We have introduced define the **complex visibility** V from the two independent (real) correlator outputs R_c and R_s as:

$$\boxed{V = R_c - iR_s = |V|e^{i\phi}}$$

R_c cosine response (even)

R_s sine response (odd)

where $|V| = \sqrt{R_c^2 + R_s^2}$ is the visibility amplitude

$\phi = tan^{-1}(R_s/R_c)$ is the visibility phase

We have seen the relation between the source brightness and the response of an interferometer:

$$\boxed{V = \int_{\Omega_{\text{source}}} I_\nu(\hat{s}) e^{-2\pi i\nu \vec{b}\cdot\hat{s}/c} d\Omega}$$

$$\tau_g = \frac{\vec{b}\cdot\hat{s}}{c}$$

geometric delay

which is a 2D Fourier transform, giving as a well established way to recover $I_\nu(\hat{s})$ from *V.*

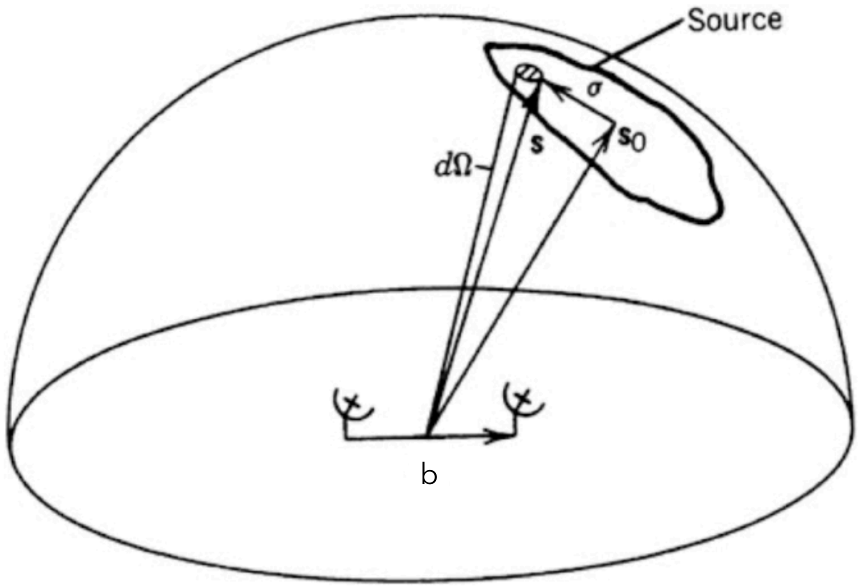
This relation is valid in the ideal case of an interferometer with isotropic response and in the quasi-monochromatic approximation.

Visibility-brightness relation for a realistic antenna response

We know consider the relation between source brightness and complex visibility (for an extended source) in the case of a realistic (direction dependent) antenna response, in the quasi-monochromatic approximation. We have:

$$\boxed{V = \int_{\Omega_{\rm source}} A(\hat{s}) I_\nu(\hat{s}) e^{-2\pi i \nu \overrightarrow{b} \cdot \hat{s}/c} d\Omega}$$

where $A(\hat{s})$ is the effective area of each (identical) antenna.



Typically, source positions are expressed as with respect to the phase-center: $\hat{s} = \hat{s}_0 + \hat{\sigma}$

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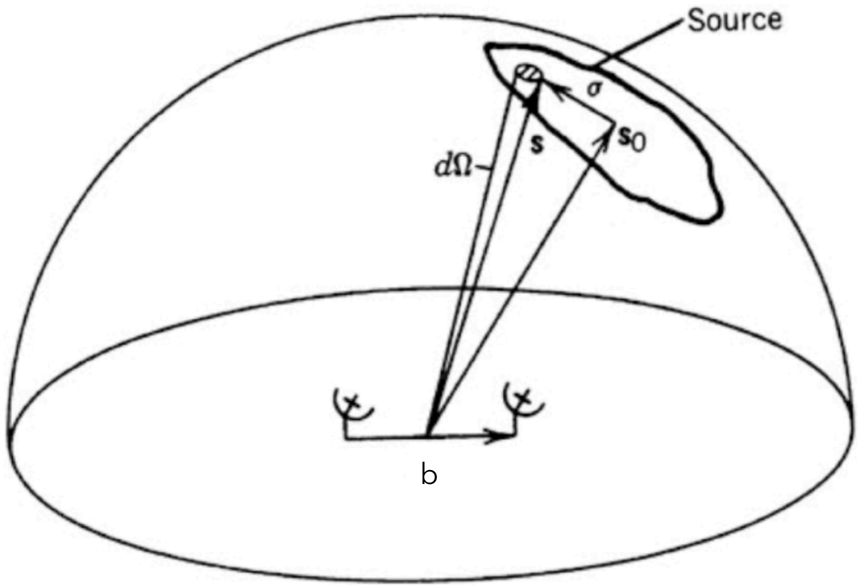

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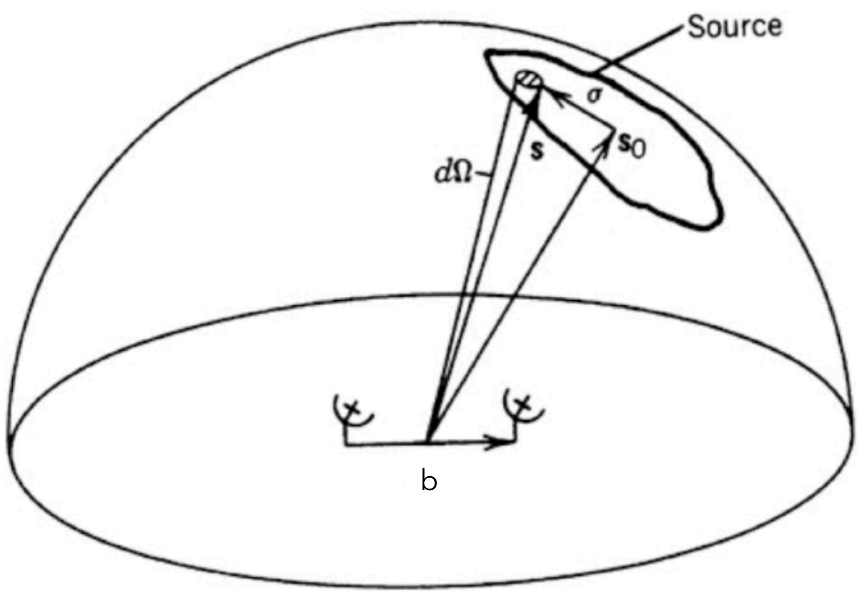
Let's now express the correlator response as a function of the visibility….

Correlator response and visibility

We have seen that the correlator response in the case of a realistic (direction dependent) antenna response, in the quasi-monochromatic approximation, is:



$$R = \int_{\Omega_{\rm source}} A(\hat{s}) I_\nu(\hat{s}) cos(2\pi\nu \vec{b} \cdot \hat{s}/c) d\Omega$$

$$= cos\left(2\pi\nu\frac{\vec{b}\cdot\hat{s}_0}{c}\right)\int_{\Omega_{\rm source}} A(\hat{\sigma}) I_\nu(\hat{\sigma}) cos(2\pi\nu \vec{b} \cdot \hat{\sigma}/c) d\Omega$$

$$-sin\left(2\pi\nu\frac{\vec{b}\cdot\hat{s}_0}{c}\right)\int_{\Omega_{\rm source}} A(\hat{\sigma}) I_\nu(\hat{\sigma}) sin(2\pi\nu \vec{b} \cdot \hat{\sigma}/c) d\Omega =$$

$$= cos\left(2\pi\nu\frac{\vec{b}\cdot\hat{s}_0}{c}\right)|V|cos\phi - sin\left(2\pi\nu\frac{\vec{b}\cdot\hat{s}_0}{c}\right)|V|sin\phi =$$

$$= |V| cos(2\pi\nu\tau_g - \phi)$$

Note:

We do not measure $r = FT(I)$

We measure *R* = something related to *V*, which resembles the $FT(I)$

Complex visibility for a finite bandwidth interferometer

The definition of visibility for quasi-monochromatic interferometers may be generalized to interferometers with finite bandwidths and integration times, which are necessary for high sensitivity.

