Equazione di Schrödinger

<u>Teor</u> (Riesz-Thorin) Sia T un operatore lineare $T: L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d) \to L^{q_0}(\mathbb{R}^d) \cap L^{q_1}(\mathbb{R}^d)$

con

$$|Tf|_{L^{q_j}} \leq M_j \, |f|_{L^{p_j}} \quad \text{per } j = 0, 1$$

$$\forall f \in L^{p_0} \cap L^{p_1}$$

Allora $\forall$ $t \in (0,1)$ posto

$$\frac{1}{p_t} := (1-t)\frac{1}{p_0} + t\frac{1}{p_1}$$

$$\frac{1}{q_t} := (1-t)\frac{1}{q_0} + t\frac{1}{q_1}$$

si ha

$$|Tf|_{L^{q_t}} \leq (M_0)^{1-t} (M_1)^t \, |f|_{L^{p_t}}$$

$$\forall f \in L^{p_0} \cap L^{p_1}$$

<u>Dim</u> Osservazioni. Se $p_t = \infty$, $p_0 = p_1 = \infty$

$$|Tf|_{L^{q_t}} = \big|\, |Tf| \,\big|_{L^{q_t}} = \big|\, |Tf|^t \, |Tf|^{1-t} \big|_{L^{q_t}}$$

$$\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1} \qquad \leq \big|\, |Tf|^{1-t} \big|_{L^{\frac{q_0}{1-t}}} \; \big|\, |Tf|^t \big|_{L^{\frac{q_1}{t}}}$$

$$= \|Tf\|_{L^{q_0}}^{1-t} \, \|Tf\|_{L^{q_1}}^{t} \le M_0^{1-t} \|f\|_{L^\infty}^{1-t} \, M_1^{t} \, \|f\|_{L^\infty}^{t}$$

$$= M_0^{1-t} M_1^{t} \, \|f\|_{L^\infty}$$

Sia ora $p_t < +\infty$. Allora

$$\|Tf\|_{L^{q_t}} = \sup \left\{ \left| \int_{\mathbb{R}^d} Tf \, g \, dx \right| : g \in L^{q_t'}, \ \|g\|_{L^{q_t'}} = 1 \right\}$$

$\int_{\mathbb{R}^d} Tf \, g \, dx$ dove $\|g\|_{L^{q_t'}} = 1 = \|f\|_{L^{p_t}}$

$$f = \sum_{j=1}^{m} a_j \chi_{E_j} \qquad a_j = e^{i\vartheta_j} |a_j|$$

$q_t' < +\infty$

$$g = \sum_{k=1}^{N} b_k \chi_{F_k} \qquad b_k = e^{i\psi_k} |b_k|$$

$$\alpha(t) = \frac{1}{p_t} \neq 0$$

$$f_z = \sum_{j=1}^{m} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{i\vartheta_j} \chi_{E_j} \qquad \alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_1}$$

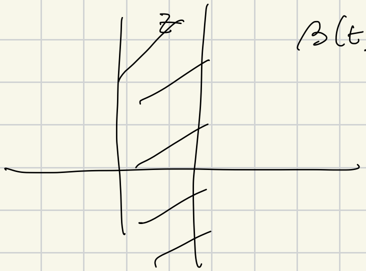
$$0 \le \operatorname{Re} z \le 1$$

$$g_z = \sum_{k=1}^{N} |b_k|^{\frac{\beta(z)}{\beta(t)}} e^{i\psi_k} \chi_{F_k} \qquad \beta(z) = \frac{1-z}{q_0} + \frac{z}{q_1}$$

$$\beta(t) = \frac{1}{q_t} \neq 0$$

$f_t = f$

$g_t = g$



$F(z) = \int T f_z \, g_z \, dx$ è una funzione olomorfa in $0 < \operatorname{Re} z < 1$

$$= \sum_{j=1}^{m} \sum_{k=1}^{N} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} |b_k|^{\frac{\beta(z)}{\beta(t)}} e^{i\theta_j} e^{i\psi_k} \int T \chi_{E_j} \chi_{F_k}$$

