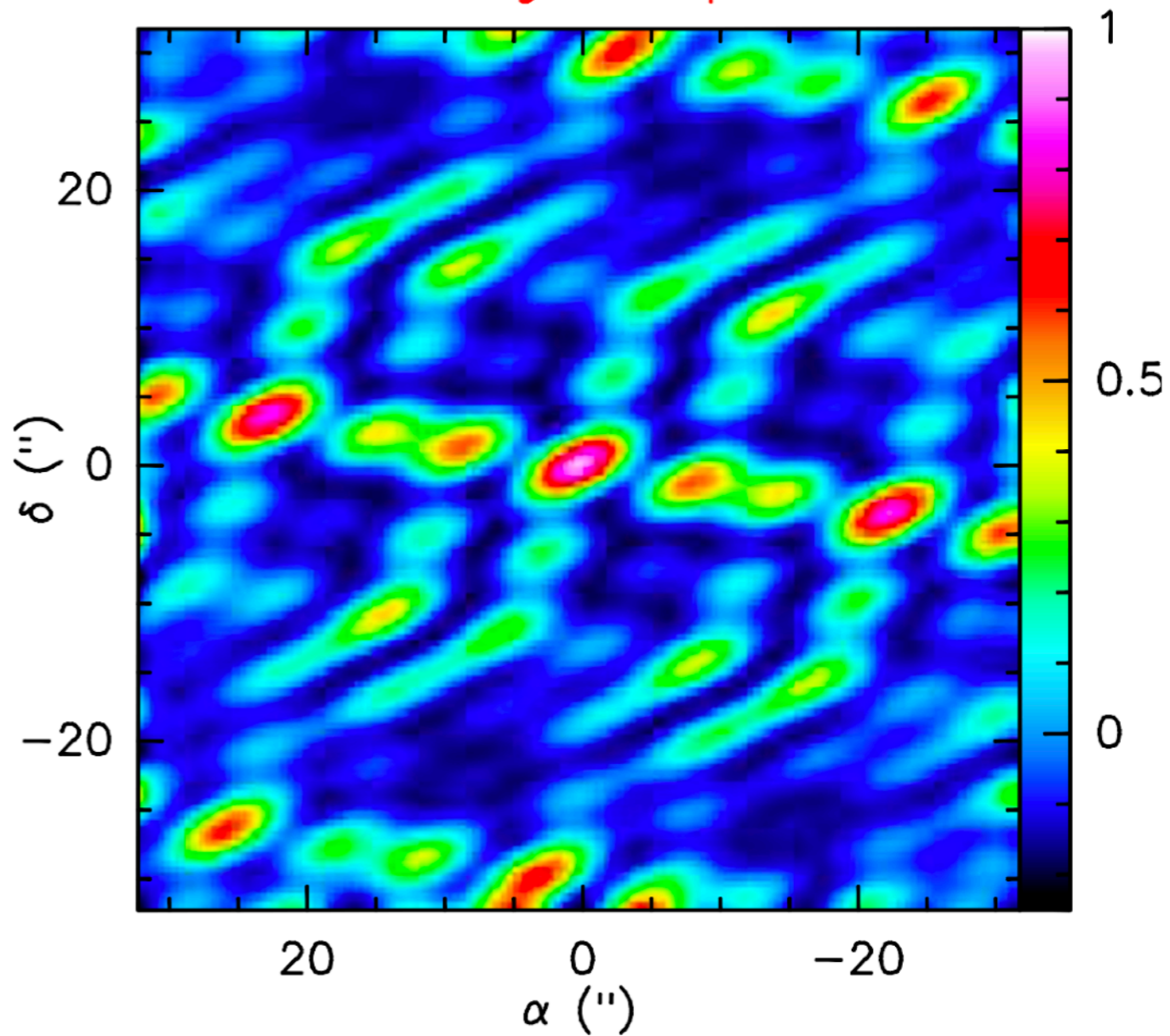
Associated image of a point source



