Chapter 1

Functions of Several Variables

1.1 Introduction

A **real valued function of n–variables** is a function $f : D \to \mathbb{R}$, where the domain D is a subset of $\mathbb{R}^n$. So: for each $(x_1, x_2, \ldots, x_n)$ in D, the value of f is a real number $f(x_1, x_2, \ldots, x_n)$. For example, the volume of a cylinder: $V = \pi r^2 h$ (i.e. $V = F(r, h)$) is a function of two variables.

If f is defined by a formula, we usually take the domain D to be as large as possible. For example, if f is a function defined by $f(x, y) = 9 - \cos(x) + \sin(x^2 + y^2)$, we have a function of 2 variables defined for all $(x, y) \in \mathbb{R}^2$. So $D = \mathbb{R}^2$. However, if f is defined by

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

Then f is a function of 3 variables, defined whenever $\sqrt{x^2 + y^2 + z^2} \neq 0$. This is all $(x, y, z) \in \mathbb{R}^3$ except for $(x, y, z) = (0, 0, 0)$.

Likewise, a **multivariable function of m–variables** is a function $f : D \to \mathbb{R}^n$, where the domain D is a subset of $\mathbb{R}^m$. So: for each $(x_1, x_2, \ldots, x_n)$ in D, the value of f is a vector $f(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

For example:

1. An object rotating around the origin in the xy-plane (say at distance 5 from the origin) will have its position described by the function $f(t) = (5\cos(t), 5\sin(t))$. This is a function from $\mathbb{R}$ to $\mathbf{R}^2$.

2. An object spiralling around the x axis (again at distance 5 from the axis) and travelling at constant speed might have its position given by $f(t) = (t, 5\cos(t), 5\sin(t))$. This function is from $\mathbb{R}$ to $\mathbf{R}^2$.

1.1.1 Surfaces

The **graph of a function** $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, is the set

$$\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}.$$

This is a *surface* in $\mathbb{R}^3$. The *height* $z = f(x, y)$ over each point gives the value of f. The equation $ax + by + cz = d$ represents a plane in $\mathbf{R}^3$. This equation can be written less elegantly, expressing z as a function of x and y, as

$$z = -\frac{a}{c}x - \frac{b}{c}y + \frac{d}{c}, \qquad \text{provided } c \neq 0.$$

When looking at functions of one variable $y = f(x)$ it is possible to plot (x, y) points to determine the shape of the graph. In the same way, when looking at a function of two variables $z = f(x, y)$, it is possible to plot the points (x, y, z) to build up the shape of a surface.

Example 1.1 Draw the graph (or surface) of the function: $z = 9 - x^2 - y^2$ (a circular paraboloid).

For

$$\begin{aligned} x = 0,\ y = 0 \quad &\rightarrow \quad z = 9, \\ x = 0,\ y = 1 \quad &\rightarrow \quad z = 8, \\ x = 1,\ y = 0 \quad &\rightarrow \quad z = 8, \\ x = 1,\ y = 1 \quad &\rightarrow \quad z = 7, \\ &\qquad\ \ \text{etc} \ldots . \end{aligned}$$

We can eventually plot enough points to find the surface in Figure 1.1.

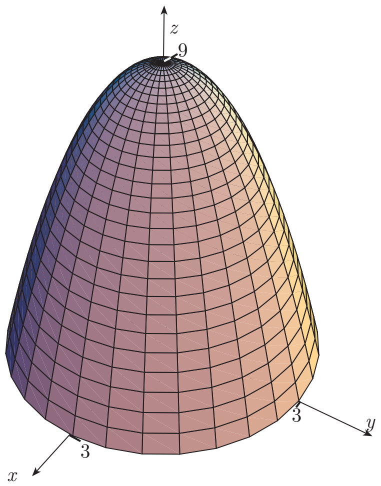


Figure 1.1: The paraboloid $z = 9 - x^2 - y^2$.

This method of drawing a surface is time consuming as we need to calculate many points before being able to plot the graph. Also some surfaces are quite complex and difficult to draw. We would like another way of representing the graph so that the surface is easier to draw or visualize.

1.1.2 Contours and level curves

Three dimensional surfaces can be depicted in two–dimensions by means of level curves or contour maps. If $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function of two variables, the **level curves of f** are the subsets of D:

$$\{(x, y) \in D : f(x, y) = c\},$$

where c=constant. If $f = \text{height}$, level curves are **contours** on a contour map. If $f = \text{air pressure}$, level curves are the **isobars** on a weather map.

The graph of f can be built up from the level sets: The slice at height $z =$ c, is the **level set** $f(x, y) = c$.

Example 1.2 For the elliptic paraboloid $z = x^2 + y^2$, for example, the level curves will consist of concentric circles. For, if we seek the locus of all points on the paraboloid for which $z = \frac{1}{2}$, we solve the equation

$$\frac{1}{2} = x^2 + y^2$$

which is of course an equation of a circle. The locus of points 0 units above the xy plane is just the origin, for if $z = 0$, the equation becomes $x^2 + y^2 = 0$, and this implies that $x = y = 0$.

Example 1.3 For the hyperbolic paraboloid $z = x^2 - y^2$ the level curves are hyperbolae except for $x^2 - y^2 = 0$ which is the union of two lines.

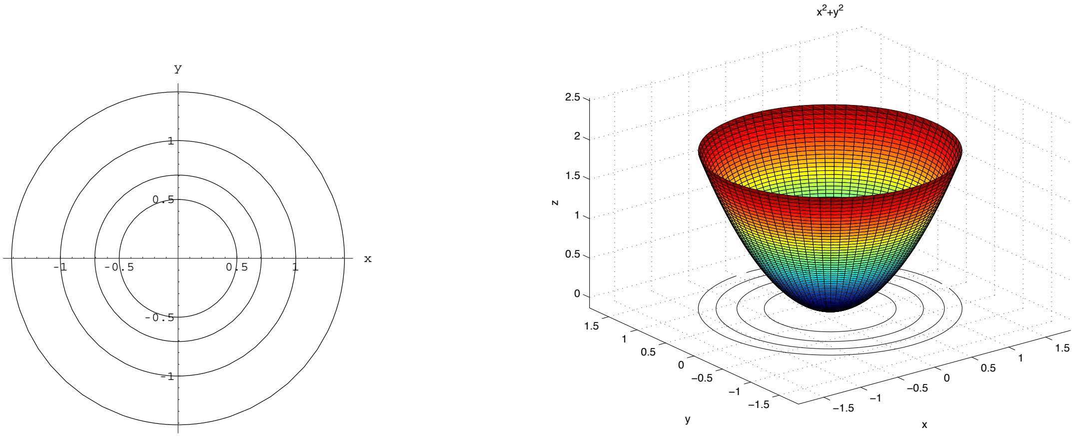


Figure 1.2: Level curves of the elliptic paraboloid $z = x^2 + y^2$.

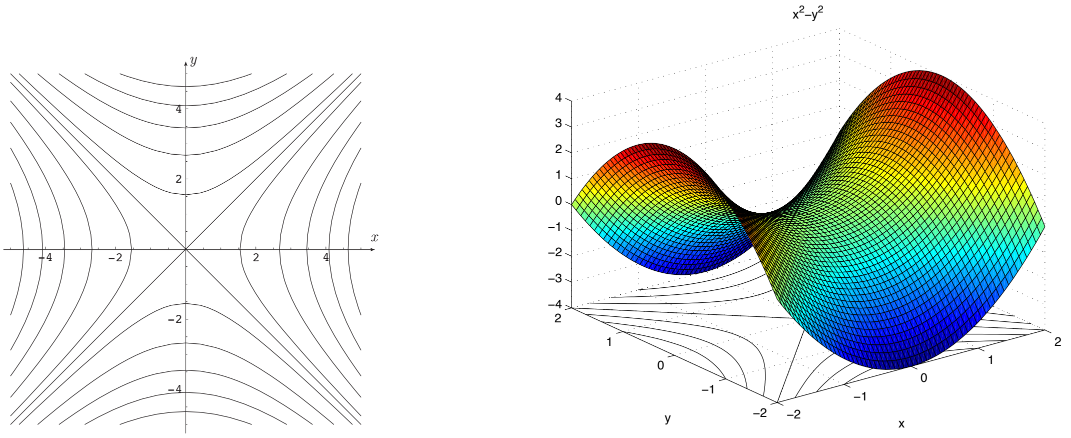


Figure 1.3: Level curves of the hyberbolic paraboloid $z = x^2 - y^2$.

1.1.3 Partial derivatives

These measure the rate of change of a function with respect to one of the variables, keeping all other variables **fixed**. Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables, and $(x_0, y_0) \in D$.