$$\boxed{V = \int_{\Omega_{\text{source}}} A(\hat{s}) I_\nu(\hat{s}) e^{-2\pi i \nu \tau_g} d\Omega}$$

quasi-monochromatic

$$\boxed{V = \int_{\Omega_{\text{source}}} \left(\int_{\nu_0 - \Delta\nu/2}^{\nu_0 + \Delta\nu/2} A(\hat{s}) I_\nu(\hat{s}) e^{-2\pi i \nu \tau_g} d\nu \right) d\Omega}$$

finite bandwidth $\Delta\nu$ centered on frequency ν_0

integral over the observation bandwidth

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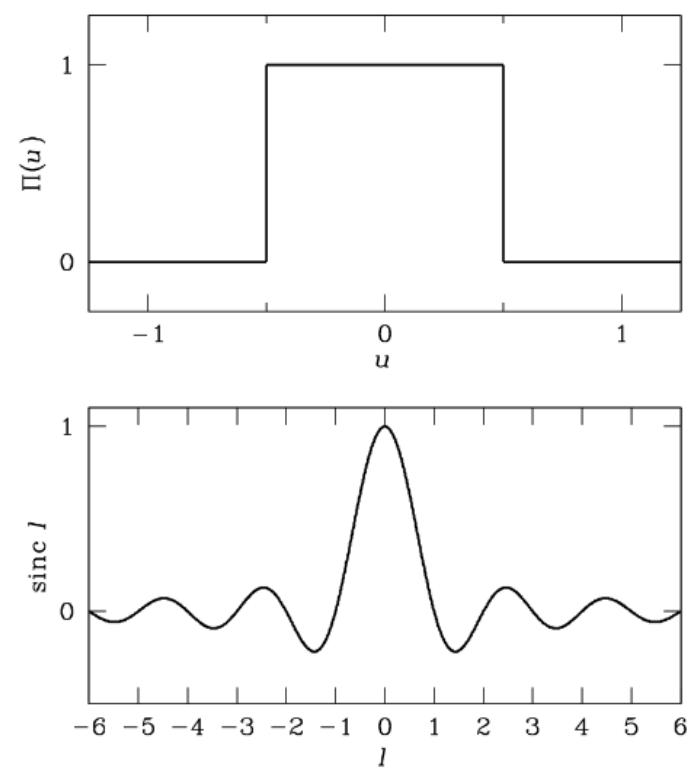
finite bandwidth $\Delta\nu$ centered on frequency ν_0

integral over the observation bandwidth

If the source brightness and response of the interferometer are ~constant over $\Delta\nu$, the integral over frequency is just the FT of a rectangle function:

$$\boxed{V \simeq \int A(\hat{s}) I_\nu(\hat{s}) sinc(\Delta\nu\tau_g) e^{-2\pi i \nu \tau_g} d\Omega}$$

For a finite bandwidth $\Delta\nu$ and delay τ_g, the fringe amplitude is attenuated by a factor $sinc(\Delta\nu\tau_g)$. This effect is called bandwidth smearing.



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$$\boxed{R = |V| cos(2\pi\nu_0\tau_g - \phi) \frac{sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g}}$$

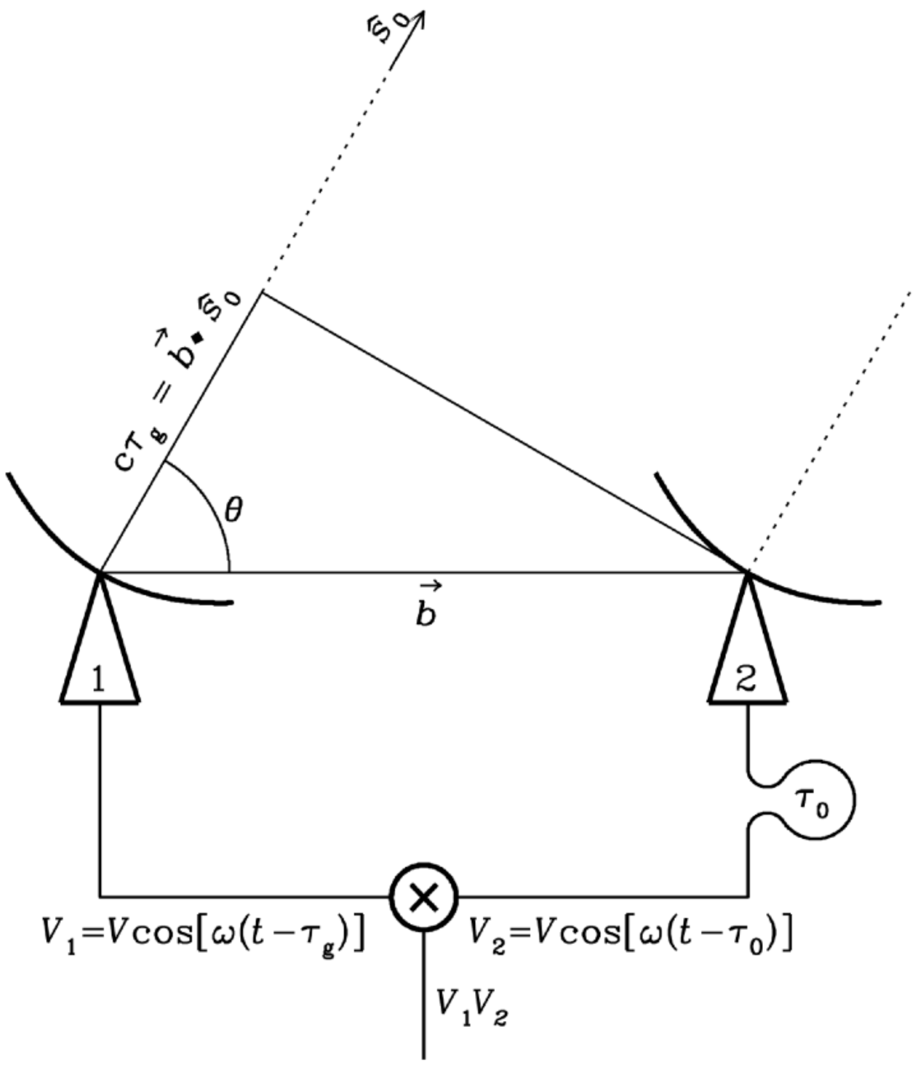
Correlator response

Complex visibility for a finite bandwidth interferometer

This attenuation can be eliminated <u>in a given direction</u> $\hat{s}_0$ (called the phase reference position or phase center) by introducing a compensating delay $\tau_0 \sim \tau_g$ in the signal path of the "leading" antenna.

This is usually done with digital electronics by introducing an instrumental delay in the correlator. As Earth rotates, τ_0 must be continuously adjusted so that $|\tau_0 - \tau_g| << \Delta\nu$. If the delay is compensated, one can measure $R = |V| cos\phi$

However: τ_g varies with direction, so delay compensation can be exact in only one direction.



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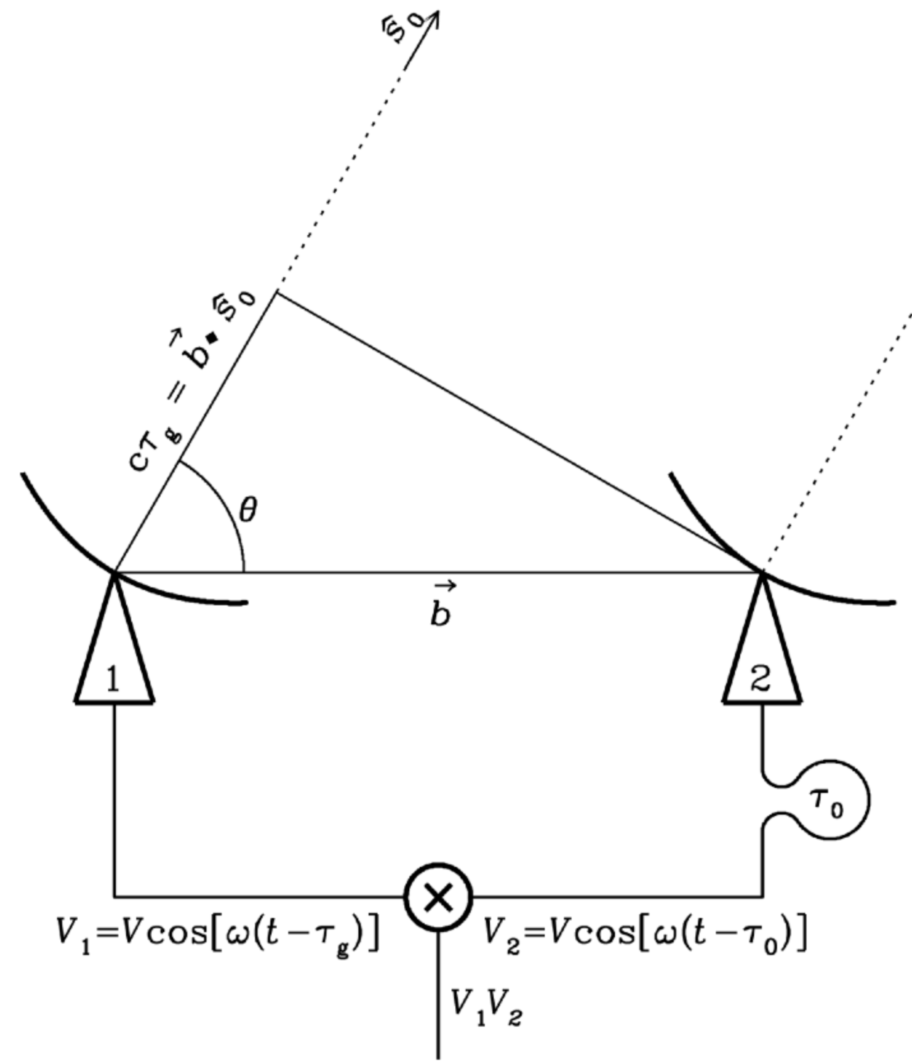