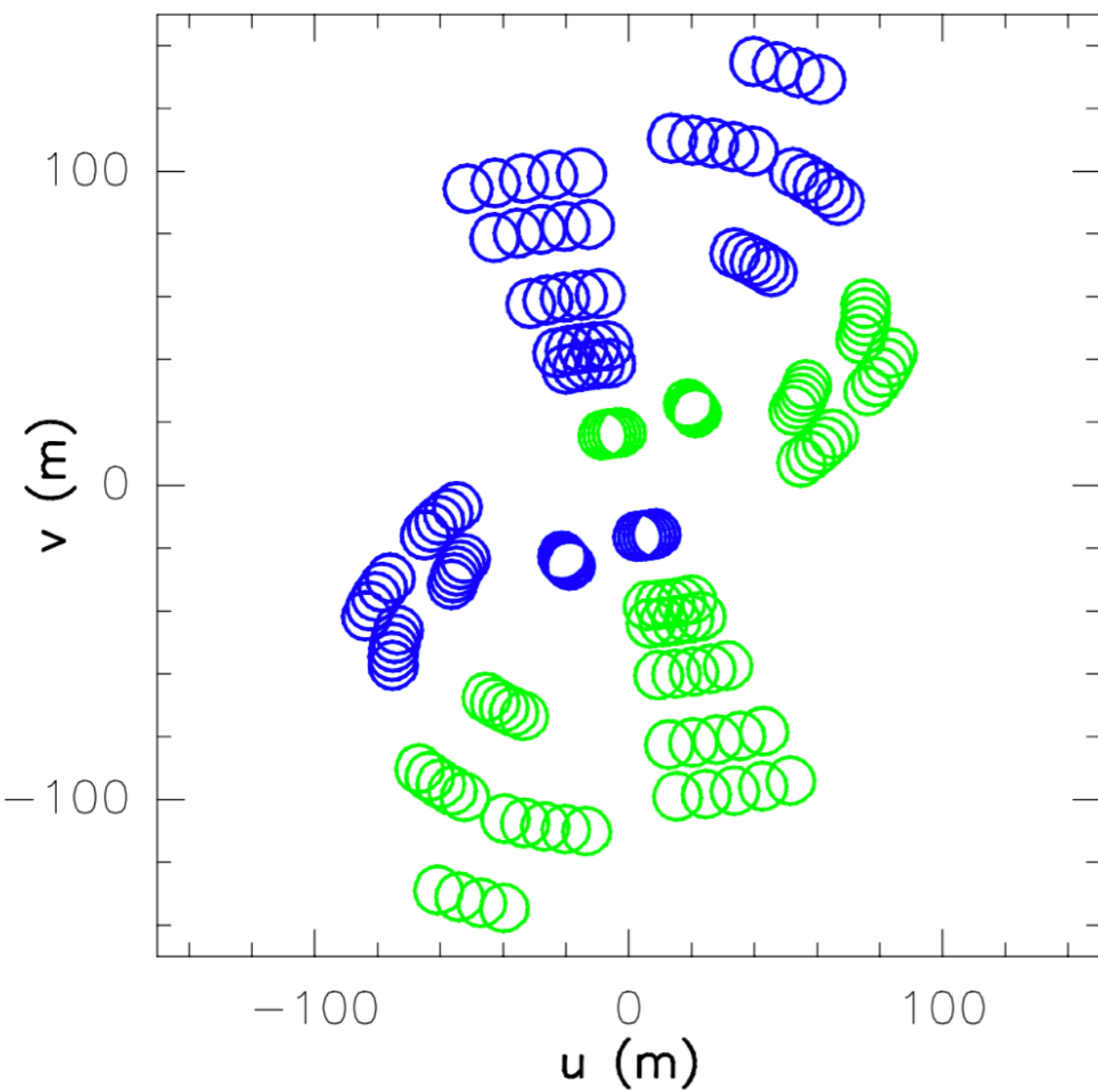
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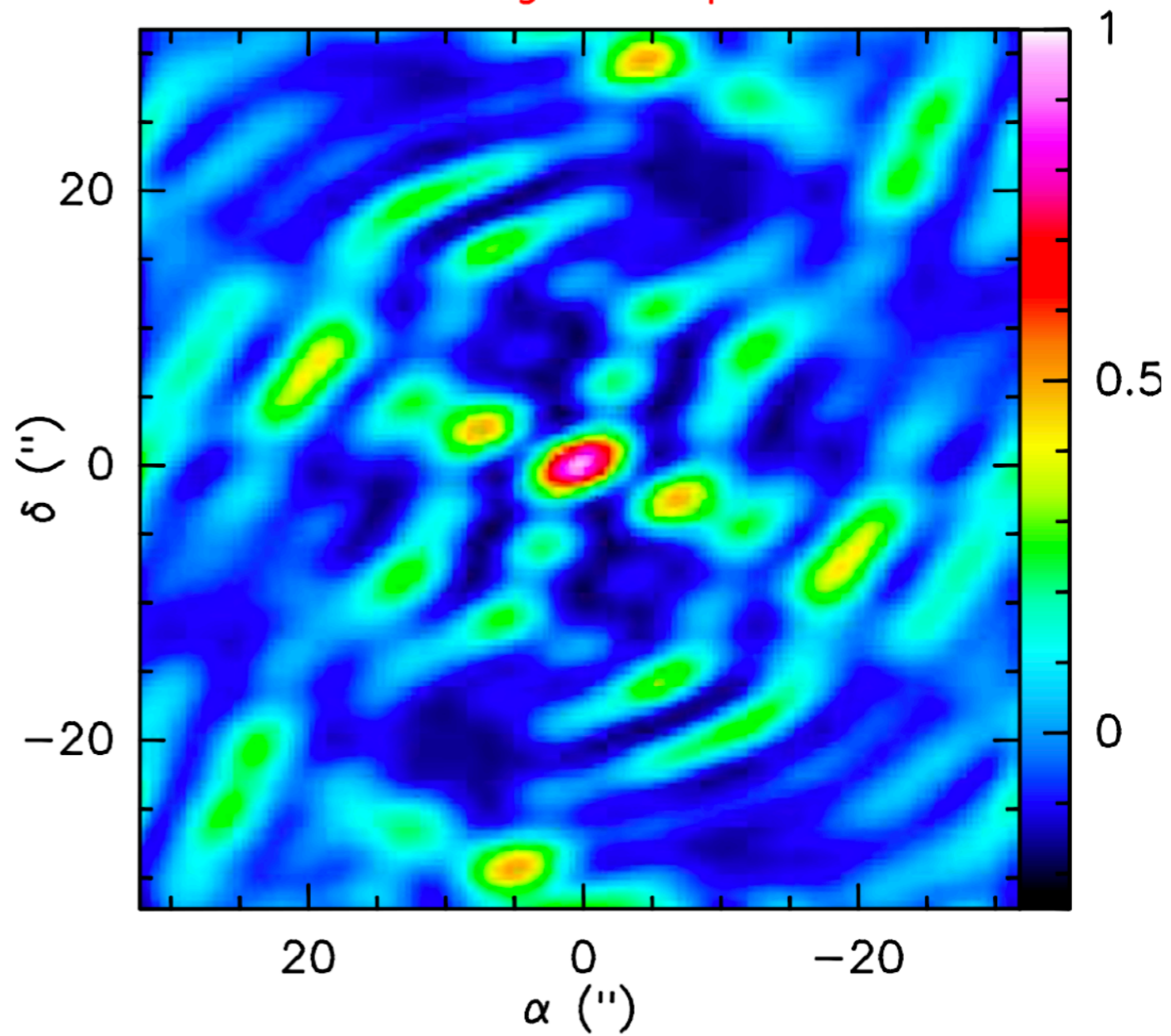
