Spin–Boson model

AI

Let us consider a spin coupled to a harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{x} - \frac{\gamma}{m\omega^2}\hat{S}_x\right)^2 - h_z\hat{S}_z, \tag{1}$$

where $\hat{S}_\alpha = \hbar\hat{\sigma}_\alpha/2$, $\alpha = x, y, z$, and $\hat{\sigma}_\alpha$ are the Pauli matrices. Here the coupling strength γ will be taken to be small.

The eigenvalues and the eigenstates of the unperturbed Hamiltonian ($\gamma = 0$) are given by

$$\hat{H}_0|n, \sigma_z\rangle = (\epsilon_n - h_z\frac{\hbar}{2}\sigma_z)|n, \sigma_z\rangle \tag{2}$$

with $\epsilon_n = \hbar\omega(n + 1/2)$, $\sigma_z = \pm 1$.

We use the Fermi's golden rule to calculate the transition rates up to the leading order in γ:

$$\omega((n, \sigma_z) \to (n', \sigma'_z)) = \frac{2\pi}{\hbar}|\langle n, \sigma_z|\gamma\hat{x}\hat{S}_x|n', \sigma'_z\rangle|^2\delta(E_{n,\sigma_z} - E_{n',\sigma'_z}), \tag{3}$$

where we have defined $E_{n,\sigma_z} = \epsilon_n - h_z\frac{\hbar}{2}\sigma_z$ the eigenvalues of the unperturbed energy eigenbasis. Recalling the equalities

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{\sigma}_x = \frac{\hbar}{2}(\hat{\sigma}^+ + \hat{\sigma}^-), \tag{4}$$

and inspecting eq. (3), we see that only transitions of the type $n \to n \pm 1$ are selected, and furthermore the constraint on the energy change results in the equality $h_z = \omega$.

Thus we have

$$\begin{aligned} &\omega((n, \sigma_z) \to (n', \sigma'_z)) = \\ &= \frac{\pi\gamma^2\hbar^2}{8m\omega}\left[n\delta_{n,n'=n-1}\delta_{\sigma_z=1,\sigma'_z=-1} + (n+1)\delta_{n,n'=n+1}\delta_{\sigma_z=-1,\sigma'_z=1}\right] \end{aligned} \tag{5}$$

We now trace on the bath quantum number, and assume that the harmonic oscillator is in a thermal state with probability

$$p_n = \mathrm{e}^{-\beta\hbar\omega(n+1/2)}/Z, \qquad Z = \left[2\sinh\left(\beta\hbar\omega/2\right)\right]^{-1}, \tag{6}$$

and obtain

$$\begin{aligned} \omega(\sigma_z = 1 \to \sigma_z' = -1) &= c\sum_{n=0}^{\infty} p_n n = \frac{c}{\mathrm{e}^{\beta\omega\hbar} - 1}, & (7)\\ \omega(\sigma_z = -1 \to \sigma_z' = 1) &= c\sum_{n=0}^{\infty} p_n (n+1) = \frac{c}{1 - \mathrm{e}^{-\beta\omega\hbar}}, & (8) \end{aligned}$$

with $c = \pi\gamma^2\hbar^2/8m\omega$. We now consider the Hamiltonian of the spin alone

$$H_s = -\frac{\hbar\omega}{2}\hat{\sigma}_z \tag{9}$$

and notice that the rates satisfy the detailed balance relation

$$\frac{\omega(\sigma_z = 1 \to \sigma_z' = -1)}{\omega(\sigma_z = -1 \to \sigma_z' = 1)} = \mathrm{e}^{-\beta\hbar\omega} = \mathrm{e}^{-\beta[H_s(\sigma_z=-1) - H_s(\sigma_z=1)]} \tag{10}$$