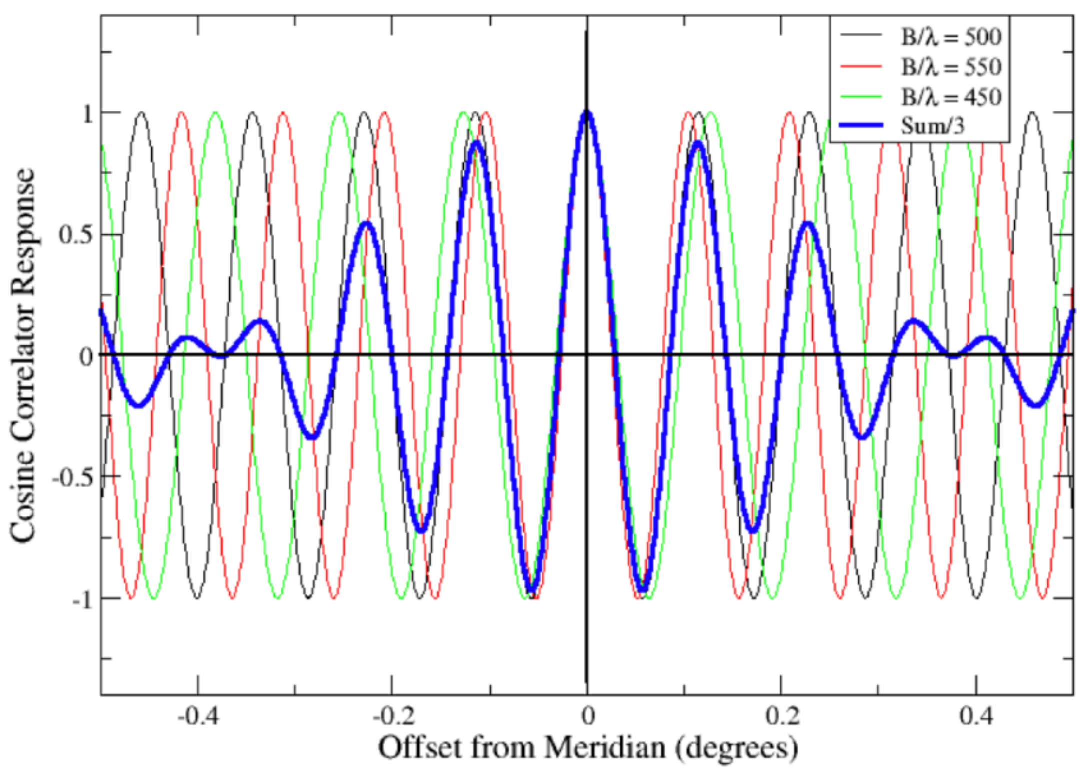

Example: three wavelength components for the same physical baseline. For each λ, the separation between the fringes is $\sim \lambda / b$. They get increasingly out of step at increasing offset from the phase center

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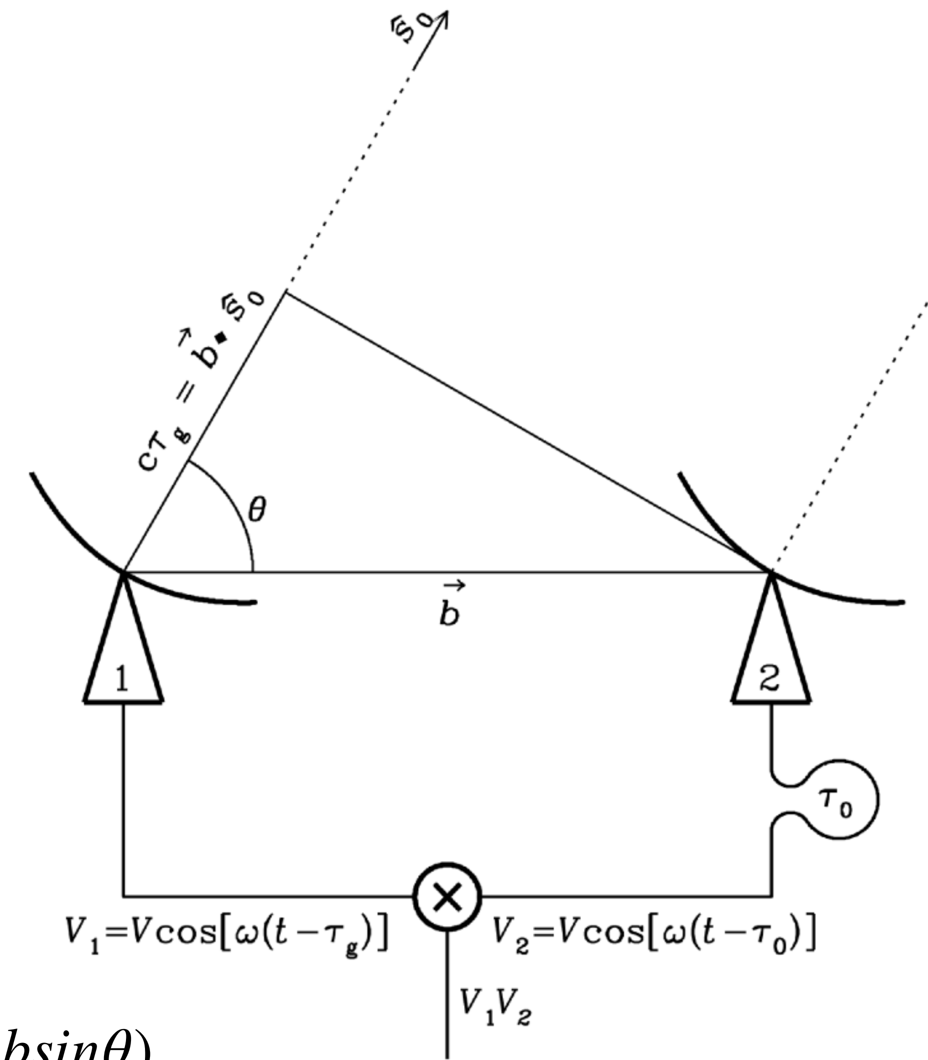


The angular radius $\Delta\theta$ of the "usable" field within which the bandwidth smearing is small can be derived by requiring that $\Delta\nu\Delta\tau_g << 1$

which implies $\boxed{\dfrac{\Delta\theta}{\theta_s} << \dfrac{\nu}{\Delta\nu}}$ *

using $|c\Delta\tau_g| = bsin\theta\Delta\theta$ and the synthesized beamwidth $\theta_s \sim \lambda/(bsin\theta)$

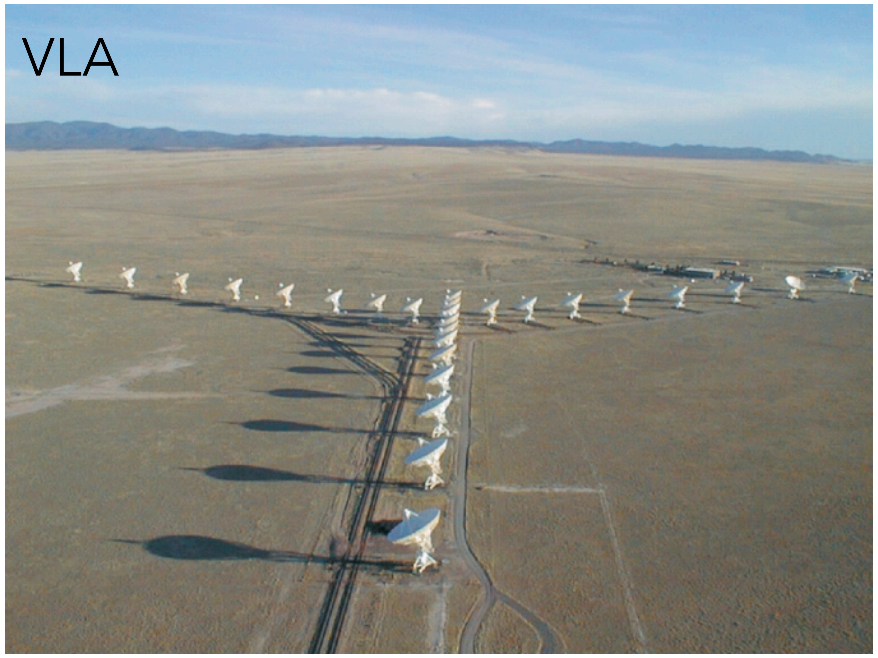
At larger angular offsets from the phase reference position, bandwidth smearing will broaden the synthesized beam by convolving it with a rectangle function of angular width $\Delta\theta\Delta\nu/\nu$.

This is why wide field images (large $\Delta\theta$) can be made only by dividing the bandwidth into a number of narrower frequency channels, each satisfying *

Complex visibility for a finite bandwidth

How small does our bandwidth need to be, in order to ensure little bandwidth smearing within a given offset $\Delta\theta$?

$$\boxed{\frac{\Delta\theta}{\theta_s} << \frac{\nu}{\Delta\nu}}$$



For example, the synthesized beam of the VLA "B" configuration (maximum b~10 km) at $\lambda \sim 20$ cm is $\theta_s \sim 0.2/10^4$ rad ~ 4 arcsec.