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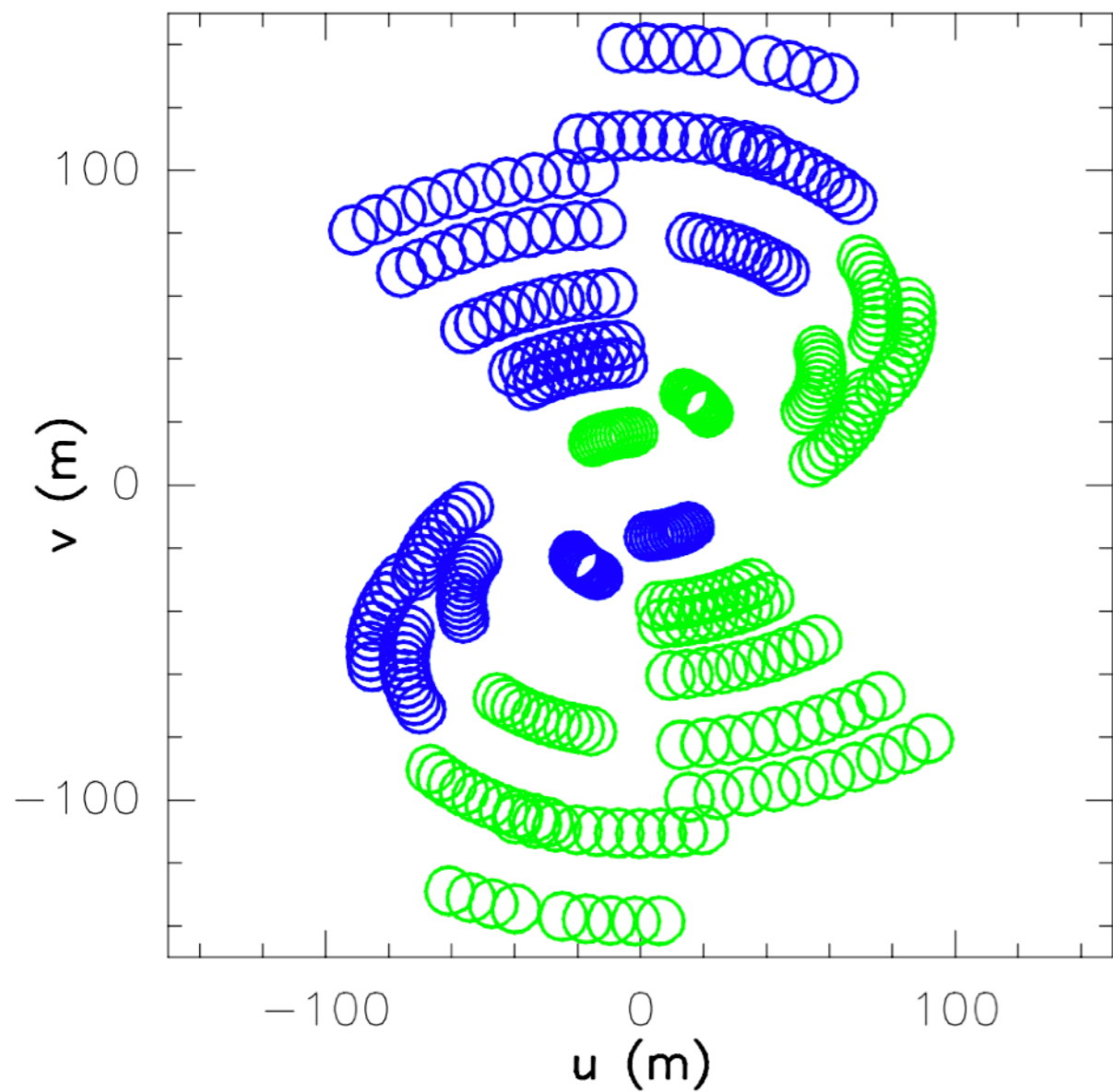
