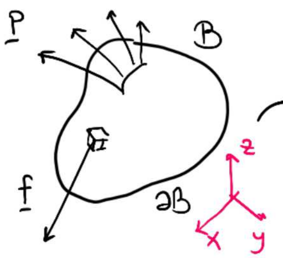
Finite Elements in Structural Mechanics

Beams and frames

Marco Rossi

STRUCTURAL THEORY of BEAMS

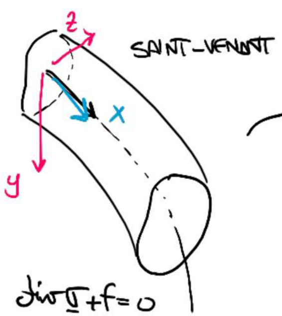


$$\mathrm{div}\,\underline{\underline{\sigma}} + \underline{f} = 0$$

GENERIC 3D ELASTIC SOLID

↓

GENERAL THEORY

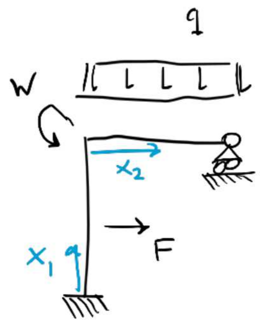


$$\mathrm{div}\,\underline{\underline{\sigma}} + \underline{f} = 0$$

3D ELASTIC SOLID ASSUMING A "BEAM GEOMETRY"

↓

FUNDAMENTAL RESULT FOR GEOMETRIC REDUCTION



$$EJ v^{IV} = q(x)$$

1D ELASTIC MODEL

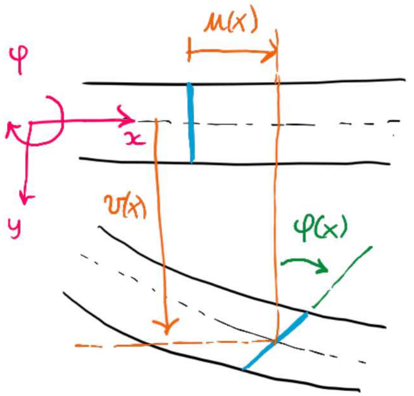
↓

1D BEAM MODEL

THE FEATURES of THE 1D MODEL ARE OBTAINED FROM THE DE SAINT-VENANT'S MODEL →

NAVIER'S HYPOTHESIS

THE CROSS-SECTION HAS A RIGID MOTION (ROTATION, TRANSLATION) AND STILL REMAINS PLANE



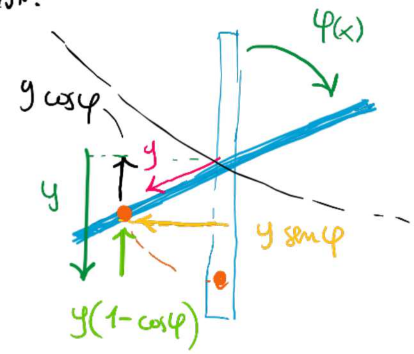
$u_x(x,y)$, $u_y(x,y)$ DISPLACEMENTS of THE GENERIC POINT (x,y) OF THE BEAM

$u(x), v(x)$: DISPLACEMENTS of THE AXIS OF THE BEAM

$\varphi(x)$: ROTATION of THE CROSS-SECTION

THE DISPLACEMENT OF A GENERIC POINT of THE CROSS-SECTION:

$$\begin{cases} u_x = u_x(x,y) = u(x) - y \sin\varphi(x) \\ u_y = u_y(x,y) = v(x) - y(1-\cos\varphi(x)) \end{cases}$$



THE DISPLACEMENT of THE GENERIC POINT CAN BE REWRITTEN AS A FUNCTION of THE DISPLACEMENTS OF THE AXIS OF THE BEAM (functions of x)

SMALL DISPLACEMENT HYPOTHESIS

$$\begin{cases} \cos\varphi(x) \approx 1 \\ \sin\varphi(x) \approx \varphi(x) \end{cases} \rightarrow \boxed{\begin{aligned} u_x(x,y) &= u(x) - y\,\varphi(x) \\ u_y(x,y) &= v(x) \end{aligned}}$$

DEFORMATION FIELDS

$$\varepsilon_x(x,y) = \frac{\partial u_x}{\partial x} = \frac{\partial u(x)}{\partial x} - y\frac{\partial \varphi(x)}{\partial x} \quad , \quad \varepsilon_y(x,y) = \frac{\partial u_y}{\partial y} = \frac{\partial v(x)}{\partial y} = 0$$