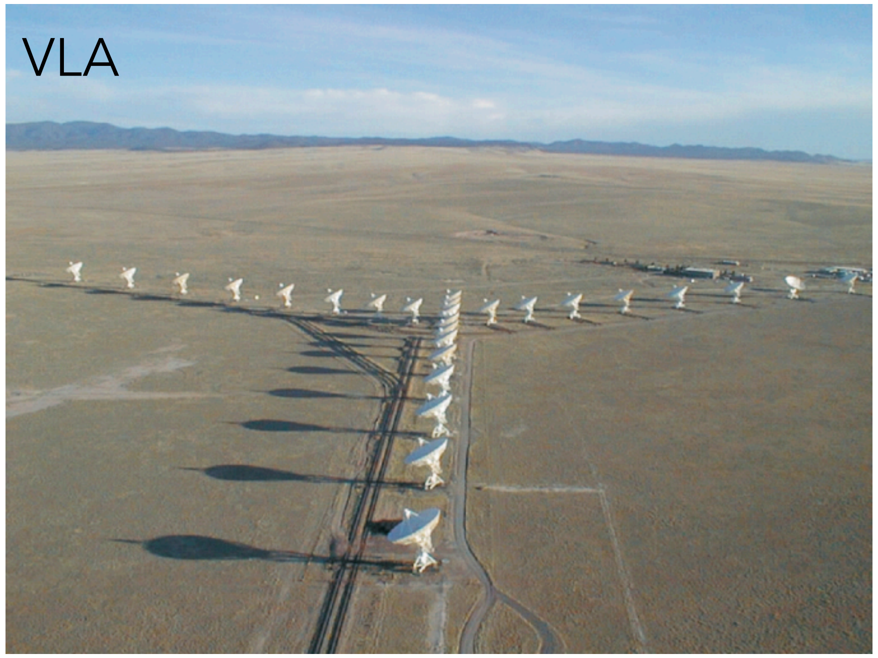
We want to image an angular radius $\Delta\theta$=15 arcmin = 900 arcsec, corresponding to ~$\theta_p/2$ (primary beam) of the VLA.

This imply a bandwidth $\Delta\nu << \frac{\nu\theta_s}{\Delta\theta} = \frac{1.5 \times 10^9\text{Hz} \cdot 4 \text{ arcsec}}{900 \text{ arcsec}} \sim 7\text{MHz}$

Complex visibility for a finite bandwidth and averaging times

How small does our bandwidth need to be, in order to ensure little bandwidth smearing within a given offset $\Delta\theta$?

$$\boxed{\frac{\Delta\theta}{\theta_s} << \frac{\nu}{\Delta\nu}}$$



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Additionally, the synthesized beam might be broadened by time smearing, due to excessively large correlator averaging times. To avoid this, the correlator averaging time Δt must be short enough that Earth's rotation will not significantly change the source position in the frame of the interferometer.

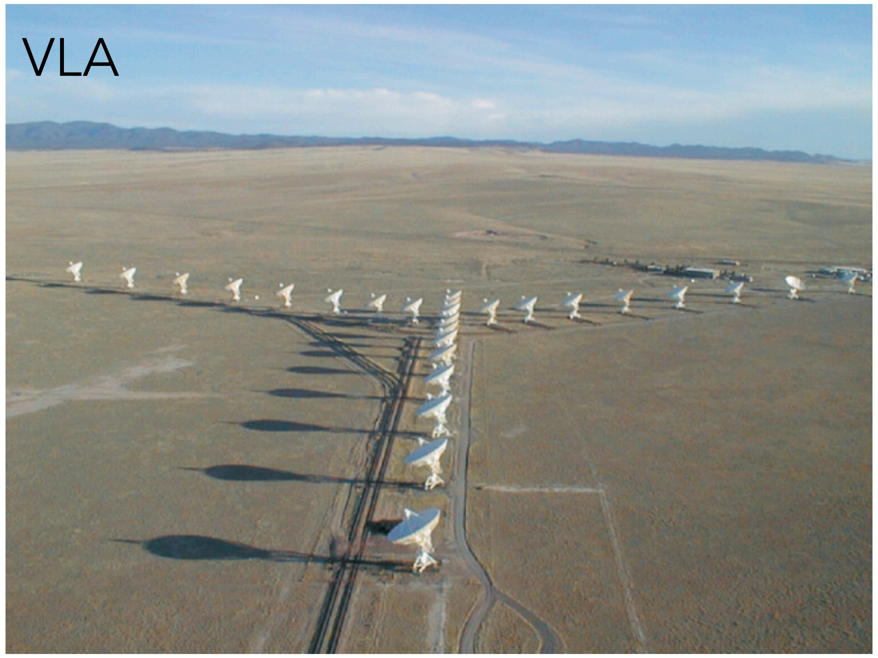
If the phase center is set to track the celestial pole, a source $\Delta\theta$ away from the celestial pole will move with an angular rate $2\pi\Delta\theta/P$ [rad/s], where P is the Earth's sidereal rotation period.

Complex visibility for a finite bandwidth and averaging times

How small does our averaging time need to be, in order to ensure little time smearing in an image of angular radius $\Delta\theta$?

$$\frac{2\pi\Delta\theta}{P}\Delta t << \theta_s \qquad P \sim 10^4 \text{ s}$$

$$\boxed{\frac{\Delta t}{1.4\times 10^4\text{s}} << \frac{\theta_s}{\Delta\theta}}$$



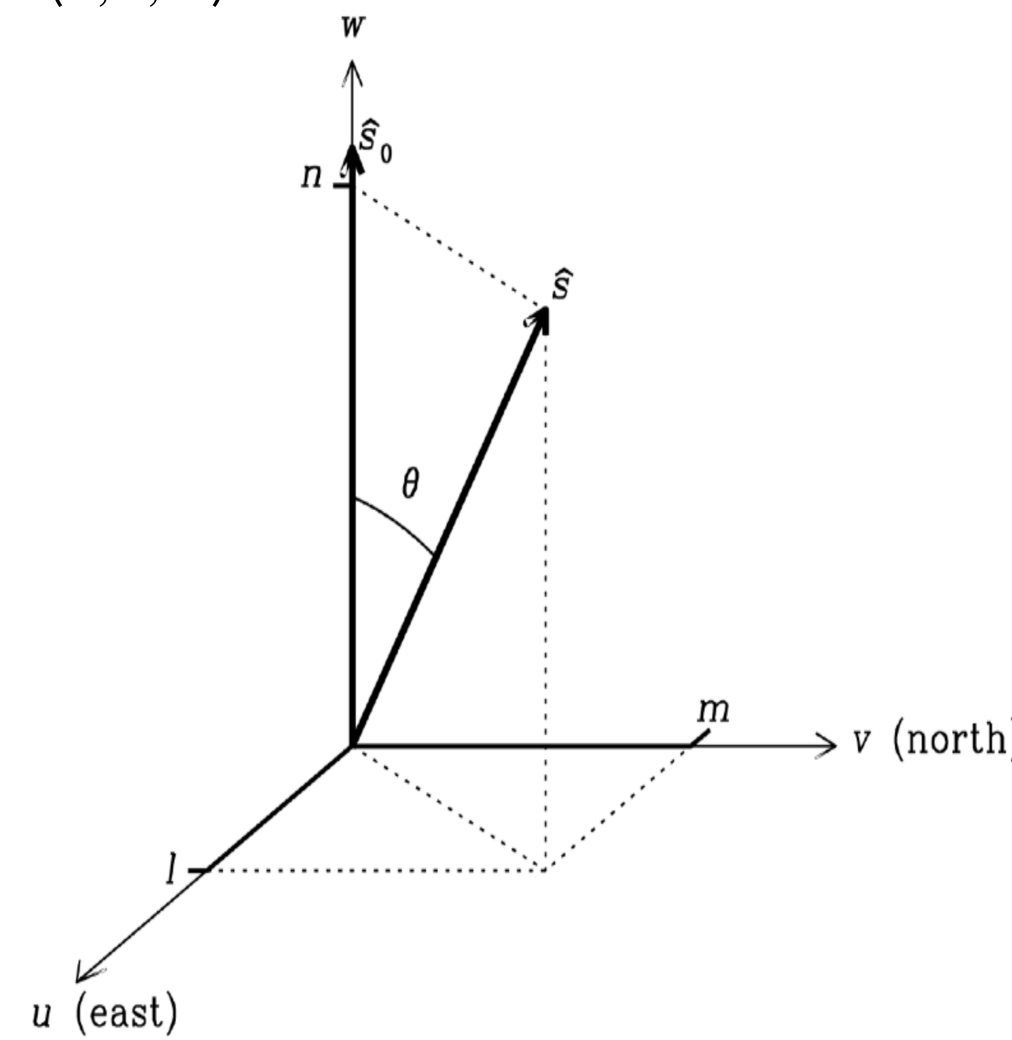
Again, for the VLA "B" configuration (maximum b~10 km) at $\lambda \sim 20$ cm we have $\theta_s \sim 0.2/10^4$ rad ~ 4 arcsec

We want to image an angular radius $\Delta\theta$=15 arcmin = 900 arcsec, corresponding to ~$\theta_p/2$ (primary beam) of the VLA.

$$\Delta t << \frac{\theta_s}{\Delta\theta}\cdot 1.4\times 10^4\text{s} = \frac{4 \text{ arcsec}}{900 \text{ arcsec}}\cdot 1.4\times 10^4 \text{ s} \sim 60\text{s}$$

Interferometers in three dimensions

In general a baseline $\overrightarrow{b}$ is described by a set of three coordinates (u, v, w). The w-axis is in the reference direction $\hat{s}_o$ usually chosen to point the source.



The u and v axes point east and north in the (u, v) plane normal to the w-axis. u,v,w are the components of $\overrightarrow{b}/\lambda$ in wavelength units.