Voglio dimostrare che

$$|F(t)| = \left| \int T f \, g \right| \leq M_0^{1-t} M_1^{t}$$

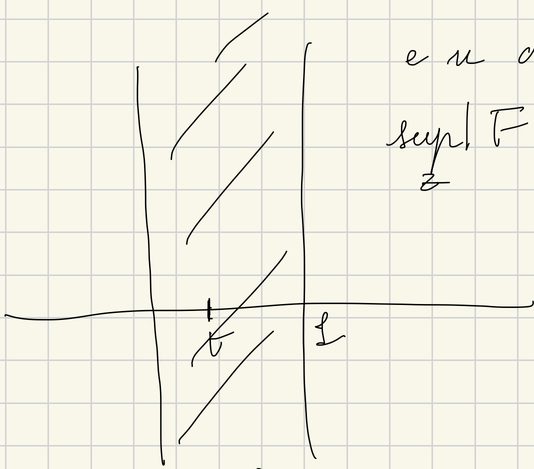
Lo dimostriamo dimostrando che

$$|F(z)| \leq M_0 \quad \text{se } \operatorname{Re} z = 0$$

$$|F(z)| \leq M_1 \quad \text{se } \operatorname{Re} z = 1$$

e si dimostra che

$$\sup_z |F(z)| = B < +\infty$$



C'è un teorema di analisi complessa, il

lemma delle tre rette,

che garantisce che

$$|F(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$$

$$\left| |a_j|^{\frac{\alpha(z)}{\alpha(t)}} \right| = \left| |a_j|^{\frac{\frac{1-z}{p_0}+\frac{z}{p_1}}{\alpha(t)}} \right| =$$

$$= |a_j|^{\frac{\frac{1-\operatorname{Re} z}{p_0}+\frac{\operatorname{Re} z}{p_1}}{\alpha(t)}} \qquad 0 \leq \operatorname{Re} z \leq 1$$

Vogliamo

$$|F(iy)| \leq M_0$$

$$\alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_1}$$

$$\alpha(t) = \frac{1}{p_t}$$

$p_0 < +\infty$

$$|f_{iy}|^{p_0} = \left| \sum_{j=1}^{m} |a_j|^{\frac{\alpha(iy)}{\alpha(t)}} e^{i\vartheta_j} \chi_{E_j} \right|^{p_0}$$

$$= \sum_{j=1}^{m} |a_j|^{\operatorname{Re}\frac{\alpha(iy)}{\alpha(t)} p_0} \chi_{E_j}$$

$$= \sum_{j=1}^{m} |a_j|^{\frac{p_0 \operatorname{Re}\left(\frac{1-\cancel{iy}}{p_0} + \frac{\cancel{iy}}{p_1}\right)}{\frac{1}{p_t}}} \chi_{E_j} =$$

$$= \sum_{j=1}^{m} |a_j|^{p_t} \chi_{E_j} = |f|^{p_t}$$

$$\int_{\mathbb{R}^d} |f_{iy}|^{p_0} dx = \int_{\mathbb{R}^d} |f|^{p_t} dx = 1$$

$$|g_{iy}|^{q_0'} = |g|^{q_t'}$$

$$\int_{\mathbb{R}^d} |g_{iy}|^{q_0'}\,dx = \int_{\mathbb{R}^d} |g|^{q_t'}\,dy = 1$$

$$|F(iy)| = \left|\int_{\mathbb{R}^d} Tf_{iy}\, g_{iy}\,dx\right| \leq$$

$$\leq \|Tf_{iy}\|_{L^{q_0}} \overbrace{\|g_{iy}\|_{q_0'}}^{=1}$$

$$\leq M_0 \|f_{iy}\|_{L^{p_0}} = M_0$$

$$|F(iy)| \leq M_0 \qquad \forall\ y \in \mathbb{R}$$

$$|F(1+iy)| \leq M_1 \qquad \forall\ y \in \mathbb{R}$$

$$|F(t)| \leq M_0^{1-t} M_1^{t}$$

$$\left|\int Tf\, g\right| \leq M_0^{1-t} M_1^{t} \implies \|T\|_{L^{p_t} \to L^{q_t}} \leq M_0^{1-t} M_1^{t}$$

$$\|f\|_{L^{p_t}} = 1 \qquad \|g\|_{L^{q_t'}} = 1$$

$$\mathcal{F}f = \frac{1}{(2\pi)^{\frac{d}{2}}} \lim_{R\to\infty} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\, \chi\!\left(\frac{x}{R}\right) dx$$

$$\left| \mathcal{F} : L^1 \to L^\infty \right| \leq \frac{1}{(2\pi)^{\frac{d}{2}}}$$

$$\left| \mathcal{F} : L^2 \to L^2 \right| \leq 1$$

Lieb

$$\left| \mathcal{F} : L^p \to L^{p'} \right| \leq (2\pi)^{-d\left(\frac{1}{2} - \frac{1}{p'}\right)}$$