(NO ELONGATION ALONG y-AXIS)

$$\gamma_{xy}(x,y) = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\varphi(x) + \frac{\partial v(x)}{\partial x}$$

$$\boxed{\begin{aligned} \varepsilon_x &= \frac{du(x)}{dx} - y\frac{d\varphi(x)}{dx} \\ \gamma_{xy} &= -\varphi(x) + \frac{dv(x)}{dx} \end{aligned}}$$

THIN (SLENDER) BEAM HYPOTHESIS

EULER-BERNOULLI MODEL

DURING THE DEFORMATION of A SLENDER BEAM THE CROSS-SECTIONS STILL REMAIN ORTHOGONAL TO THE BEAM AXIS

↓

THE ROTATION OF THE CROSS-SECTION IS THE SAME OF THE BEAM AXIS !!

$$\boxed{\varphi(x) = \frac{dv(x)}{dx}}$$

EULER – BERNOULLI HYPOTHESIS $\longrightarrow \varphi(x) = \dfrac{dv(x)}{dx} \rightarrow \boxed{\gamma_{xy} = \varphi - \dfrac{dv}{dx} = 0}$

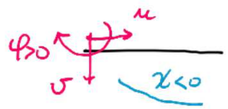
ACCORDING TO THIS HYPOTHESIS, THE SHEAR STRAIN VANISHES, $\gamma_{xy} = 0$, BUT FROM SAINT'S VENANT THEORY WE KNOW THAT:

$$d\gamma_{xy} = \frac{dT}{GA_s} \rightarrow \underbrace{dT}_{\text{FINITE VALUE}} = \underbrace{GA_s}_{\text{IT MUST BE INFINITY}} \underbrace{d\gamma_{xy}}_{\text{NULL}}$$

$\infty \cdot 0 = k$

IN EULER-BERNOULLI BEAM, THE SHEAR RIGIDITY TENDS TO INFINITY, $GA_s \to \infty$,

IN SLENDER BEAM ONE NEGLECTS SHEAR DEFORMATIONS



IN GENERAL

$$\varepsilon_x(x) = \underbrace{\frac{du}{dx}}_{\substack{\eta(x) \\ \text{ELONGATION OF THE BEAM AXIS}}} - y \underbrace{\frac{d\varphi}{dx}}_{\substack{-\chi(x) \\ \text{CURVATURE}}}, \quad \varepsilon_y(x) = 0, \quad \gamma_{xy}(x) = \underbrace{-\varphi(x) + \frac{dv(x)}{dx}}_{\substack{t(x) \\ \text{SHEAR DEFORMATION} \\ \text{(VANISHING FOR EB MODEL)}}}$$

HENCE

$$\begin{cases} \varepsilon_x(x) = \eta(x) + y\,\chi(x) \\ \gamma_{xy}(x) = t(x) \end{cases} \longrightarrow$$

LET'S USE THE PRINCIPLE OF VIRTUAL WORK TO ASSOCIATE THE CORRECT STATIC QUANTITIES

INTERNAL WORK:

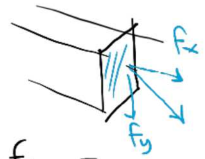
$$\mathcal{L}_{VI} = \int_V \underline{\underline{\sigma}} \cdot \underline{\underline{\varepsilon}} = \int_V \sigma_x \varepsilon_x + \tau_{xy}\gamma_{xy} = \int_0^\ell \left[\int_A \sigma_x \left(\eta(x) + y\chi(x) \right) + \tau_{xy} t(x)\, dA \right] dx$$

$$= \int_0^\ell \left\{ \eta(x) \underbrace{\int_A \sigma_x}_{\substack{N(x) \\ \text{AXIAL FORCE}}} + \chi(x) \underbrace{\int_A \sigma_x y}_{\substack{M(x) \\ \text{BENDING MOMENT}}} + t(x) \underbrace{\int_A \tau_{xy}}_{\substack{T(x) \\ \text{SHEAR FORCE}}} \right\} dx = \int_0^\ell \eta(x)N(x) + \chi(x)M(x) + t(x)T(x)$$

ONE CAN UNDERSTAND THE LINK BETWEEN STATIC AND KINEMATIC QUANTITIES

EXTERNAL WORK:

$\underline{f} = \{f_x, f_y\}$ IS THE VOLUME FORCE $\quad [f] = \dfrac{F}{L^3}$



$$\mathcal{L}_{VE} = \int_V u_x f_x + u_y f_y = \int_V \left(u(x) - y\varphi(x) \right) f_x + v(x) f_y =$$

$$= \int_0^\ell \left\{ u(x) \underbrace{\int_A f_x\, dA}_{\substack{p(x) \\ \text{DISTRIBUTED AXIAL LOAD}}} - \varphi(x) \underbrace{\int_A y f_x}_{\substack{-m(x) \\ \text{DISTRIBUTED UNIT MOMENT}}} + v(x) \underbrace{\int_A f_y}_{\substack{q(x) \\ \text{TRANSVERSAL LOAD}}} \right. = \int_0^\ell u(x)p(x) + \varphi(x)m(x) + v(x)q(x)$$

HENCE THE PVW FOR A BEAM IS

$$\int_0^\ell \eta(x)N(x) + \chi(x)M(x) + t(x)T(x) = \int_0^\ell u(x)\,p(x) + v(x)\,q(x) + \varphi(x)\,m(x)$$

KINEMATIC MODEL:

$$\begin{cases} u_x(x,y) = u(x) - y\,\varphi(x) \\ u_y(x,y) = v(x) \end{cases}$$

COMPATIBILITY:

$$\begin{cases} \varepsilon(x) = \eta(x) + y\,\chi(x) \\ \gamma(x) = t(x) \end{cases}$$

INDEFINITE EQUILIBRIUM + STATIC EQUIVALENCE

FIND THE EQUILIBRIUM EQUATION FOR THE SAINT-VENANT SOLID

SAINT-VENANT'S HYPOTHESIS

$$\begin{cases} \operatorname{div}(\underline{\underline{\sigma}}) + \underline{f} = 0 \\ \underline{\underline{\sigma}}\,\underline{m} = 0 \end{cases} \leftarrow$$

i) $\sigma_y = \sigma_z = 0$, NEGLECTABLE

ii) NO LOADS ON THE LATERAL SURFACE OF THE SOLID

IN THE DERIVATION, LET'S CONSIDER ALSO τ_{xz}

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & 0 \\ \tau_{xz} & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\sigma}}\,\underline{m} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & 0 \\ \tau_{xz} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \overset{=0}{m_1} \\ m_2 \\ m_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

BOUNDARY CONDITIONS $\quad \tau_{xy} m_2 + \tau_{xz} m_3 = 0$

INDEFINITE EQUILIBRIUM EQUATIONS

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + f_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + f_y = 0 \end{cases}$$

... COMING FROM TRANSVERSAL EQUILIBRIUM EQUATIONS...

LET'S FIND ALSO THE ROTATIONAL EQUILIBRIUM EQUATIONS W.R.T. Z-AXIS

$$\downarrow$$

$$\frac{\partial \sigma_x}{\partial x} \cdot y + \frac{\partial \tau_{xy}}{\partial y} \cdot y + \frac{\partial \tau_{xz}}{\partial z} \cdot y + f_x \cdot y = 0$$

SEMI-INDEFINITE FORM OF EQUILIBRIUM EQUATIONS:

TRANSLATION

$$\begin{cases} \displaystyle\int_A \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x \right) = 0 \quad (1) \\ \displaystyle\int_A \left(\frac{\partial \tau_{xy}}{\partial x} + f_y \right) = 0 \quad (2) \end{cases}$$

ROTATION

$$\int_A \left(\frac{\partial \sigma_x}{\partial x} y + \frac{\partial \tau_{xy}}{\partial y} y + \frac{\partial \tau_{xz}}{\partial z} y + f_x \cdot y \right) = 0 \quad (3)$$

EQUAT. (1)

$$\frac{\partial}{\partial x} \underbrace{\int_A \sigma_x}_{N(x)} + \int_A \left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) + \underbrace{\int_A f_x}_{p(x)} = 0$$

$$\downarrow$$

$$\boxed{\frac{dN(x)}{dx} = -p(x)}$$

AXIAL FORCE EQUILIBRIUM EQUATION

USE GAUSS THEOREM

$$\int_A \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \int_A \operatorname{div} \begin{Bmatrix} 0 \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix} = \int_{\partial A} \begin{Bmatrix} 0 \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix} \cdot \begin{Bmatrix} m_1 \\ m_2 \\ m_3 \end{Bmatrix} =$$

$$= \int_{\partial A} \tau_{xy} m_2 + \tau_{xz} m_3 = 0$$

ACCORDING TO BOUNDARY CONDITIONS

EQUAT. (2)

$$\int_A \frac{\partial \tau_{xy}}{\partial x} + \int_A f_y = \frac{\partial}{\partial x} \underbrace{\int_A \tau_{xy}}_{T(x)} + \underbrace{\int_A f_y}_{q(x)} = 0 \qquad \boxed{\frac{dT(x)}{dx} = -q(x)}$$