Definition 1.1 *The* **partial derivative** *of f with respect to x is:*

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

(i.e. differentiate f with respect to x, treating y as a constant). The **partial derivative** *of f with respect to y is:*

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

(i.e. differentiate f with respect to y, treating x as a constant). Provided, of course, the limits exist.

Referring to Figure 1.4, let us intersect the surface by a vertical plane $y = \text{constant}$, say $y = y_0$, to give a curve of intersection AP_0B. Then $\dfrac{\partial f}{\partial x}(x_0, y_0)$ gives the slope of the tangent P_0T to this curve at the point (x_0, y_0, z_0).

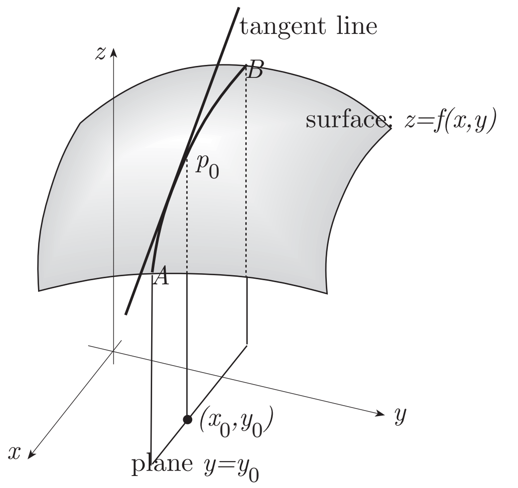


Figure 1.4: Tangent to surface in the x direction.

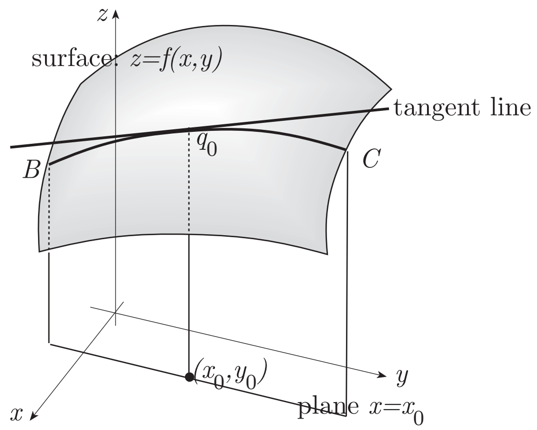


Figure 1.5: Tangent to surface in the y direction.

Similarly, $\dfrac{\partial f}{\partial y}(x_0, y_0)$ gives the slope of the tangent Q_0T to the curve of intersection BQ_0C of the surface with a vertical plane $x = x_0$ (see figure 1.5). Therefore $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ give the slope of a surface at a point in the directions of the x and y axes respectively.

Similarly, for a function f of n variables $x_1, \ldots x_n$ we can define partial derivatives,

$$\frac{\partial f}{\partial x_1} = f_{x_1}, \frac{\partial f}{\partial x_2} = f_{x_2}, \ldots, \frac{\partial f}{\partial x_n} = f_{x_n}.$$

Exactly the same rules of differentiation apply as for a function of one variable. If we have a function of two variables $f(x, y)$ we treat y as a constant when calculating $\dfrac{\partial f}{\partial x}$, and treat x as a constant when calculating $\dfrac{\partial f}{\partial y}$.

1.1.4 Higher partial derivatives

Notice that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are themselves functions of two variables, so they can also be partially differentiated.

For a function of two variables $f : D \to \mathbb{R}$ there are four possiblilties:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}, \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}, \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}, \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}.
\end{aligned}$$

Higher order partial derivatives are defined similarly:

$$\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = f_{xyx}.$$

Usually, but certainly not always,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

For the functions we will be encountering the mixed partial derivatives will generally be equal. In fact, this is true whenever f_{xy} and f_{yx} are continuous.

Example 1.4 Find all the second partial derivatives of

$$f(x, y) = x^3 e^{-2y} + y^{-2}\cos(x)$$

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 3x^2 e^{-2y} - y{-}2\sin(x), \\
\frac{\partial f}{\partial y} &= -2x^3 e^{-2y} - 2y^{-3}\cos(x),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2e^{-2y} - y{-}2\sin(x)\right) = 6xe^{-2y} - y{-}2\cos(x)\,,\\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-2x^3e^{-2y} - 2y^{-3}\cos(x)\right) = 4x^3e^{-2y} + 6y^{-4}\cos(x)\,,\\
\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-2x^3e^{-2y} - 2y^{-3}\cos(x)\right) = -6x^2e^{-2y} + 2y^{-3}\sin(x)\,,\\
\frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2e^{-2y} - y{-}2\sin(x)\right) = -6x^2e^{-2y} + 2y{-}3\sin(x)\,.
\end{aligned}$$

1.1.5 Differentiable functions

For a function of one variable $f(x)$ is **differentiable at** x_0 means:

1. The graph of f has a **tangent line** at $x = x_0$ (see Figure 1.6). The equation of this tangent line is

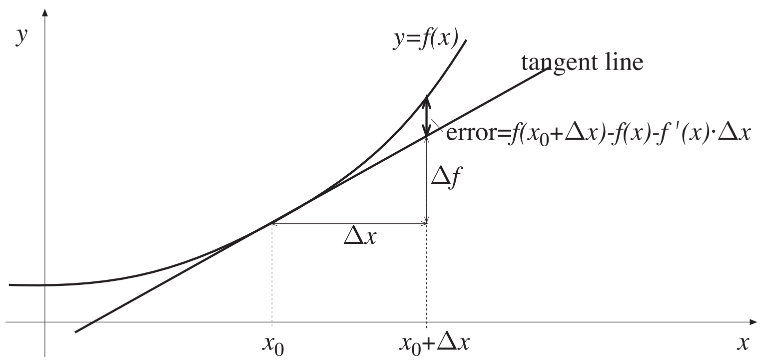


Figure 1.6: A differentiable function has a tangent line.

$$y - y_0 = f'(x_0)(x - x_0).$$

2. The change in f near x_0 is well approximated by a linear function:

$$\Delta f = f'(x_0)\Delta x + \text{error},$$

where the error is small compared with Δx or more precisely: $\frac{\text{error}}{\Delta x} \to 0$ as $\Delta x \to 0$.

These ideas can be generalized to a function of two variables. A function f is differentiable at (x_0, y_0) means that f has a well defined **tangent plane** at (x_0, y_0). All tangents to the surface at (x_0, y_0) lie in the one plane (see Figure 1.7). Or more formally

Definition 1.2 *A function $f(x, y)$ if* **differentiable** *at (x_0, y_0) if the change in f near (x_0, y_0) is well approximated by a linear function:*

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \text{error},$$

where $\dfrac{\text{error}}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \to 0$ *as* $(\Delta x, \Delta y) \to (0, 0)$. *Here*

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

The equation of the **tangent plane** to $z = f(x, y)$ at (x_0, y_0) is

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y$$

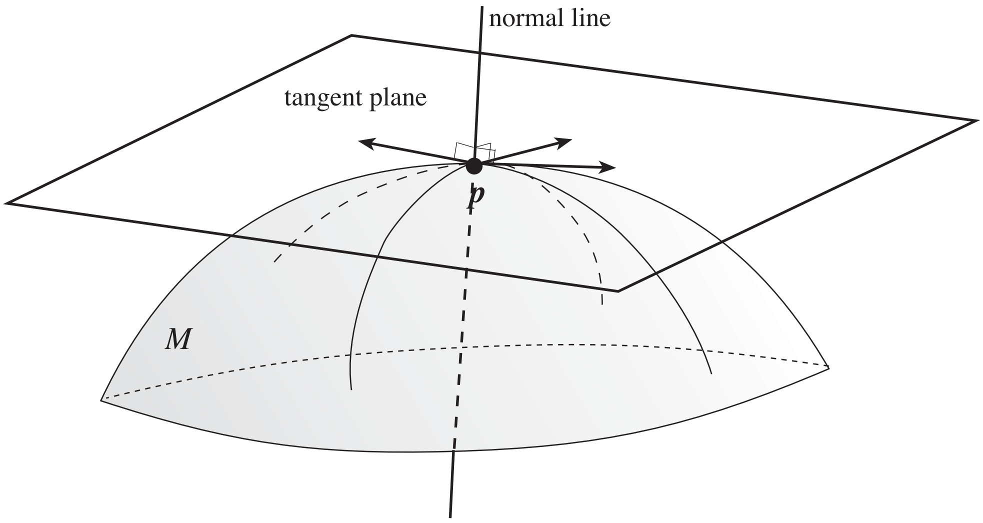


Figure 1.7: The tangent plane and normal line to the surface $z = f(x, y)$.

or

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Example 1.5 Find the cartesian equation of the tangent plane to the surface

$$z = 1 - x^2 - y^2$$

at the point $(1, 2, -4)$.