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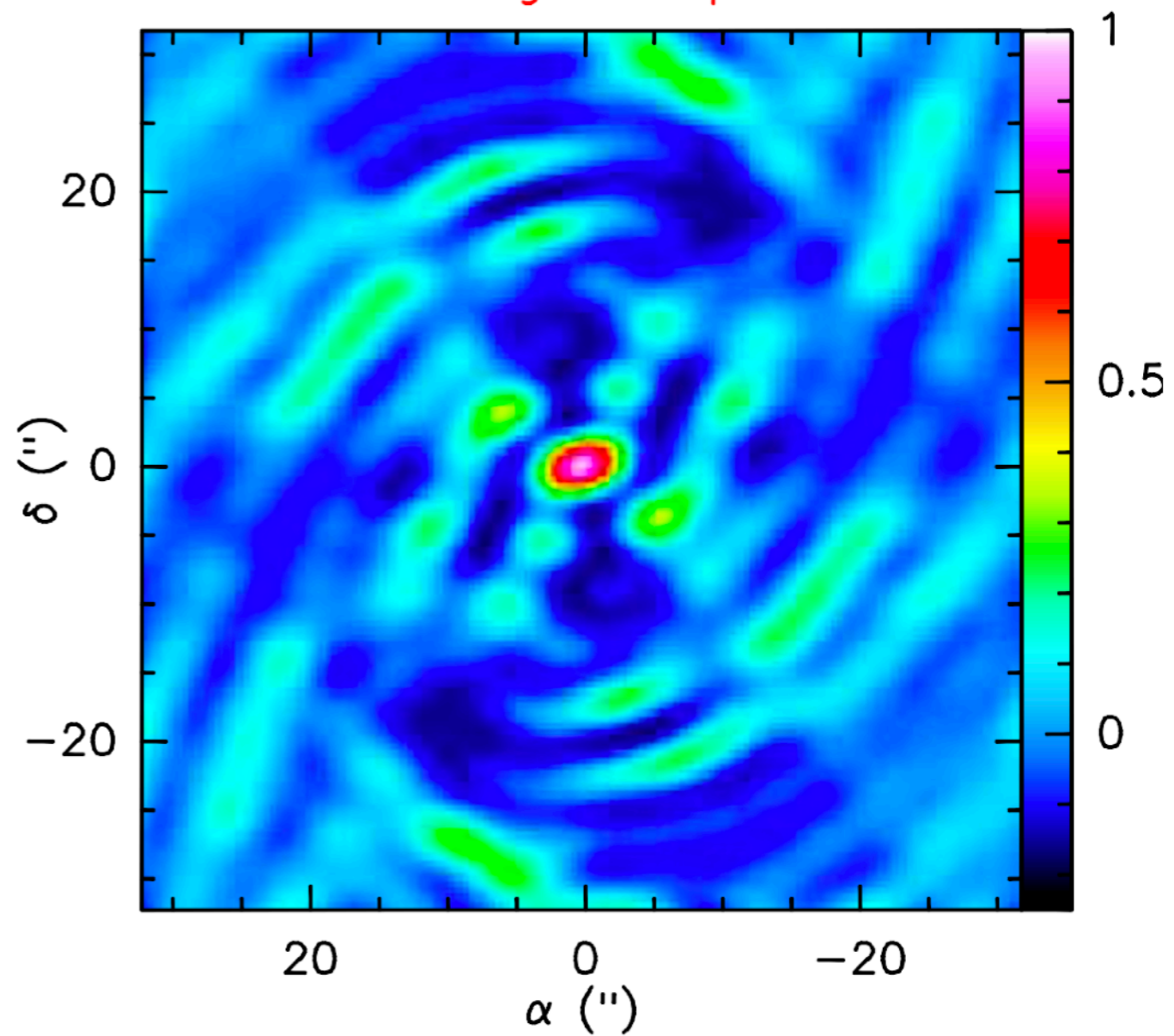