TRANSVERSAL EQUILIBRIUM EQUATION

EQUAT. (3)

$$\int_A \frac{\partial}{\partial x}(\sigma_x y) - \int_A \sigma_x \overset{=0}{\frac{\partial y}{\partial x}} + \int_A \frac{\partial}{\partial y}(\tau_{xy} \cdot y) - \underbrace{\int_A \tau_{xy}}_{T(x)} \cdot \overset{=1}{\frac{\partial y}{\partial y}} + \int_A \frac{\partial}{\partial z}(\tau_{xz} y) - \int_A \tau_{xz} \overset{=0}{\frac{\partial y}{\partial z}} + \underbrace{\int_A f_x y}_{m(x)} =$$

$$\frac{\partial}{\partial x} \underbrace{\int_A \sigma_x y}_{M(x)} + \int_{\partial A} y \overset{=0}{(\tau_{xy} m_2 + \tau_{zx} m_3)} - T(x) + m(x) = 0 \qquad \boxed{\frac{dM(x)}{dx} = T(x) - m(x)}$$

ROTATION EQUILIBRIUM

WE DERIVED THE EQUILIBRIUM EQUATIONS OF THE 1D-MODEL BEAM, DIRECTLY FROM SAINT-VENANT MODEL !!

LET'S DEFINE THE CONSTITUTIVE MODEL

ACCORDING TO STATIC AND KINEMATIC ASSUMPTIONS :

$$\sigma_{xy} = \sigma(x) = E\left(\eta(x) + y\,\chi(x)\right)$$

$$\tau_{xy} = \tau_{xy}(x) = G\,t(x)$$

NOW USE CONSTITUTIVE EQUATIONS TOGHETHER WITH STATIC EQUIVALENCE

$$N(x) = \int_A \sigma(x)\,dA = \int_A E\eta(x) + \int_A y\,E\chi(x) =$$

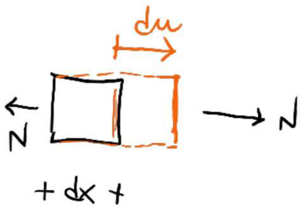
$$= E\eta(x)\int_A dA + E\chi(x)\underbrace{\int_A y\,dA}_{S_z = 0} = EA\,\eta(x)$$

$S_z = 0$ VANISHING FIRST MOMENT of INERTIA

$$\boxed{N(x) = EA\,\eta(x)}$$

MOREOVER $\eta(x) = \dfrac{du}{dx} \rightarrow \boxed{du = \dfrac{N}{EA}\,dx}$

INFINITESIMAL AXIAL DISPLACEMENT of THE CROSS-SECTION



$$M(x) = \int_A \sigma_x \cdot y\,dA = \int_A E\eta(x)\cdot y + \int_A E y^2 \chi(x) = E\eta(x)\underbrace{\int_A y\,dA}_{S_z = 0} + E\chi(x)\underbrace{\int_A y^2 dA}_{J_z} = EJ\chi(x)$$

$S_z = 0$ VANISHING FIRST MOMENT of AREA

$$\boxed{M(x) = EJ\chi(x)}$$

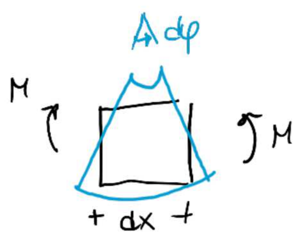
MOMENT - CURVATURE FORMULA

$\chi(x) = -\dfrac{d\varphi(x)}{dx} \leadsto \boxed{d\varphi = -\dfrac{M(x)}{EJ}\,dx}$ INFINITESIMAL ROTATION OF THE CROSS-SECTION

ACCORDING TO EULER-BERNOULLI MODEL :

$\varphi(x) = \dfrac{dv(x)}{dx}$ ROTATION OF CROSS-SECTION IS EQUAL TO THE ROTATION OF BEAM AXIS

$$\chi(x) = -v''(x)$$

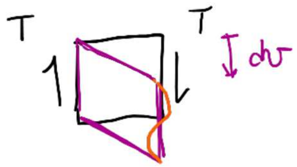


EQUATION FLEXURAL DISPLACEMENT EULER-BERNOULLI

$$\boxed{v''(x) = -\frac{M(x)}{EJ}}$$

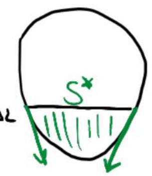
$\chi(x)$ IS THE BEAM CURVATURE

SHEAR DEFORMATION IS MORE COMPLICATED $\longrightarrow$ SHEAR FACTOR CORRECTION



LET'S ASSUME JOURAWSKY HYPOTESIS :

$$\tau = \frac{T}{J}\frac{S^*}{b} \rightarrow$$ PROBLEMS ON THE LATERAL SURFACE...