An arbitrary unitary vector $\hat{s}$ has components (l, m, n), where $n = cos\theta = \sqrt{1 - m^2 - l^2}$

Because $d\Omega = \dfrac{dldm}{\sqrt{1 - m^2 - l^2}}$

the three-dimensional generalization of the complex visibility definition $V = \int_{\Omega_{\text{source}}} A(\hat{s})I_\nu(\hat{s})e^{-2\pi i\nu\tau_g}d\Omega$ is

$$V(u, v, w) = \int\int \frac{I_\nu(l, m)}{\sqrt{1 - m^2 - l^2}} exp[-i2\pi(ul + vm + wn)]dldm$$

which is NOT a FT unless $w = 0$

Interferometers in three dimensions

$$V(u,v,w) = \int\int \frac{I_\nu(l,m)}{\sqrt{1-m^2-l^2}} exp[-i2\pi(ul+vm+wn)]dldm$$

For any interferometer, if we consider only directions close to $\hat{s}_o$ we have $n = cos\theta \sim 1-\theta^2/2$ and

$$V(u,v,w) \sim exp(-i2\pi w)\int\int \frac{I_\nu(l,m)}{\sqrt{1-m^2-l^2}} exp[-i2\pi(ul+vm-w\theta^2/2)]dldm$$

The factor $exp[i2\pi w\theta^2/2]$ can be kept ~1 if $w\theta^2 << 1$, that is by imaging a field of view whose radius $\theta << w^{-1/2} \sim (\lambda/b)^{1/2}$

For example, a $\theta << 0.01$ rad is sufficiently small for a baseline of $10^4\lambda$. Then

$$\boxed{V(u,v) = \int\int \frac{I_\nu(l,m)}{\sqrt{1-m^2-l^2}} exp[-i2\pi(ul+vm)]dldm}$$

<u>Approximation ok in (sub)mm domain, problems start at wavelengths > cm</u>.
A wider field image can be imaged with two-dimensional Fourier transforms by breaking it up into smaller facets, much like a fly's eye, and merging the facets to make the final image.

Interferometer field of view

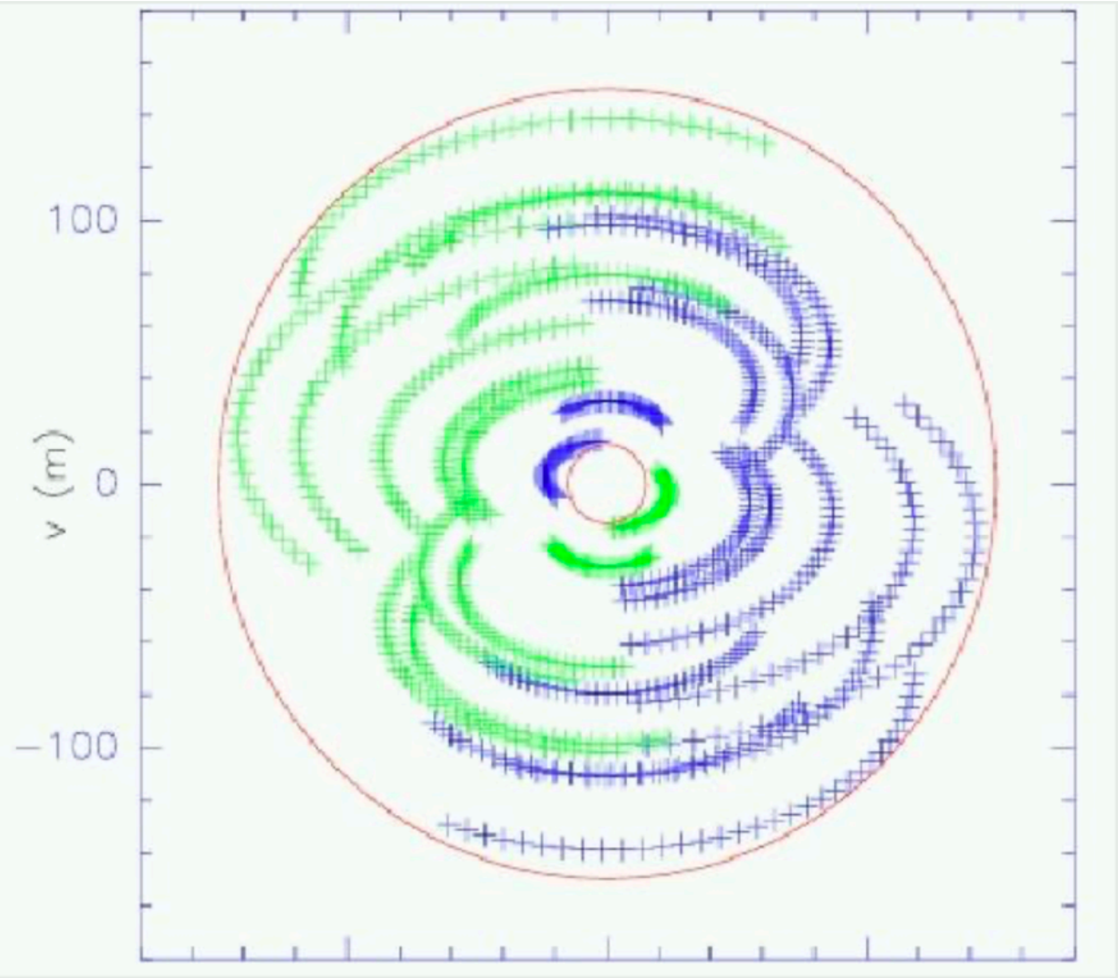
- Field of view is limited by
  - the **antenna primary beam**: the interferometer measures $A \times I$
  - the **2D visibility approximation**
  - the frequency averaging (bandwidth)
  - the time averaging (integration)

Values for Plateau de Bure

θ_s	ν (GHz)	2-D Field	0.5 GHz Bandwidth	1 Min Averaging	Primary Beam
$5''$	80	$5'$	$80''$	$2'$	$60''$
$2''$	80	$3.5'$	$30''$	$45''$	$60''$
$2''$	230	$3.5'$	$1.5'$	$45''$	$24''$
$0.5''$	230	$1.7'$	$22''$	$12''$	$24''$

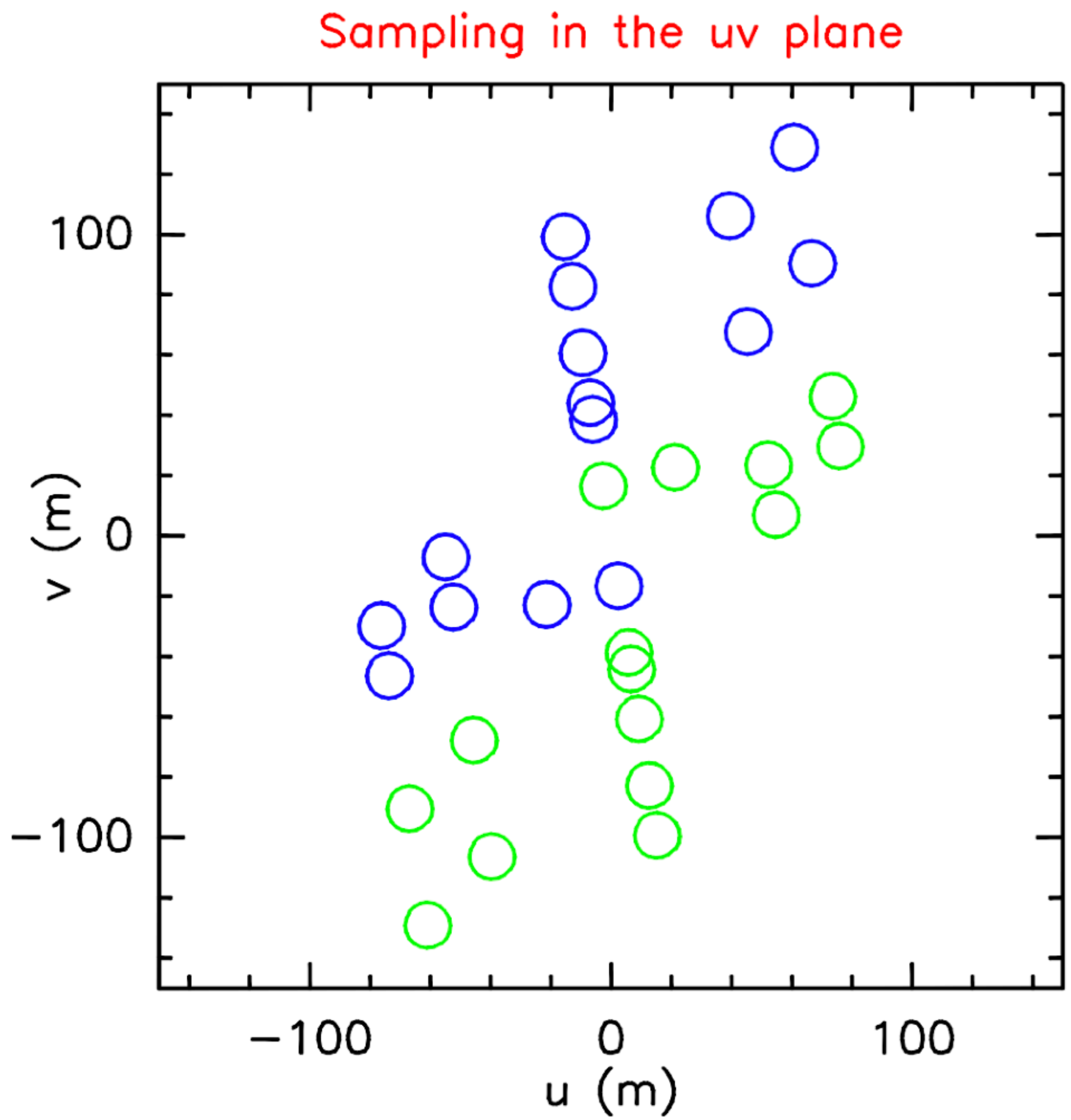
Measurements = uv plane sampling $\times$ visibilities
After FT: dirty map = dirty beam $*$ (prim. beam $\times$ sky)