$1 \leq p \leq 2$

$$\frac{1}{p} = \frac{1-t}{1} + \frac{t}{2}$$

$$\frac{1}{p'} = \frac{1-t}{\infty} + \frac{t}{2}$$

$$t = \frac{2}{p'}$$

$$\frac{1}{p'} = 1 - \frac{1}{p} = (1-t)+t - \left(\frac{1-t}{1} + \frac{t}{2}\right) = (1-t)\left(1-\frac{1}{1}\right) + t\left(1 - \frac{1}{2}\right.$$

$$\left| \mathcal{F} : L^p \to L^{p'} \right| \leq \left| \mathcal{F} : L^1 \to L^\infty \right|^{1-t} \left| \mathcal{F} : L^2 \to L^2 \right|^t$$

$$= (2\pi)^{-\frac{d}{2}\left(1 - \frac{2}{p'}\right)}$$

$$= (2\pi)^{-d\left(\frac{1}{2} - \frac{1}{p'}\right)}$$

<u>Lemma</u> Sia

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)\, f(y)\, dy$$

$$M_1 := \sup_{x \in \mathbb{R}^d} \int |K(x,y)|\, dy < +\infty$$

$$M_2 := \sup_{y \in \mathbb{R}^d} \int |K(x,y)|\, dx < +\infty$$

allora $\quad \|T\|_{L^p \to L^p} \leq M_1^{\frac{1}{p'}} \quad M_2^{\frac{1}{p}}$

<u>Dim</u> $\quad T : L^1 \to$

$$\int dx \, |Tf(x)| \leq \int dx \int |K(x,y)| \, |f(y)| \, dy =$$

$$= \int dy \, |f(y)| \int dx \, |K(x,y)|$$

$$\leq \underbrace{\sup_{y' \in \mathbb{R}^d} \int dx \, |K(x,y)|}_{M_1} \quad \|f\|_{L^1}$$

$$\|T : L^1 \to L^1\| \leq M_1 \, , \qquad \|T : L^\infty \to L^\infty\| \leq M_2$$

$1 < p < \infty \qquad \frac{1}{p} = \frac{1-t}{1} + \frac{t}{\infty} = 1 - t$

$$t = 1 - \frac{1}{p} = \frac{1}{p'}$$

$$\|T : L^p \to L^p\| \leq M_1^{\frac{1}{p'}} \quad M_2^{\frac{1}{p}}$$

Equazione di Schrodinger

$$\begin{cases} i \partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases} \qquad \partial_t u = i \Delta u$$

$$\begin{cases} i\,\partial_t \hat{u}(t,\xi) - |\xi|^2 \hat{u}(t,\xi) = 0 \\ \hat{u}(0,\xi) = \hat{u}_0(\xi) \end{cases}$$

$$e^{it|\xi|^2}$$

$$\partial_t \hat{u}(t,\xi) + i\,|\xi|^2 \hat{u}(t,\xi) = 0$$

$$\partial_t \left(e^{it|\xi|^2} \hat{u}(t,\xi) \right) = 0$$

$$e^{it|\xi|^2} \hat{u}(t,\xi) = \hat{u}_0(\xi)$$

$$\hat{u}(t,\xi) = e^{-it|\xi|^2} \hat{u}_0(\xi)$$

$$= \hat{G}(t,\xi)\, \hat{u}_0(\xi)$$

$$u(t,x) = (2\pi)^{-\frac{d}{2}}\, G(t,\cdot) * u_0(x)$$

$$e^{-it|\xi|^2} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} (2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}}\, dx$$

$$e^{-z\frac{|\xi|^2}{2}} = (2\pi z)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\, e^{-\frac{|x|^2}{2z}}\, dx$$