Solution: We first find the partial derivatives

$$\begin{array}{rll} & \dfrac{\partial z}{\partial x} = -2x, & \dfrac{\partial z}{\partial y} = -2y, \\ \text{At } (1, 2, -4) & \dfrac{\partial z}{\partial x} = -2, & \dfrac{\partial z}{\partial y} = -4. \end{array}$$

Therefore by using the formula the tangent plane is

$$\begin{aligned} z &= -4 + (x - 1)(-2) + (y - 2)(-4), \\ &= -4 - 2x + 2 - 4y + 8, \\ z &= 6 - 2x - 4y. \end{aligned}$$

If we rearrange this equation into the usual form for a plane we have

$$x\frac{\partial f}{\partial x}(x_0, y_0) + y\frac{\partial f}{\partial y}(x_0, y_0) - z = -f(x_0, y_0) + x_0\frac{\partial f}{\partial x}(x_0, y_0) + y_0\frac{\partial f}{\partial y}(x_0, y_0)$$

so the **normal vector for the tangent plane** is

$$\mathbf{n} = (f_x(x_0, y_0), f_y(x_0, y_0), -1)\,.$$

From this we can find the equation to the normal line to the tangent plane at (x_0, y_0) as

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}$$

or

$$(x, y, z) = (x_0, y_0, z_0) + t\,(f_x(x_0, y_0), f_y(x_0, y_0), -1)\,.$$

1.1.6 Errors and approximations

If the function $z = f(x, y)$ is differentiable at (x_0, y_0), then for (x, y) near (x_0, y_0)

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y.$$

where $\Delta x = x - x_0,\ \Delta y = y - y_0$. The right hand side of this equation is called the linear approximation to f near (x_0, y_0). This equation may also be written as

$$\Delta f \approx f_x \Delta x + f_y \Delta y$$

which is sometimes written as

$$df = f_x\, dx + f_y\, dy$$

which is called the **differential of f.**

Some applications of this **linear approximation** to f are

1. Estimating values of functions.
2. Estimating errors in measurements.

Example 1.6 Estimate the value of

$$f(x,y) = \sqrt{1 - x + 2y} \qquad \text{when } x = 0.01,\ y = 0.02.$$

Solution: Use the linear approximation to f at $(0, 0)$:

$$\begin{aligned} f_x = \frac{\partial f}{\partial x} &= \frac{1}{2}(1 - x + 2y)^{-1/2}(-1) \\ f_y = \frac{\partial f}{\partial y} &= \frac{1}{2}(1 - x + 2y)^{-1/2}(2) \end{aligned}$$

At $(0,0):\quad f_x = -\frac{1}{2},\quad f_y = 1.$
The linear approximation for (x, y) near $(0, 0)$ is

$$\begin{aligned} f(x,y) &\approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ &= 1 - \frac{1}{2}x + y \end{aligned}$$

Taking $x = 0.01,\ y = 0.02$ gives

$$\begin{aligned} f(0.01, 0.02) &\approx 1 - \frac{1}{2}(0.01) + (.02) \\ &= 1.015 \end{aligned}$$

The size of the error depends on the second order derivatives of f.

1.2 Space Curves

1.2.1 Vector valued functions

A **curve** in $\mathbb{R}^n$ is a vector valued function

$$\mathbf{c} : [a, b] \to \mathbb{R}^n$$

where $\mathbf{c}(t) = (c_1(t), c_2(t), \ldots, c_n(t))$. You can think of $\mathbf{c}(t)$ as the position at time t. For example the vector function $\mathbf{c}(t) = (\cos(t), \sin(t))$ describes the unit circle in $\mathbb{R}^2$. This is equivalent to the parametric equations $x = \cos(t),\ y = \sin(t),\ t \in \mathbb{R}$.

The derivative of **c** is

$$\frac{d\mathbf{c}}{dt} = \mathbf{c}'(t) = (c_1'(t), c_2'(t), \ldots, c_n'(t))$$

that is we differentiate each component of **c**. The derivative can also be defined as the limit

$$\mathbf{c}'(t) = \lim_{\Delta t} \frac{\mathbf{c}(t+\Delta t) - \mathbf{c}(t)}{\Delta t}.$$

The derivative gives the **tangent vector** to the curve at the point $\mathbf{c}(t)$.

If $\mathbf{c}(t)$ represents the position at time t,

then $\mathbf{c}'(t) = \frac{d\mathbf{c}}{dt} =$ **velocity vector**,

and $\mathbf{c}''(t) = \frac{d^2\mathbf{c}}{dt^2} =$ **acceleration vector**,

at time t.

For example if $\mathbf{c}(t) = (\cos(t), \sin(t))$ represents the unit circle in $\mathbb{R}^2$ then

$$\begin{array}{lcl} \mathbf{c}'(t) = (-\sin(t), \cos(t)) & = & \text{velocity,} \\ \mathbf{c}''(t) = (-\cos(t), -\sin(t)) & = & \text{acceleration.} \end{array}$$

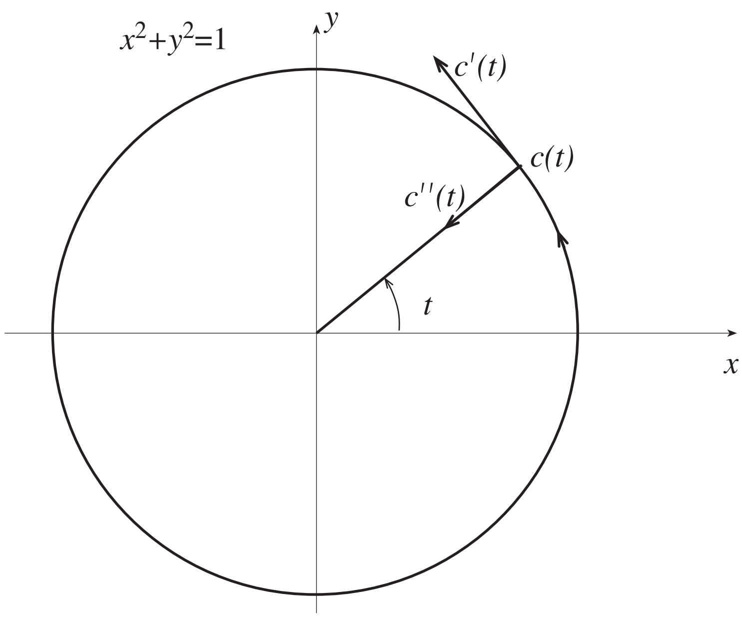


Figure 1.8: The velocity and acceleration of $\mathbf{c}(t) = (\cos(t), \sin(t))$.

1.2.2 The chain rule

If y is a differentiable function of x i.e. $y = f(x)$ and x is a differentiable function of t i.e. $x = g(t)$, then y is a function of t

$$y = f(g(t)) \quad \text{and} \quad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

If z is a function of x and y i.e. $z = f(x, y)$ and x and y are both functions of the **same** variable t, i.e.

$$f = f(x, y) \qquad \text{and} \qquad x = g(t),\ y = h(t)$$

then z is a function of t:

$$z = f\big(g(t), h(t)\big).$$

For example, if $z = x^2 - y^2, \quad x = \sin(t)$ and $y = \cos(t)$ then $z = \sin^2(t) - \cos^2(t)$.

We would like to be able to find the rate of change of f with respect to t. This can be found from the **chain rule** for two variables

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example 1.7 If $z = x^2 - y^2, \quad x = \sin(t)\,,\ y = \cos(t)$, find $\dfrac{dz}{dt}$ where $t = \frac{\pi}{3}$.

Solution: We can do this by two methods the second can be used to check our first answer.

Method 1. Using the chain rule

$$\begin{aligned} \frac{\partial z}{\partial x} &= 2x, & \frac{\partial z}{\partial y} &= -2y, \\ \frac{dx}{dt} &= \cos(t)\,, & \frac{dy}{dt} &= -\sin(t)\,, \end{aligned}$$

At $t = \frac{\pi}{3}$,

$$x = \sin(t) = \frac{\sqrt{3}}{2} \text{ and } y = \cos(t) = \frac{1}{2}.$$

Therefore

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2x)(\cos(t)) + (-2y)(-\sin(t)) \\ &= 2x\cos(t) + 2y\sin(t) \\ &= 2\frac{\sqrt{3}}{2} \times \frac{1}{2} + 2\frac{1}{2} \times \frac{\sqrt{3}}{2} \\ &= \sqrt{3}. \end{aligned}$$

Method 2. Using substitution to find $z = z(t)$

$$z = x^2 - y^2 = \sin^2(t) - \cos^2(t)\,.$$

Therefore

$$\frac{dz}{dt} = 2\sin(t)\cos(t) - 2\cos(t)\left(-\sin(t)\right) = 4\sin(t)\cos(t) = 4\frac{\sqrt{3}}{2} \times \frac{1}{2} = \sqrt{3},$$

which is the same answer we obtained by method 1.