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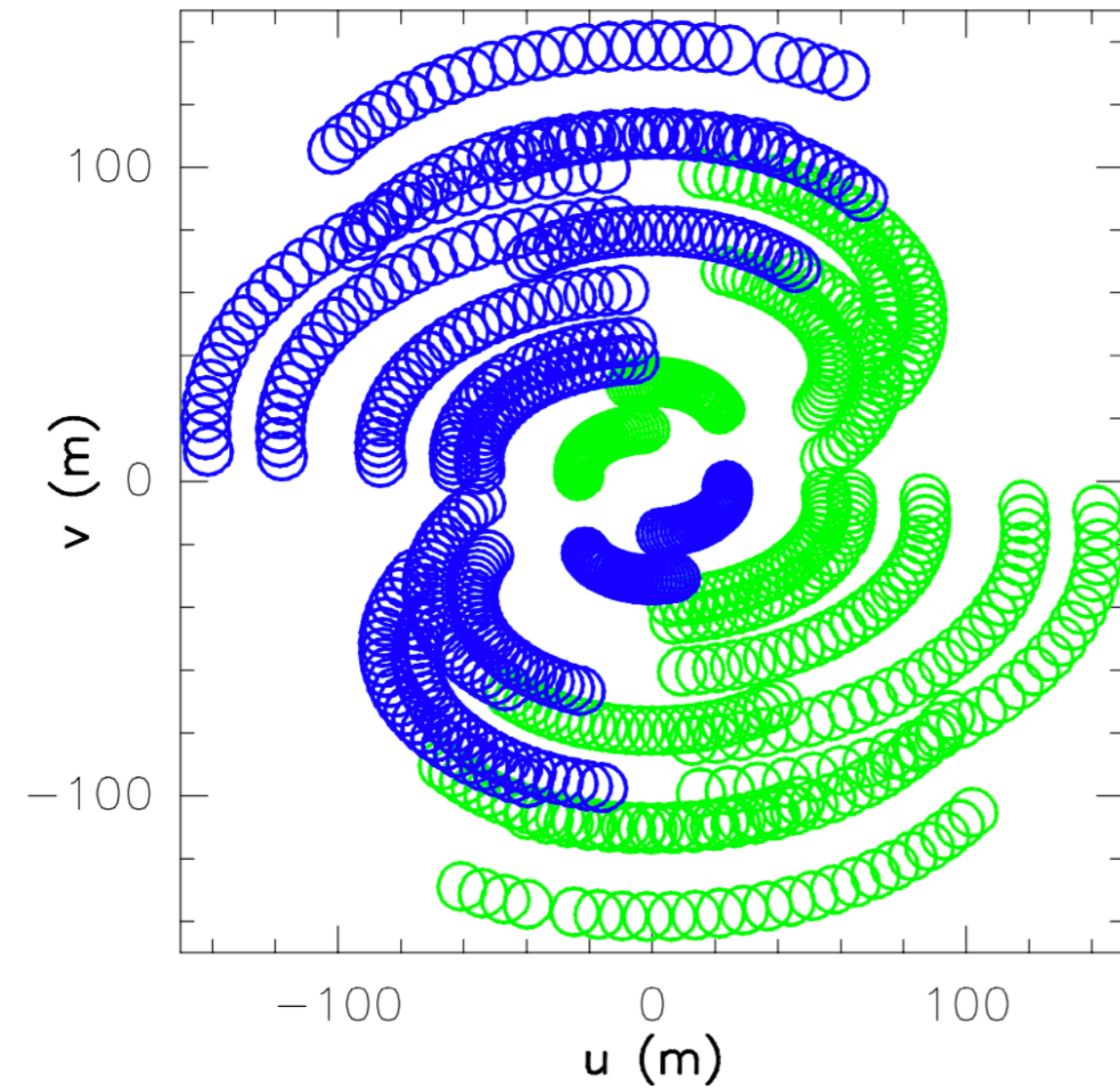