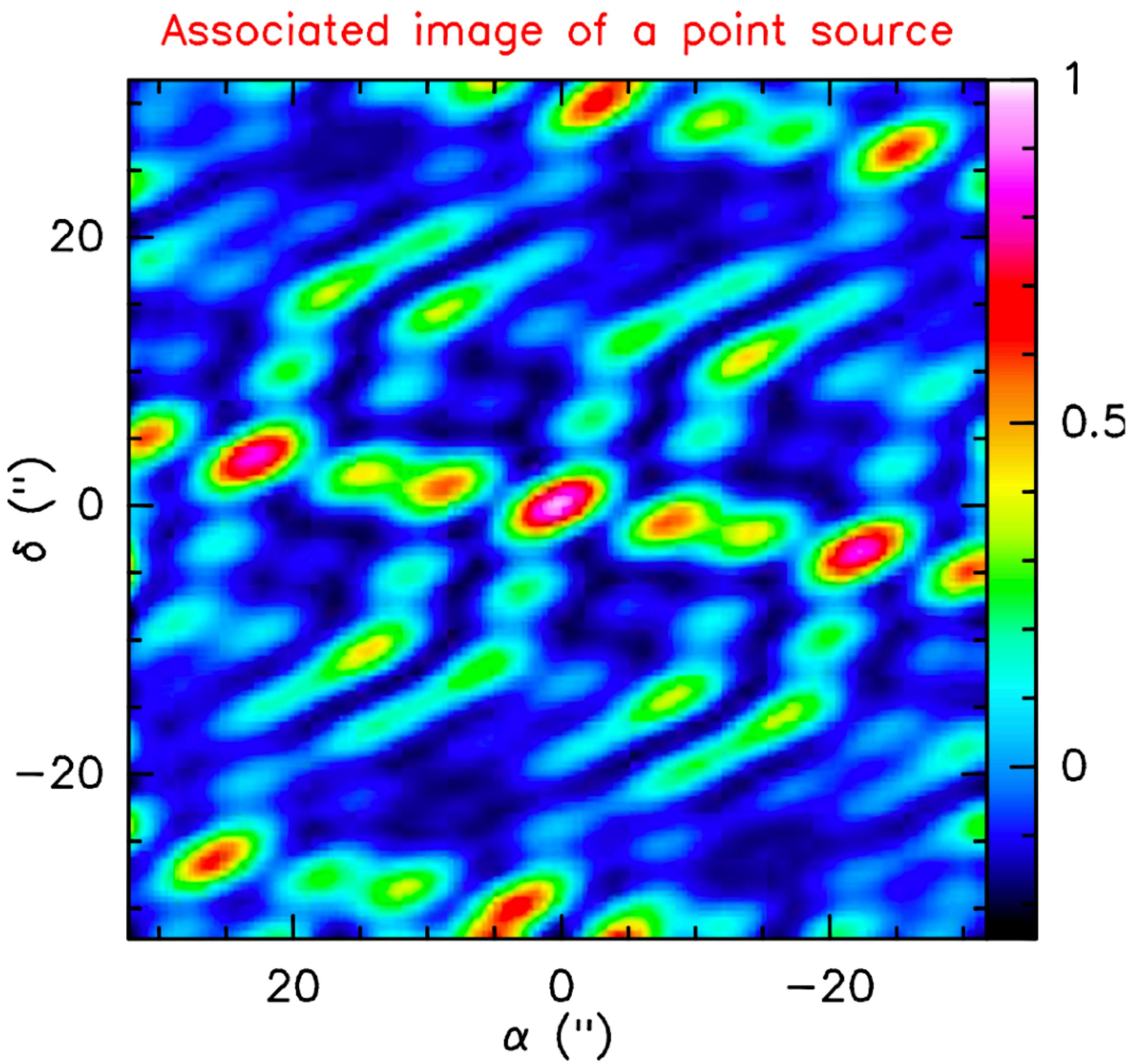
The FT of the *uv* plane coverage gives the dirty beam = the PSF of the observations



Max. baseline gives the angular resolution

Examples of realistic fringes





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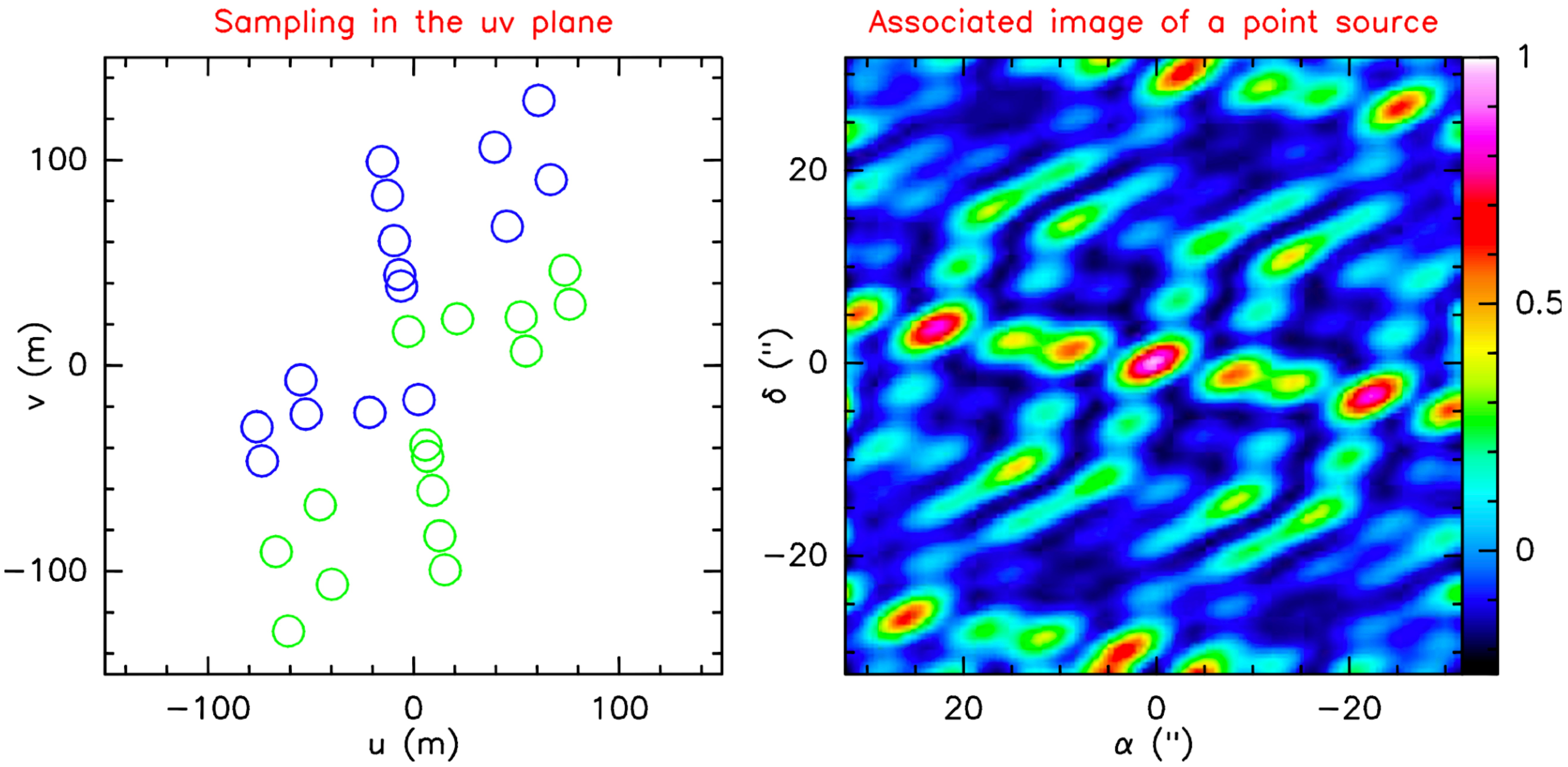