The chain rule can be used for functions of more than two variables:

Given a function $f(x_1, x_2, \ldots, x_n)$ defined at points of $\mathbb{R}^n$, consider the values of f along a curve

$$x_1 = x_1(t),\ x_2 = x_2(t), \ldots,\ x_n = x_n(t).$$

Here $t \in \mathbb{R}$ is a parameter along the curve (e.g. time or arc length).

Let $w = f(x_1, x_2, \ldots, x_n) =$ function of t. If $f,\ x_1,\ x_2,\ \ldots, x_n$ are differentiable, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx}{dt} + \cdots \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}$$

where each $\dfrac{\partial w}{\partial x_i}$ is evaluated at $(x_1, x_2, \ldots, x_n)$.

1.3 Gradient Vectors and Directional Derivatives

1.3.1 Gradient and Gradient vector

Consider the function of two variables $f(x,y)$. Its graph represents a surface in three dimensions. If x and y are themselves a function of another variable t then $(x(t), y(t))$ is a curve $C = \mathbf{c}(t)$ in the xy plane. The function $f(x(t), y(t))$ will then represent a curve on the surface of $f(x,y)$ directly above curve $C = \mathbf{c}(t)$ in the xy plane.

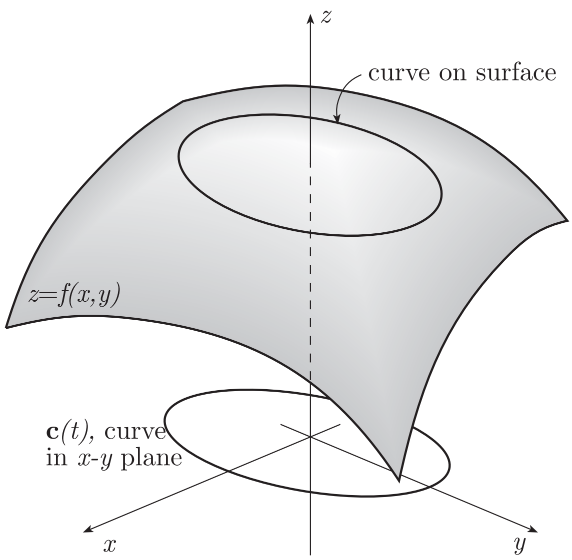


Figure 1.9: The curve on the surface of $z = f(x,y)$ directly above the curve $C = \mathbf{c}(t)$ in the xy plane.

The chain rule expresses $\dfrac{df}{dt}$ along the curve C as the dot product of the two vectors

$$\mathbf{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \quad \text{and} \quad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

then

$$\mathbf{v} \cdot \nabla f = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{df}{dt}$$

i.e. the chain rule. The two vectors $\mathbf{v}$ and ∇f have only two components therefore they lie in the xy plane.

$\mathbf{v}$ is tangent to the curve $(x(t), y(t))$ and is called the **tangent vector.**

∇f is called the **gradient vector of f.**

These ideas can also be extended for functions of more than two variables.

Definition 1.3 *If $f(x_1, x_2, \ldots, x_n)$ is a function of n variables then the* **gradient** *of f is the vector valued function*

$$\nabla f(x_1, x_2, \ldots, x_n) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Notice that if $f : \mathbb{R}^n \to \mathbb{R}$, then $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$. This is an example of a "vector field" on $\mathbb{R}^n$.

1.3.2 Directional derivatives

The partial derivatives f_x and f_y give the rate of change of f in directions parallel to the x and y axis. What about other directions?

Let $P_0(x_0, y_0)$ be a fixed point in the xy–plane. Let l be a line in the xy–plane that passes through P_0. The point $P(x, y)$ moves along the line l. Directly above it, the point Q moves along the surface $z = f(x, y)$, tracing out a curve C. What is the rate at which the z–coordinate of Q changes with the distance s between P_0 and P?

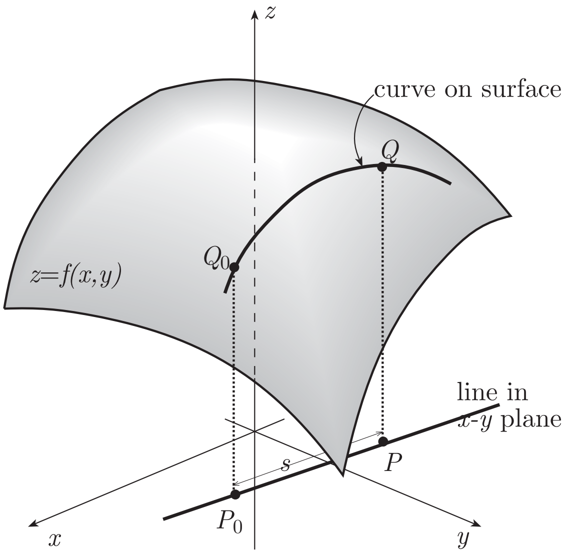


Figure 1.10: Curve on the surface of $z = f(x, y)$ above the line l.

Let **u** be the unit vector

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$$

with initial point $P_0(x_0, y_0)$ and pointing in the direction of motion of $P(x, y)$. Then

$$\begin{aligned}\overrightarrow{P_0P} &= s\mathbf{u}\\ (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} &= su_1\mathbf{i} + su_2\mathbf{j}\end{aligned}$$

therefore

$$x = x_0 + su_1 \quad \text{and} \quad y = y_0 + su_2.$$

Hence

$$\begin{aligned}z &= f(x, y)\\ &= f(x_0 + su_1, y_0 + su_2).\end{aligned}$$

By the chain rule,

$$\begin{aligned}\frac{dz}{ds} &= \frac{\partial z}{\partial x}\frac{dw}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}\\ &= f_x u_1 + f_y u_2\\ &= \nabla f \cdot \mathbf{u}.\end{aligned}$$

Thus, the rate of change of $f(x, y)$ at (x_0, y_0) in the direction of the unit vector **u** is given by the **directional derivative**

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

This can be extended to functions of n variables.

Let

- $f(x_1, \ldots, x_n)$ be a differentiable function of n variables,
- $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ and
- P be a point in $\mathbb{R}^n$.

Definition 1.4 *The* **directional derivative** *of f in the direction of* **u** *at P is:*

$$\begin{aligned} D_{\mathbf{u}}f(P) &= \lim_{t \to 0} \frac{f(P + t\mathbf{u}) - f(P)}{t} \\ &= \frac{d}{dt} f(P + t\mathbf{u})\Big|_{t=0} \\ &= \textit{rate of change of } f \textit{ in the direction of } \mathbf{u}. \end{aligned}$$

Taking $\mathbf{x}(t) = P + t\mathbf{u}$, $\frac{d\mathbf{x}}{dt} = \mathbf{u}$ *in the chain rule*

$$D_{\mathbf{u}}f = \frac{d}{dt} f(\mathbf{x}(t)) = \nabla f \cdot \frac{d\mathbf{x}}{dt} = \nabla f \cdot \mathbf{u}.$$

Thus

$$D_{\mathbf{u}}f(P) = \nabla f(P) \cdot \mathbf{u}.$$

Example 1.8 Find the rate of change of

$$f = 1 - \frac{x^2}{4} - \frac{y^2}{4}$$

at the point $(1, 0)$ in the direction of the vectors:

(i) $\mathbf{a}$: a unit vector $45°$ to the x–axis,

(ii) $\mathbf{b} = (0, 1)$.

From the previous example the gradient vector at $(1, 0)$ is $\nabla f = (-\frac{1}{2}, 0)$

(i) Unit vector: $\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{1}{\sqrt{2}}(1, 1)$. Therefore the directional derivative is

$$D_{\mathbf{u}}f = \frac{1}{\sqrt{2}}(1, 1) \cdot (-\frac{1}{2}, 0) = -\frac{1}{2\sqrt{2}}$$

(ii) Unit vector: $\mathbf{u} = \mathbf{b} = (0, 1)$. Therefore the directional derivative is

$$D_{\mathbf{u}}f = (0, 1) \cdot (-\frac{1}{2}, 0) = 0$$

i.e. there is no change in f in this direction.