Sampling in the uv plane



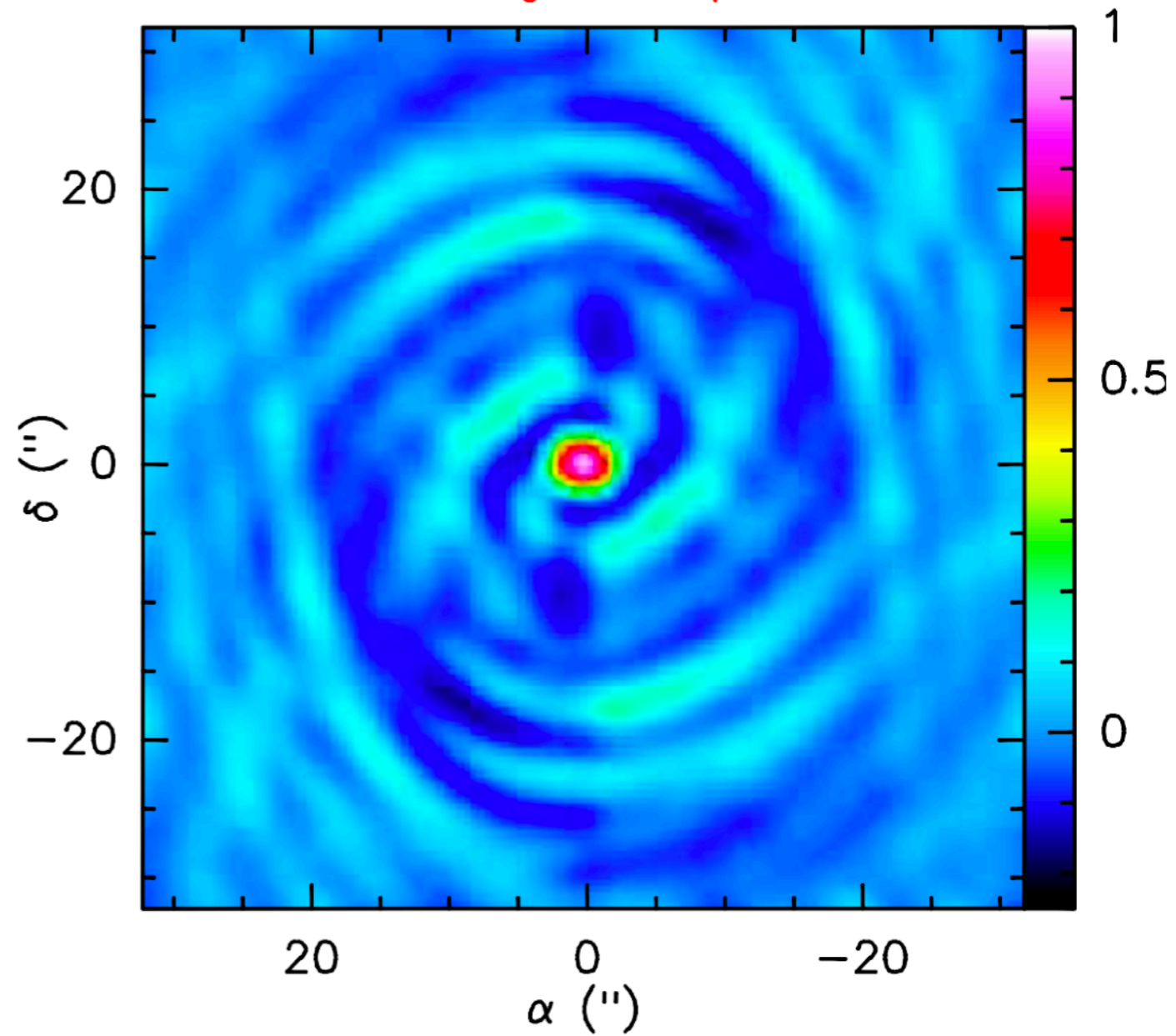
Associated image