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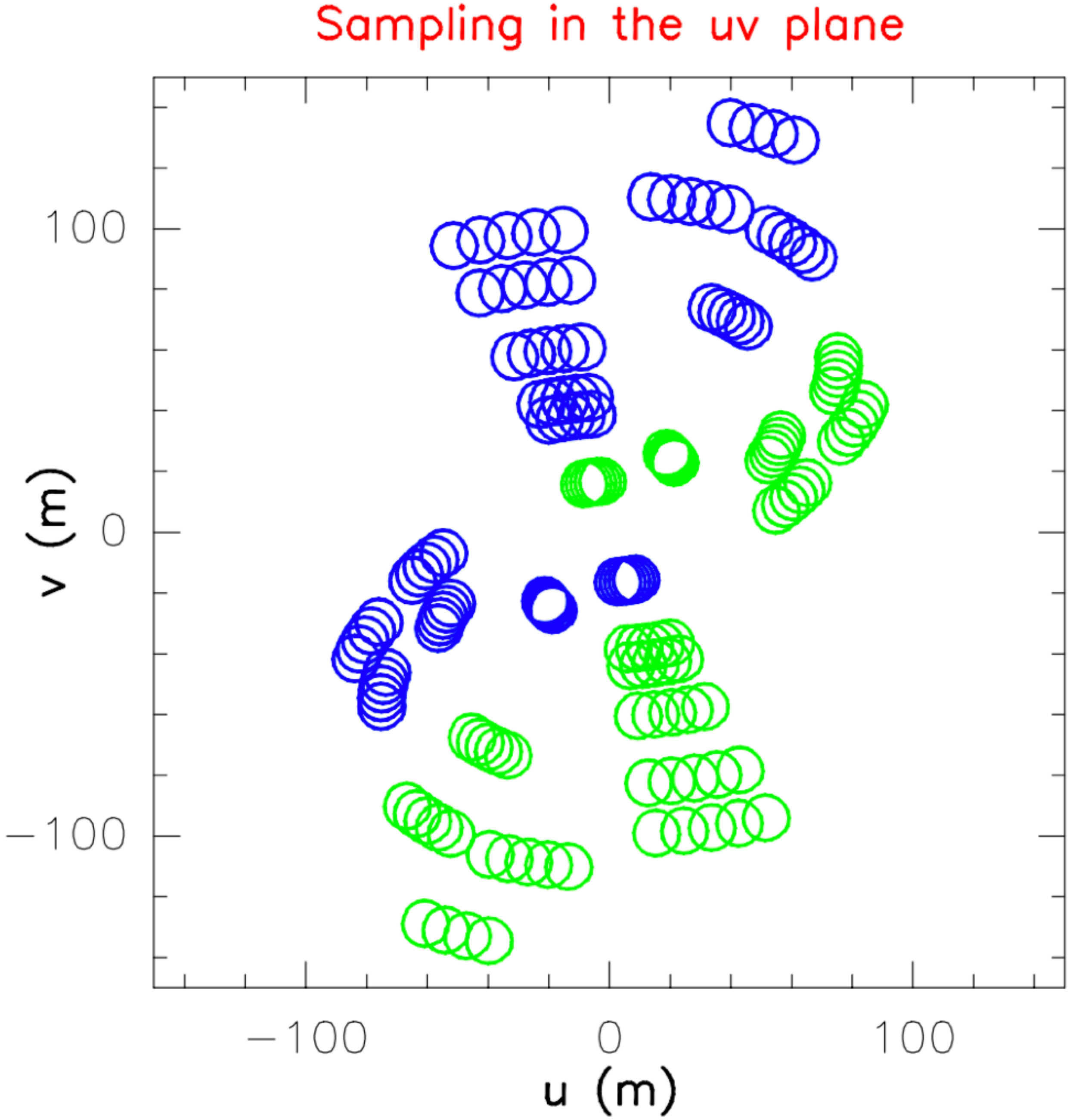

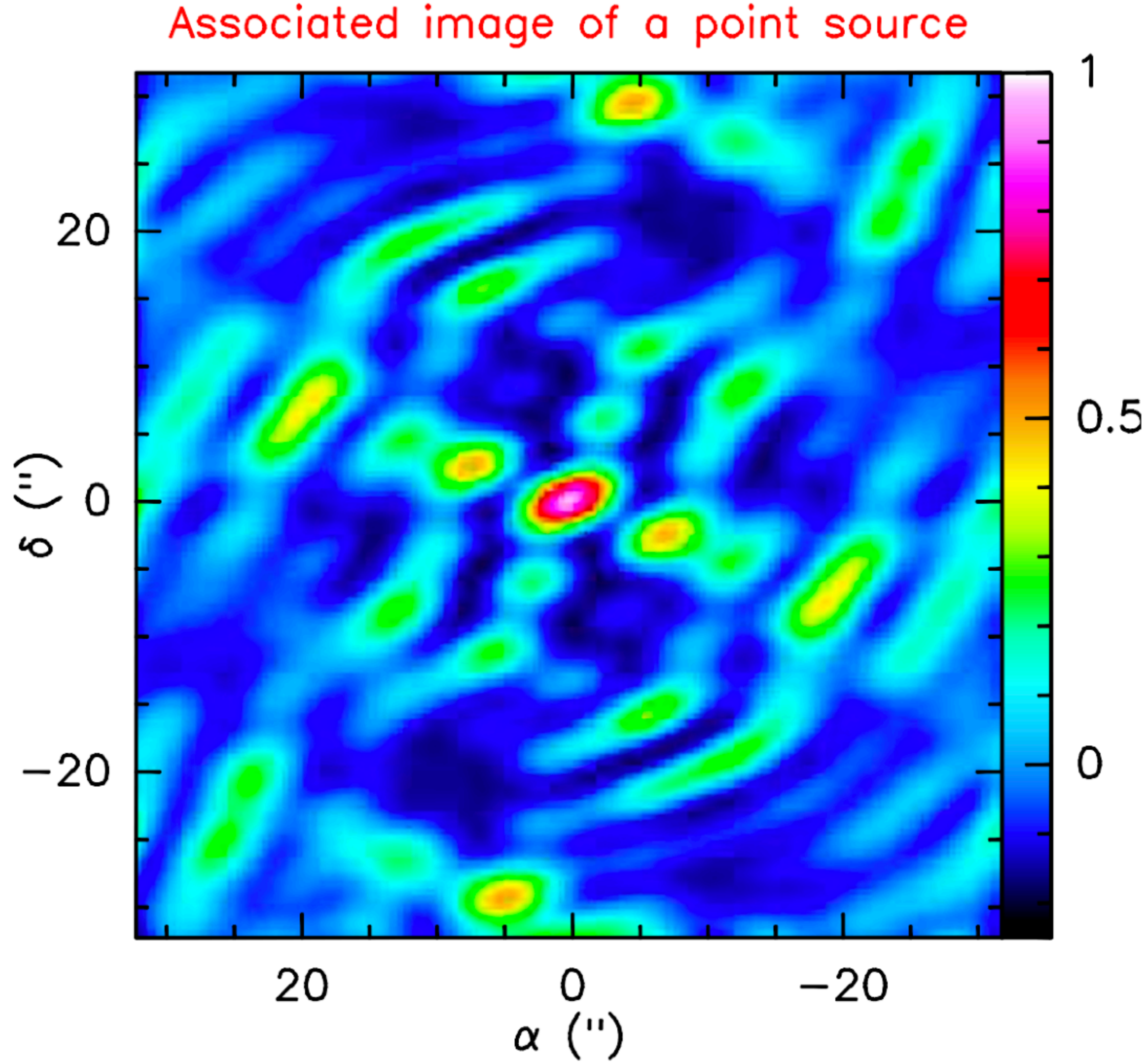

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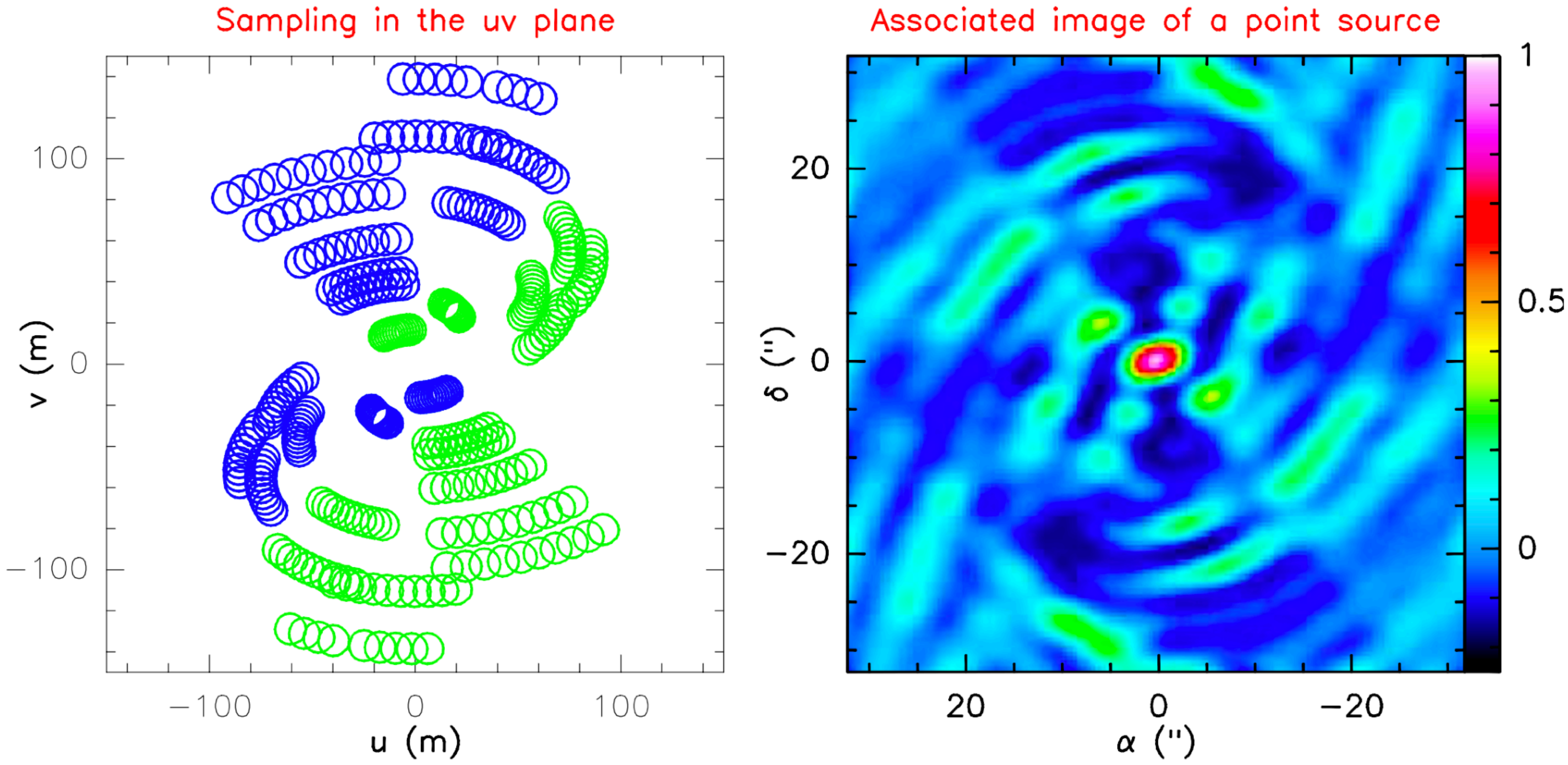