1.3.3 Directions of fastest increase and decrease

Given $f(x_1, \ldots, x_n)$ a differentiable function of n variables, and a point P in $\mathbb{R}^n$, we can consider $D_{\mathbf{u}}f$ as a function of the unit vector **u**.

If $\nabla f \neq 0$, then the directional derivative at$P$ has a maximum value of $||\nabla f||$ in the direction of ∇f. That is the maximum rate of increase in f is $||\nabla f||$ in the direction of $\mathbf{u} = \dfrac{\nabla f}{||\nabla f||}$.

The directional derivative at P has a minimum value of $-||\nabla f||$ in the direction of $-\nabla f$. That is the maximum rate of decrease of f at P is given by $-||\nabla f||$ and is in the dirction of $-\nabla f$ or $\mathbf{u} = \dfrac{-\nabla f}{||\nabla f||}$.

1.3.4 Level curves and gradient

Let $x = x(t)$, $y = y(t)$ be a level curve of $f(x, y)$ so that

$$f(x(t), y(t)) = \text{const}.$$

Then by the chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0.$$

So

$$\nabla f \cdot \mathbf{v} = 0$$

hence $\nabla f(x_0, y_0)$ is normal to the level curve at (x_0, y_0).

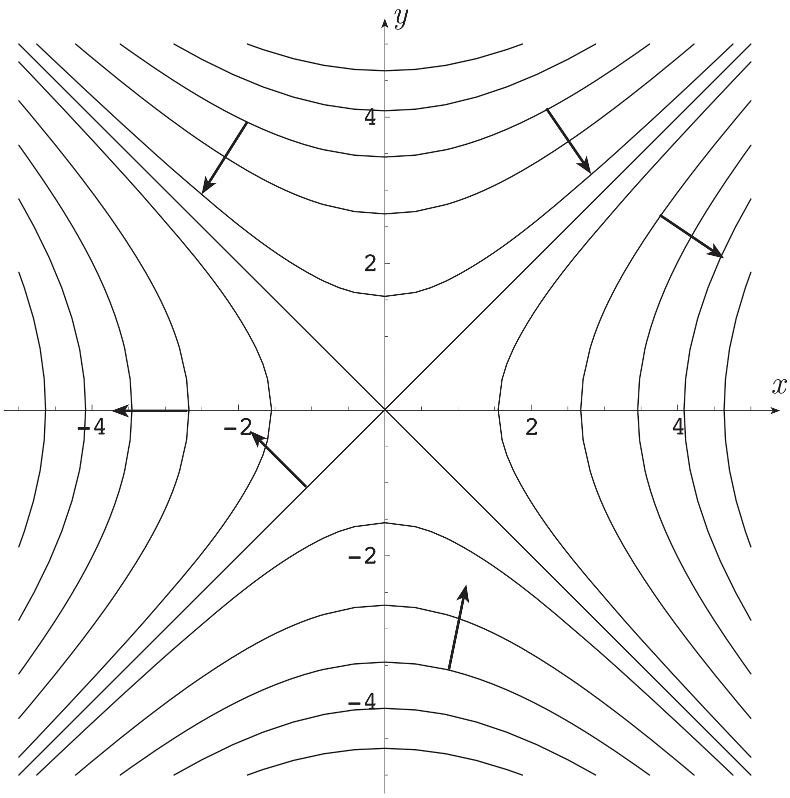


Figure 1.11: The gradient vector ∇f is perpendicular to the level curves.

Example 1.9 Find the **level curve** of $f = x^2 - y^2$ corresponding to $f = 1$ and sketch the gradient vector at the points $(1,0)$, $(-1,0)$. *Solution:* The level curve is $1 = x^2 - y^2$ or $y = \pm\sqrt{x^2-1}$

$$\begin{array}{lrl} & & \nabla f = (2x, -2y), \\ \text{at} & (1,0) & \nabla f = (2,0) = 2\mathbf{i}, \\ & (-1,0) & \nabla f = (-2,0) = -2\mathbf{i}. \end{array}$$

The gradient vector is sketched in Figure 1.12.

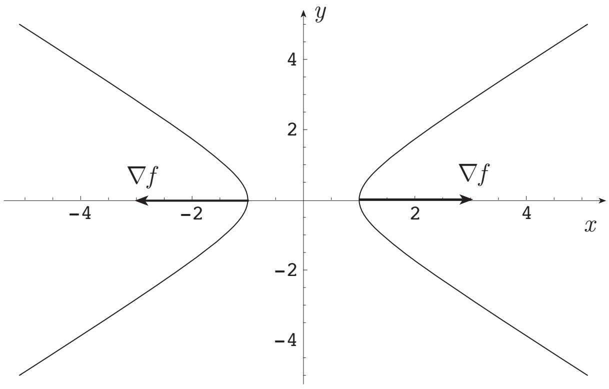


Figure 1.12: The level curves and gradient vector ∇f of $f = x^2 - y^2$.

In general, if $f(x_1, \ldots, x_n)$ is differentiable at a point P and $f(P) = c$, then $\nabla f(P)$ is **perpendicular** to the level set $f(x) = c$ containing P.

1.3.5 Tangent planes to surfaces

Theorem 1.5 *If $F(x,y,z)$ is differentiable, $\nabla F(P) \neq \mathbf{0}$, and $F(P) = c$, then the level set $F(x,y,z) = c$ has well–defined* **tangent plane** *at P which is orthogonal to $\nabla F(P)$.*

If $\nabla F = 0$ or ∇F is not defined at P, there generally won't be a nice tangent plane. For example the point at the tip of the cone in Figure 1.13.

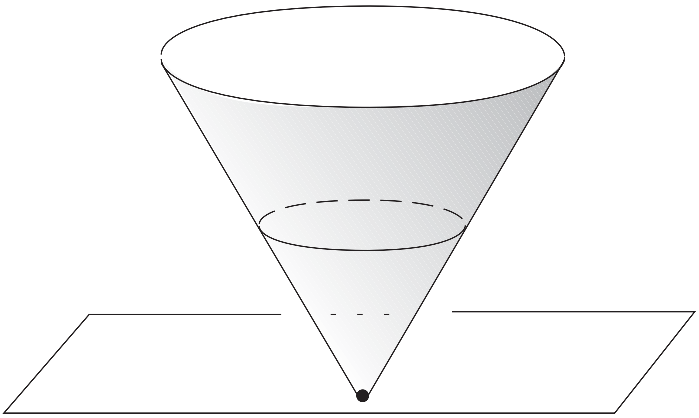


Figure 1.13: A point where there is no nice tangent plane.

Therefore as ∇F is orthogonal to the tangent plane, the equation for the tangent plane at P is given by

$$\nabla F(P) \cdot (\mathbf{x} - P) = 0$$

and the equation for the normal line $S = \{(x, y, z) : F(x, y, z) = c\}$ at P is

$$\mathbf{x}(t) = P + t\nabla F(P).$$

These equations also work in the n–dimensional case where

$$S = \text{hypersurface : } F(x_1, \ldots, x_n) = c \quad (\text{dimension } n - 1).$$

In the three dimensional case for $F(x, y, z) =$ constant these equations can also be written:

The tangent plane at $P = (x_0, y_0, z_0)$ is

$$\frac{\partial f}{\partial x}(P)(x - x_0) + \frac{\partial f}{\partial y}(P)(y - y_0) + \frac{\partial f}{\partial z}(P)(z - z_0) = 0.$$

The normal line at P is

$$\begin{aligned} x(t) &= x_0 + t\frac{\partial f}{\partial x}(P) \\ y(t) &= y_0 + t\frac{\partial f}{\partial y}(P) \\ z(t) &= z_0 + t\frac{\partial f}{\partial z}(P). \end{aligned}$$

1.4 Extreme Values and Saddle Point

1.4.1 Maxima and minima: Critical points

Let $f(x_1, \ldots, x_n) = f(\mathbf{x})$ be a function of n variables. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Definition 1.6 *f has a local maximum at $\mathbf{x}_0$ if $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for all $\mathbf{x}$ near $\mathbf{x}_0$.*

f has a local minimum at $\mathbf{x}_0$ if $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x}$ near $\mathbf{x}_0$.