of a point source



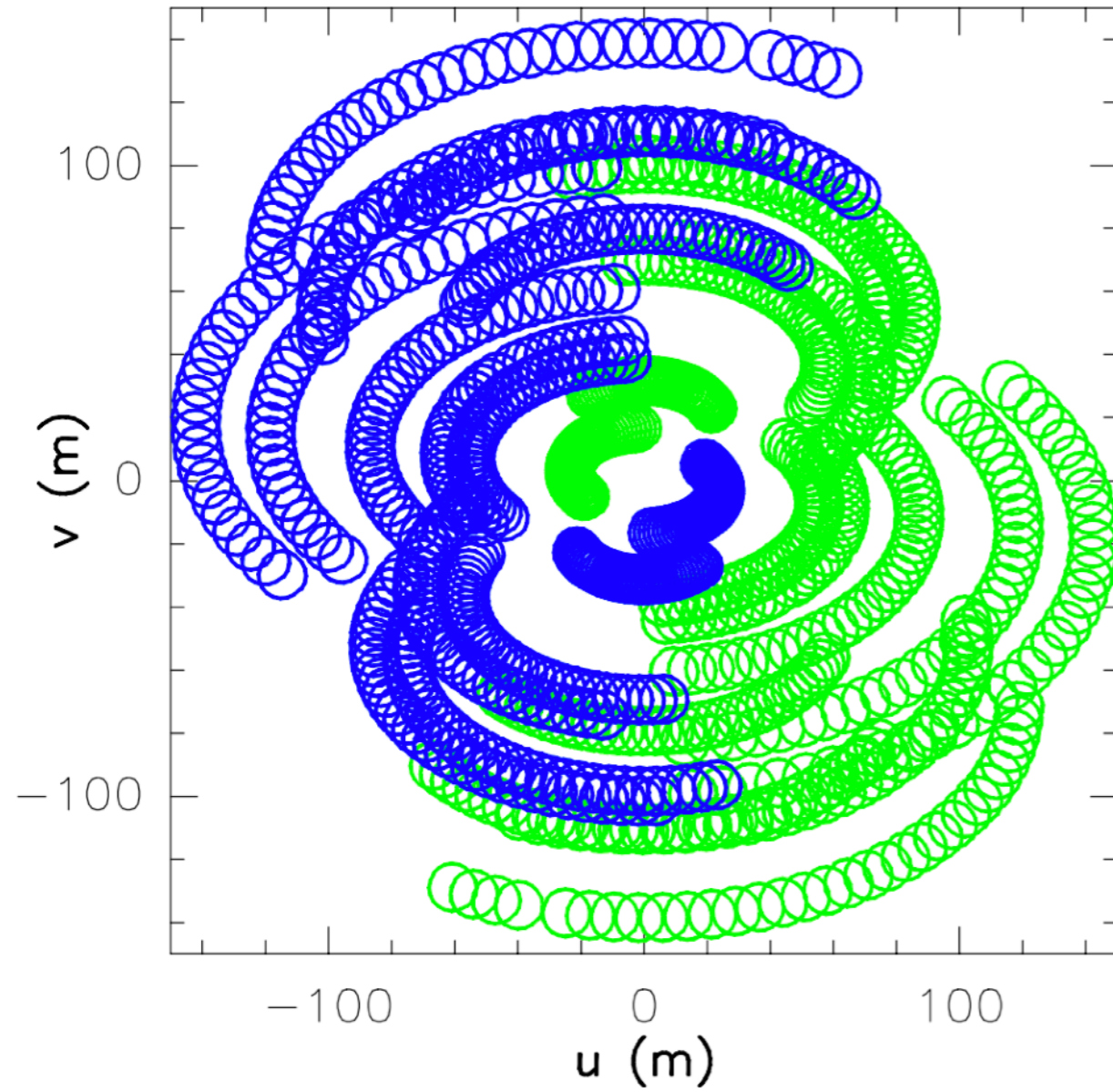
Sampling in the uv plane



Associated image of a point source



Sampling in the uv plane



Associated image of a point source