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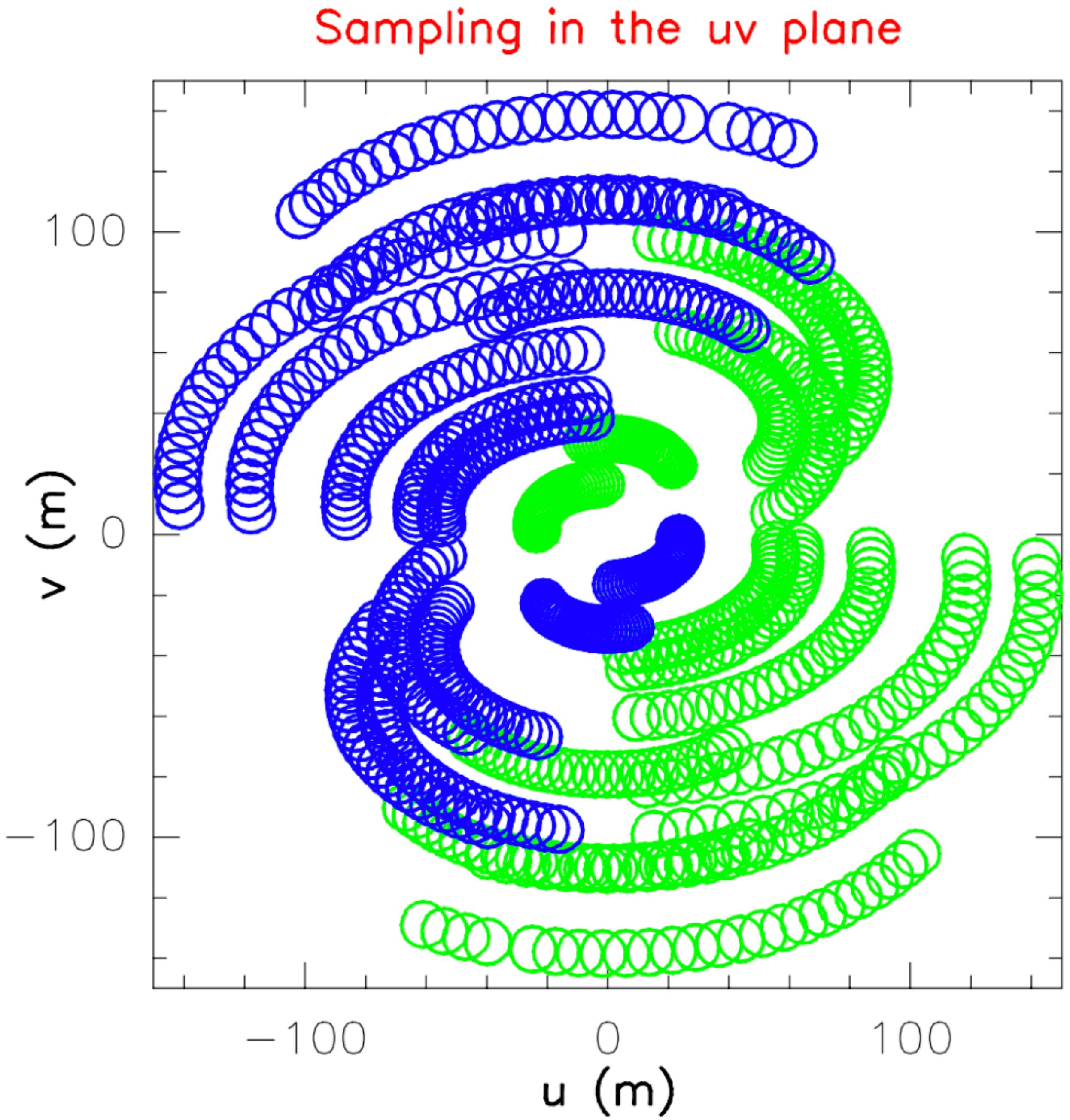

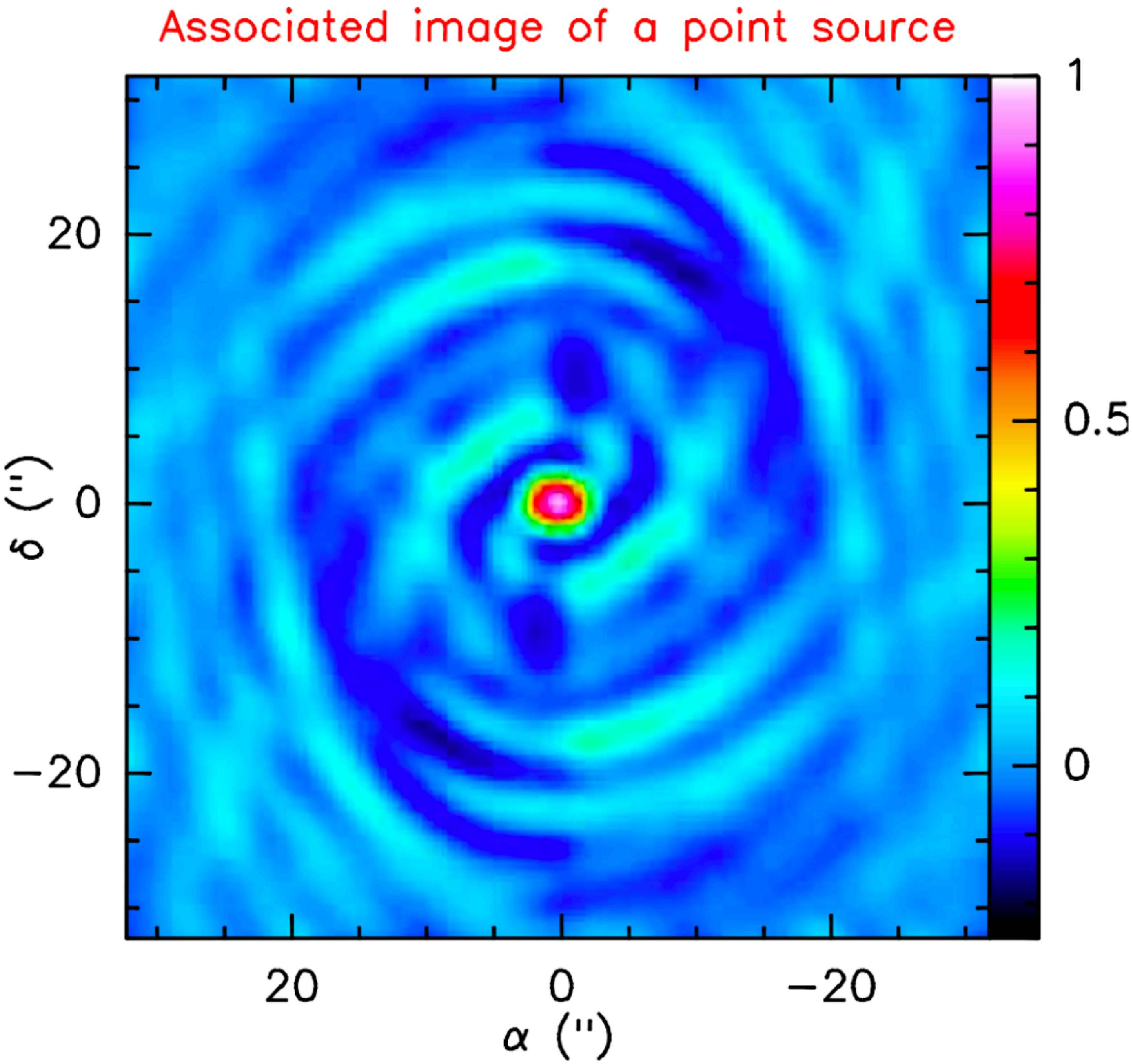

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