Theorem 1.7 *If $f(x_1, \ldots, x_n)$ has a local maximum or local minimum at P, then either:*

(i) $\dfrac{\partial f}{\partial x_1} = \dfrac{\partial f}{\partial x_2} = \ldots = \dfrac{\partial f}{\partial x_n} = 0$ *at P (i.e. $\nabla f(P) = \mathbf{0}$), or*

(ii) One or more of the partial derivatives $\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n}$ *does not exist at P.*

Definition 1.8 *We say P is a* **critical point** *of f if $\nabla f(P) = 0$.*

Note: Not all critical points are local maxima and minima.

e.g. (i) For the function $f(x, y) = x^2 + y^2$ the critical points satisfy:

$$f_x = 0 \text{ and } f_y = 0$$

so

$$2x = 0 \text{ and } 2y = 0.$$

Therefore $x = 0$ and $y = 0$ is the only critical point of f and this is a global minimum.

(ii) For the functions such as $f(x, y) = y^2 - x^2$ the critical points satisfy:

	$f_x = 0,$	$f_y = 0$	
so	$-2x = 0,$	$2y = 0.$	
Therefore	$x = 0,$	$y = 0$	is the only critical point of f and this is not a local maximum or local minimum but a saddle p

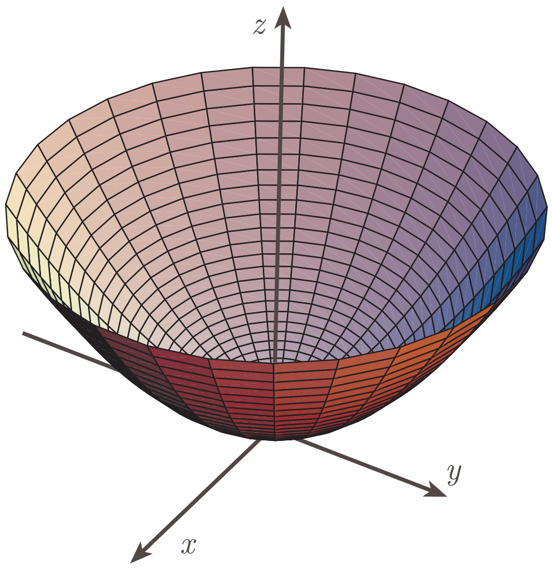


Figure 1.14: Paraboloid $f(x, y) = x^2 + y^2$.

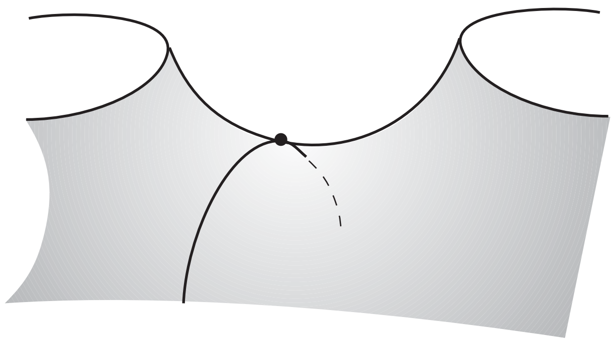


Figure 1.15: Typical saddle point.

Example 1.10 Find the critical point(s) of the function:

$$g(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y.$$

Solution: We need to solve

$$\begin{aligned} \frac{\partial g}{\partial x} &= 0 \\ 2x + 6y + 2 &= 0 \\ x + 3y &= -1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \frac{\partial g}{\partial y} &= 0 \\ 8y + 6x - 4 &= 0 \\ 4y + 3x &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 3 & -1 \\ 3 & 4 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & -5 & 5 \end{array}\right].$$

Therefore

$$-5y = 5 \quad \Rightarrow \quad y = -1,$$

and

$$x + 3y = -1 \quad \Rightarrow \quad x = 2.$$

Therefore there is a critical point at $(2, -1)$.

1.4.2 Classification of critical points

Let (x_0, y_0) be a critical point of $f(x, y)$ so that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

and assume f has continuous second order derivatives near (x_0, y_0).

$$\mathbf{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}, \quad \text{evaluated at } (x_0, y_0) = \mathbf{X_0},$$

and is called the **Hessian** of f at (x_0, y_0).

In order to determine the type of critical points we need to look more closely at $\mathbf{H}$.

Second Derivative Test: Let (x_0, y_0) be a critical point of $f(x, y)$ so that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

and assume f has continuous second order derivatives near (x_0, y_0). Then there are four cases depending upon $det(\mathbf{H})$ and $f_{xx}(x_0, y_0)$:

1. If $\det(\mathbf{H}) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} > 0$ at (x_0, y_0) and $f_{xx}(x_0, y_0) > 0$, then a **minimum** occurs at (x_0, y_0).
2. If $\det(\mathbf{H}) > 0$ at (x_0, y_0) and $f_{xx}(x_0, y_0) < 0$, then a **maximum** occurs at (x_0, y_0).
3. If $\det(\mathbf{H}) < 0$ then a **saddle point** occurs at (x_0, y_0).
4. If $\det(\mathbf{H}) = 0$ then the test is inconclusive and the nature of the critical point (x_0, y_0) can't be determined from the second derivatives. In order to determine the type of critical point it is necessary to study higher order derivatives. The functions $f(x, y) = x^4 + y^4,\ f(x, y) = -x^4 - y^4,\ f(x, y) = x^4 - y^4$ all have a critical points at $(0, 0)$ with $\det \mathbf{H} = 0$. But these points are a local minimum, a local maximum and a saddle point respectively.

Example 1.11 Find and classify the critical points of

$$f(x, y) = x^2y - x^2 - \frac{1}{3}y^3.$$

Solution: First finding the critical points

$$\begin{array}{lcl} f_x = 2xy - 2x, & & f_y = x^2 - y^2, \\ 2x(y-1) = 0, & \text{and} & x^2 - y^2 = 0, \\ x = 0 \quad \text{or} \quad y = 1 & \text{and} & x^2 = y^2. \end{array}$$

Therefore for $x = 0, \quad y^2 = 0 \Rightarrow y = 0$ and for $y = 1, \quad x^2 = 1 \Rightarrow x = \pm 1$.
Thus we have three critical points $(0, 0),\ (1, 1)$ and $(-1, 1)$.
Checking for type of critical points

$$\begin{array}{l} f_{xx} = 2y - 2, \\ f_{yy} = -2y, \\ f_{xy} = 2x. \end{array} \quad \Rightarrow \mathbf{H} = \begin{bmatrix} 2y - 2 & -2y \\ -2y & 2x \end{bmatrix}$$

At $(0, 0)$,	$\det \mathbf{H} = (-2)(0) - 0^2 = 0$	therefore the test is inconclusive.
At $(1, 1)$,	$\det \mathbf{H} = (0)(-2) - 2^2 = -4$	therefore (1,1) is a saddle point.
At $(-1, 1)$,	$\det \mathbf{H} = (0)(-2) - (-2)^2 = -4$	therefore (-1,1) is a saddle point.

1.4.3 Existence of maxima and minima

In general, **global** maxima and **global** minima need not exist.

For functions of one variable: A continuous function $f : [a, b] \to \mathbb{R}$ defined on a **closed, bounded** interval has a global maximum and global minimum in $[a, b]$. If f is differentiable, the maximum or minimum may occur at a **critical point** or **boundary point** a or b.

Similarly **for functions of n variables:** If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, and K is a **closed, bounded** subset of $\mathbb{R}^n$, then f has a global maximum and minimum on K. If f is differentiable then the maximum and minimum may occur at a **critical point** or **boundary point** of K. (The proof of this can be found in real analysis textbooks.)

Definition 1.9 *1. $K \subset \mathbb{R}^n$ is* **bounded** *if it lies within a finite distance from the origin. (i.e. There exists c, such that $||x|| < c$ for all $x \in K$.)*

2. $x \in \mathbb{R}^n$ is a **boundary point** *of K if there are points arbitrarily close to x which are in K, and points arbitrarily close to x which are not in K (see Figure 1.16).*

3. K is **closed** *if it contains all its boundary points.*

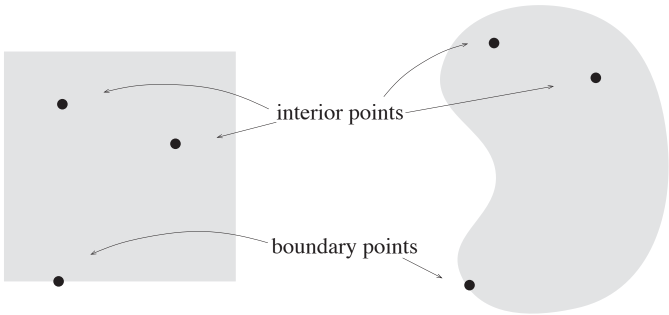


Figure 1.16: Boundary points in a set.