$z > 0$ $\qquad$ $\operatorname{Re} z > 0$

$z \to 2i$

$$u(t,x) = (4\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}}\, u_0(y)\, dy$$

$$e^{it\Delta} u_0(x)$$

$$\left\| e^{it\Delta} u_0 \right\|_{L^\infty(\mathbb{R}^d)} \leq (4\pi t)^{-\frac{d}{2}} \quad \| u_0 \|_{L^1}$$

$$\left\| e^{it\Delta} u_0 \right\|_{L^2(\mathbb{R}^d)} = \left\| e^{-it|\xi|} \hat{u}_0 \right\|_{L^2(\mathbb{R}^d)} = \| \hat{u}_0 \|_{L^2(\mathbb{R}^d)}$$

$$\left\| e^{it\Delta} \right\|_{L^2 \to L^2} = 1$$

$$1 < p \cancel{<} 2$$

$$\frac{1}{p} = \frac{1-t}{1} + \frac{t}{2}$$

$$\frac{1}{p'} = 0 + \frac{t}{2} \qquad t = \frac{2}{p'}$$

$$\left\| e^{it\Delta} \right\|_{L^p \to L^{p'}} \leq \left\| e^{it\Delta} \right\|_{L^1 \to L^\infty}^{1-\frac{2}{p'}} = (4\pi t)^{-\frac{d}{2}\left(1-\frac{2}{p'}\right)} =$$

$$= (4\pi t)^{-d\left(\frac{1}{2}-\frac{1}{p'}\right)}$$

$$L^p(\mathbb{R}^d) \longrightarrow L^{p'}(\mathbb{R}^d)$$

$$1 \leq p \leq 2 \leq p' \leq +\infty$$

$$\tau_D f = f(x-D)$$

$$\tau_D e^{it\Delta} = e^{it\Delta} \tau_D$$

$$i\partial_t u + \Delta u = f \qquad u(0) = u_0 \in H^1(\mathbb{R}^d, \mathbb{C})$$

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-t')\Delta} f(t')\, dt'$$

Strichartz $\qquad (x, t)$

$d \qquad |x| = |t|$

(q, r) è ammissibile se

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

$$2 \leq r \leq \frac{2d}{d-2} \qquad \text{se } d \geq 3$$

$d = 1 \qquad 2 \leq r \leq +\infty \qquad q = \infty \quad r = 2$

$d = 2 \qquad 2 \leq r < +\infty$

$u \qquad u_0$

$u_\lambda \qquad u_0(\lambda\, \cdot) \qquad u_\lambda(t, x) = u(\lambda^2 t, \lambda x)$

Teor

1) $\forall\, u_0 \in L^2(\mathbb{R}^d)$ si ha che

$$e^{it\Delta} u_0 \in L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \cap C^0(\mathbb{R}, L^2(\mathbb{R}^d))$$

$\forall\, (q, r)$ ammissibile ed inoltre $\exists\, C_{q,r}$ t.c.

$$\| e^{it\Delta} u_0 \|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C_{q r} \| u_0 \|_{L^2(\mathbb{R}^d)}$$

2) I un intervallo, $t_0 \in \bar{I}$. Se (γ, ρ) è una coppia ammissibile e $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))$ allora

per ogni coppia ammissibile (q, r)

$$Tf(t) = \int_{t_0}^{t} e^{i(t-s)\Delta} f(s)\, ds$$

appartiene a $L^q(I, L^r(\mathbb{R}^d)) \cap \cancel{C}\, C^0(I, L^2(\mathbb{R}^d))$

ed $\exists$ $C = C(q, r, \gamma, \rho)$ t.c.

$$\| Tf \|_{L^q(I, L^r(\mathbb{R}^d))} \le C \, \| f \|_{L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))}$$

Foschi

$$\begin{cases} i\, \partial_t u = -\Delta u + \lambda |u|^{p-1} u & t \in \mathbb{R}^d \\ u_{|t=0} = u_0 \in H^1(\mathbb{R}^d) & \lambda = 1, -1 \end{cases}$$

focusing

$1 < p < d^*$

$$d^* = \begin{cases} +\infty & d = 1, 2 \\ \dfrac{d+2}{d-2} & d \ge 3 \end{cases}$$