Examples:

1. Closed disc $x^2 + y^2 \leq r^2$ is closed and bounded (see Figure 1.17).

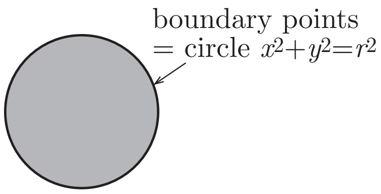


Figure 1.17: Closed disc $x^2 + y^2 \leq r^2$.

2. Open disc $x^2 + y^2 < r^2$ is bounded but not closed (see Figure 1.18).

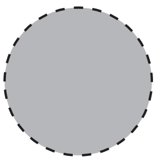


Figure 1.18: Open disc $x^2 + y^2 < r^2$.

3. Half–plane $\{(x, y) : y \geq 0\}$ is closed but not bounded (see Figure 1.19).

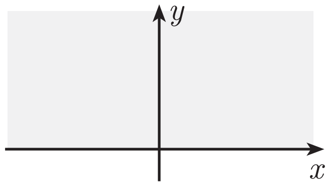


Figure 1.19: Half–plane $\{(x, y) : y \geq 0\}$.

1.5 Multiple Integrals

1.5.1 One Variable

Recall from that

$$\int_a^b f(x)\,dx$$

gives the area between the graph $y = f(x)$ and the x-axis for $a \leq x \leq b$ (if $f(x) \geq 0$). We approximate the area under the graph by rectangular strips.

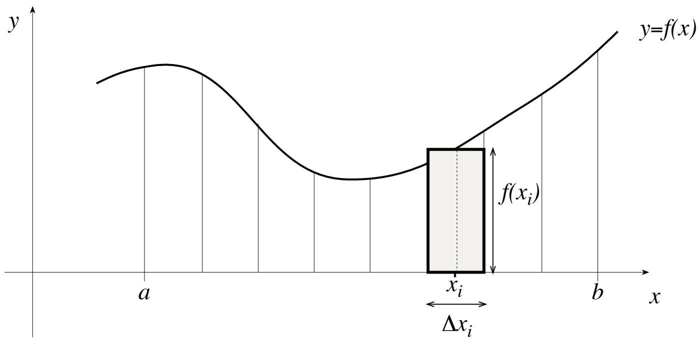


Figure 1.20: Dividing a region into rectangular strips.

Definition 1.10 *The sum*

$$\sum_i f(x_i)\Delta x_i$$

is called a Riemann sum.

Then

$$\int_a^b f(x)\,dx = \lim_{max|\Delta x_i| \to 0} \left(\sum_i f(x_i)\Delta x_i \right).$$

Question How can we calculate the volume of the region in $\mathbb{R}^3$ lying above a set D in the x–y plane ($z = 0$) and below the graph $z = f(x, y)$?

One approach is to divide the region into small rectangular boxes and add up the volumes of all the boxes. Then take a limit. The result will be called **integral of f over D**, and written

$$\iint_D f\,dA.$$

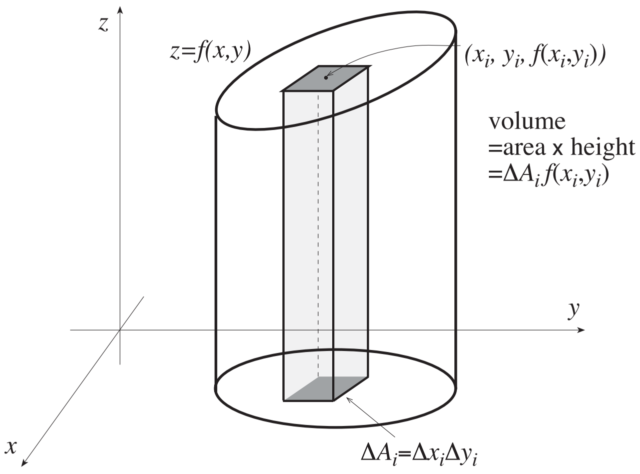


Figure 1.21: Dividing a region into rectangular boxes

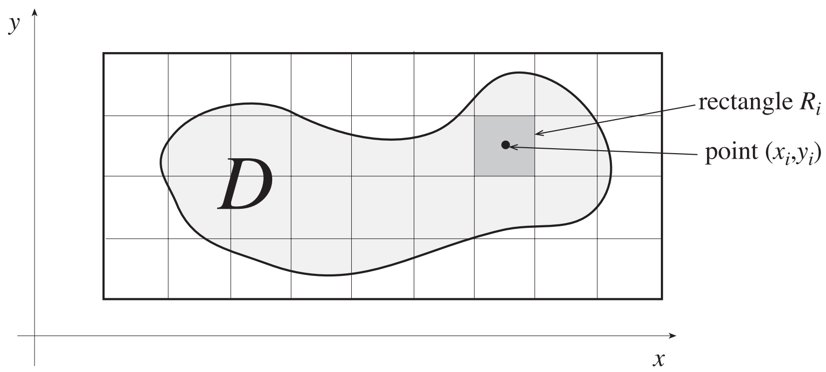


Figure 1.22: Dividing D into rectangles.

1.5.2 Definition of double integral

Let D be a bounded region in $\mathbb{R}^2$, $f : D \to \mathbb{R}$ a function of two variables defined on D. Define the integral of f over D using Riemann sums in the following way.

1. Cover D by a rectangular grid.
2. Let $A_1, A_2, \dots, A_k$ be the rectangles inside D. Then

$$\Delta A_i = \text{area of } A_i = \Delta x_i \Delta y_i$$

$$(x_i, y_i) = \text{point inside } A_i.$$

3. Now look at Riemann sums

$$\sum_{i=1}^{k} f(x_i, y_i)\Delta A_i = \sum_{i=1}^{k} f(x_i, y_i)\Delta x_i \Delta y_i$$

Let $|A|$ be the maximum length of edges in the rectangles A_i.

Theorem 1.11 *If f is* **continuous** *on D, and D is bounded by curves of finite total length, then all such Riemann sums approach the same limit provided $|A| \to c$.*

So we can define

$$\iint_D f\,dA = \lim_{|A|\to 0} \sum_{i=1}^{k} f(x_i, y_i)\Delta A_i.$$

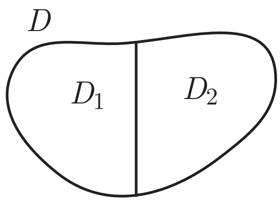
This is called **Riemann integral of f** over the region D, also written

$$\iint_D f\,dx\,dy.$$

1.5.3 Properties of double integrals

1. $\displaystyle\iint_D (f+g)\,dA = \iint_D f\,dA + \iint_D g\,dA.$
2. $\displaystyle\iint_D cf\,dA = c\iint_D f\,dA$, where c is a constant.
3. $\displaystyle\iint_D f\,dA \geq \iint_D g\,dA$, if $f \geq g$ at all points of D.
4. $\displaystyle\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$, if D is the union of two **non–overlapping** regions D_1 and D_2.

e.g.



1.5.4 Interpretations of double integrals

1. If $f : D \to \mathbb{R}$ is continuous, $f(x, y) \geq 0$ then $\displaystyle\iint_D f\,dA$ is the volume of solid lying above region D in the xy–plane and below the graph of f.
2. If $f(x, y) = 1$ for all $x,\ y$ then we obtain the area of D.

$$\text{Area}(D) = \iint_D 1\,dx\,dy.$$

3. Integral of density is the total mass
   Integral of charge density is the total charge.

1.5.5 Double Integrals over Rectangular Regions

The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Consider the reverse of differentiation, partial integration. The symbol

$$\int_a^b f(x, y)\,dx$$

is a partial definite integral with respect to x. It is evaluated by holding y fixed and integrating with respect to x.

This being the case, we can consider the following types of calculations:

$$\int_c^d \left[\int_a^b f(x,y)\,dx\right]\,dy$$

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right]\,dx$$

which are respectively written

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

These are examples of iterated (in this case, double) integrals.

Example 1.12 Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle

$$R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}$$

Solution: The volume we want is shown in Figure 1.23. The volume is given by

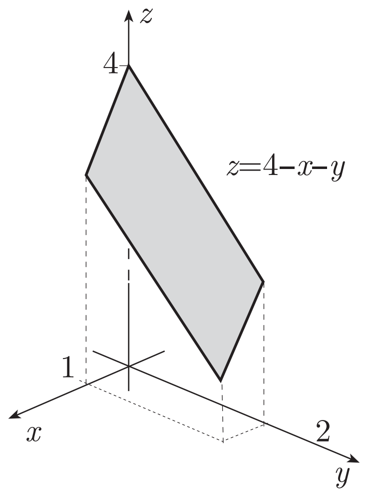


Figure 1.23: Volume under the plane $z = 4 - x - y$.

$$\begin{aligned} V &= \int_0^2 \int_0^1 (4-x-y)\,dx\,dy \\ &= \int_0^2 \left[4x - \frac{1}{2}x^2 - xy\right]_0^1 dy \\ &= \int_0^2 \left(\frac{7}{2} - y\right)\,dx\,dy \\ &= \left[\frac{7}{2}y - \frac{1}{2}y^2\right]_0^1 = 5 \end{aligned}$$

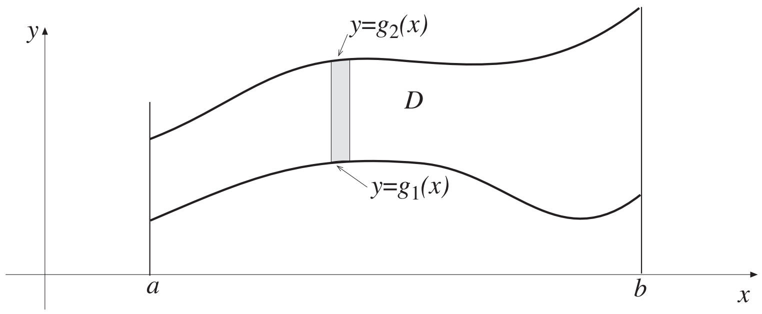
Alternatively,

$$V = \int_0^1 \int_0^2 (4-x-y)\,dy\,dx = 5.$$

1.5.6 Double Integrals over Non-rectangular Regions

The regions involved in double integrals can be divided into groups according to their boundaries.

Type 1 Region:



We see that D is the region defined by

$$g_1(x) \leq y \leq g_2(x) \text{ where } a \leq x \leq b.$$

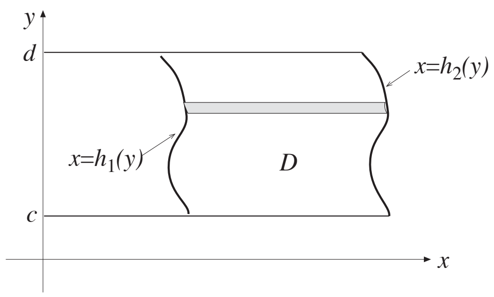
Here

$$\iint_D f\,dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

i.e.

1. Integrate $f(x,y)$ with respect to y, keeping x fixed.
2. Integrate the result with respect to x from a to b.

Type 2 Region:



N ow D is the region defined by

$$h_1(y) \leq x \leq h_2(y) \text{ where } c \leq y \leq d.$$

Here

$$\iint_D f\,dA = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy$$

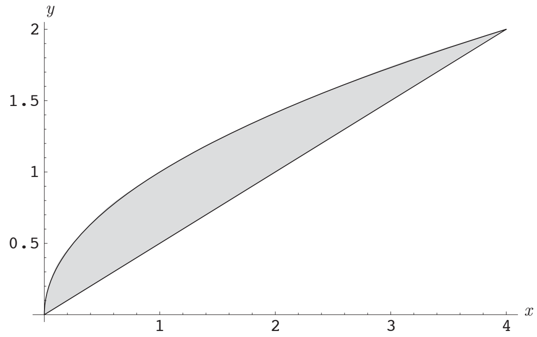
i.e.

1. Integrate $f(x,y)$ with respect to x, keeping y fixed.
2. Integrate the result with respect to y from c to d.

Example 1.13 Evaluate $\displaystyle\iint_D (y^3 + 4x)\,dA$ where D is the region enclosed by the graphs $x = y^2$ and $x = 2y$.

Solution: First, find the intersection points of these curves and sketch D.

$$\begin{aligned} x &= y^2 = 2y \\ y^2 - 2y &= 0 \\ y(y-2) &= 0 \\ y = 0 \quad &\text{or} \quad y = 2 \\ x = 0, \quad &x = 4 \quad \text{respectively.} \end{aligned}$$



This can be evaluated in two ways:

1. As a type 2 region where the region D is described as

$$y^2 \le x \le 2y, \quad 0 \le y \le 2$$

   So that the integral becomes

$$\begin{aligned} \iint_D (y^3 + 4x)\, dA &= \int_0^2 \left(\int_{y^2}^{2y} (y^3 + 4x)\, dx \right) dy \\ &= \int_0^2 \left[xy^3 + 2x^2 \right]_{x=y^2}^{x=2y} dy \\ &= \int_0^2 \left(2y^4 + 8y^2 - y^5 - 2y^4 \right) dy \\ &= \int_0^2 \left(8y^2 - y^5 \right) dy \\ &= \left[\frac{8}{3}y^3 - \frac{y^6}{6} \right]_0^2 = \frac{32}{3} \end{aligned}$$

2. As a type 1 integral

$$\begin{aligned} &\tfrac{1}{2}x \le y \le \sqrt{x}, \\ &0 \le x \le 4. \end{aligned}$$

   The integral is then (using opposite order of integration)

$$\int_0^4 \left(\int_{\frac{1}{2}x}^{\sqrt{x}} (y^3 + 4x)\, dy \right) dx.$$

1.5.7 Changing the Order of Integration

As we have seen double integrals can be evaluated by integrating with respect to x and then y or with respect to y and then x. It may be that one of these methods gives a simpler expression to integrate and we may wish to change the order of integration to take advantage of this.

Example 1.14 Evaluate the integral

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$$

Solution: We can't evaluate the inner integral $\int_y^1 e^{x^2}\, dx$ explicitly. So we need to convert this to a double integral and change the order of integration. First find the region of integration D.

We have $y \leq x \leq 1, \quad 0 \leq y \leq 1$. So D is the triangle bounded by $y = 0$ below, by $x = 1$ on the right, and by $x = y$ as the hypotenuse on the left. We can rewrite the region as

$$0 \leq y \leq x, \quad 0 \leq x \leq 1.$$

Therefore the integral is

$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

This is a much simpler integral since x can be considered a constant in the inner integral. Therefore we have

$$\begin{aligned} \int_0^1 \int_0^x e^{x^2}\, dy\, dx &= \int_0^1 \left[y e^{x^2} \right]_0^x dx \\ &= \int_0^1 \left(x e^{x^2} \right) dx \end{aligned}$$

This can now be done using the substitution $u = x^2$ and $\mathrm{d}u = 2x\, dx$.

Therefore $\displaystyle\int x e^{x^2}\, dx = \int \frac{1}{2} e^u\, du = e^u = e^{x^2}$

$$\begin{aligned} &= \left[\frac{1}{2} e^{x^2} \right]_0^1 \\ &= \frac{1}{2}(e - 1). \end{aligned}$$

1.5.8 Triple Integrals

Just as a double integral can be evaluated by two single integrations, a triple integral can be evaluated by three single integrations. If a region D in $\mathbb{R}^3$ is defined by the inequalities

$$\begin{gathered} a \leq x \leq b \\ c \leq y \leq d \\ k \leq z \leq l \end{gathered}$$

and f is continuous on the region, then

$$\begin{aligned} \iiint_D &= \int_a^b \int_c^d \int_k^l f(x,y,z)\, dz\, dy\, dx \\ &= \int_c^d \int_a^b \int_k^l f(x,y,z)\, dz\, dx\, dy \\ &= \int_a^b \int_k^l \int_c^d f(x,y,z)\, dy\, dz\, dx \\ &= \int_k^l \int_a^b \int_c^d f(x,y,z)\, dy\, dx\, dz \\ &= \int_k^l \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz \\ &= \int_c^d \int_k^l \int_a^b f(x,y,z)\, dx\, dz\, dy \end{aligned}$$

Example 1.15 Evaluate $\displaystyle\iiint_G 12xy^2z^3\, dV$ over the region G defined by the inequalities

$$\begin{gathered} -1 \leq x \leq 2 \\ 0 \leq y \leq 3 \\ 0 \leq z \leq 2 \end{gathered}$$

Solution: There are 6 possible forms of the integral to use. We choose

$$\begin{aligned}
\int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12xy^2z^3\,dz\,dy\,dx &= \int_{-1}^{2}\int_{0}^{3} \left[3xy^2z^4\right]_0^2\,dy\,dx \\
&= \int_{-1}^{2}\int_{0}^{3} 48xy^2\,dy\,dx \\
&= \int_{-1}^{2} 432x\,dx \\
&= \left[216x^2\right]_{-1}^{2} = 648
\end{aligned}$$

The other 5 possible forms give the same result.

We can also evaluate triple integrals over more general regions. In fact if $f(x, y, z) = 1$ then the triple integral $\iiint_G f(x, y, z)\,dV$ will give the volume of the region G.