$\leadsto \tau_{xz}, \tau_{yz}$ ARE NOT UNCORRELATED, HENCE THIS DEPENDENCY ENTERS IN THE ENERGY

CLAPEYRON'S THEOREM:

$$\frac{1}{2}T\,dv = \frac{1}{2}\int_V \underline{\underline{\sigma}}\cdot\underline{\underline{\varepsilon}} = \frac{1}{2}\int_V \underline{\underline{\sigma}}\cdot\mathbb{C}^{-1}\underline{\underline{\sigma}} =$$

$$= \frac{1}{2G}\int_V \tau_{xy}^2 + \tau_{xz}^2 = \frac{dz}{2G}\int_A (\tau_{2x}^2 + \tau_{2y}^2)\,dz$$

$$T\frac{dv}{dz} = \frac{1}{G}\int_A \tau_{xy}^2\left(1 + \alpha(y)\right) = \frac{T^2}{G}\underbrace{\frac{\chi}{A}}_{A_s}$$

$$\boxed{\gamma_{xy} = \frac{T}{GA_s}}$$

SHEAR DEFORMATION

EB MODEL :

$\gamma = 0$, T IS FINITE

$GA_s \to \infty$

EULER-BERNOULLI BEAM MODEL

- $v'(x) = \varphi(x) \longrightarrow$ NULL SHEAR DEFORMATION
- ROTATION OF THE CROSS-SECTION IS IDENTICAL TO THE ROTATION OF THE BEAM AXIS ($\chi(x)$ IS ALSO THE CURVATURE OF THE AXIS)
- LET'S NEGLECT $m(x)$ FOR SIMPLICITY
- LET'S CONSIDER ONLY FLEXURAL DISPLACEMENTS ($\eta(x)$ COMES FROM AXIAL BEHAVIOUR, UNCOUPLED)

KINEMATIC MODEL:

$$\begin{cases} u_x(x,y) = -y\,v'(x) \\ u_y(x,y) = v(x) \end{cases}$$

COMPATIBILITY:

$$\begin{cases} \varepsilon(x) = -y\,v''(x) \\ \gamma(x) \equiv 0 \end{cases}$$

CONSTITUTIVE MODEL + STATIC EQUIVALENCE

$$v''(x) = -\frac{M(x)}{EJ}$$

EQUILIBRIUM

$$\begin{cases} \dfrac{dT}{dx} = -q(x) \\ \dfrac{dM}{dx} = T(x) \end{cases} \longrightarrow \frac{d^2M}{dx^2} = \frac{dT}{dx} \;\downarrow\; \frac{d^2M(x)}{dx^2} = -q(x)$$

4^{th} ORDER EQUILIBRIUM EQUATION:

$$\frac{d^2}{dx^2}\left[-EJ\,\frac{d^2v(x)}{dx^2}\right] = -q(x) \longrightarrow \left[EJ\,v''(x)\right]'' = q(x), \quad x \in [0,\ell]$$

4^{th} ORDER DIFFERENTIAL EQUATION

IF $EJ = cost$

$$\boxed{EJ\,v^{IV}(x) = q(x)}$$

EQUILIBRIUM EQUATION FOR AN HOMOGENEOUS THIN ELASTIC BEAM ACCORDING TO EULER-BERNOULLI MODEL

LET'S CONSIDER THE BOUNDARY CONDITIONS

- $v(x)$, $\varphi(x) = v'(x) \longrightarrow$ ESSENTIAL BOUNDARY CONDITIONS (KINEMATIC QUANTITIES)
- $M(x) = -EJv''(x)$, $T(x) = -EJv'''(x) \longrightarrow$ NATURAL BOUNDARY CONDITIONS (STATIC QUANTITIES)

$$\begin{cases} v(0) = v_0 \\ v'(0) = v_0' \\ v''(0) = v_0'' \\ v'''(0) = v_0''' \end{cases} \qquad \begin{cases} v(L) = v_L \\ v'(L) = v_L' \\ v''(L) = v_L'' \\ v'''(L) = v_L''' \end{cases}$$

"SOME OF THOSE" CONDITIONS MUST BE FIXED

WEAK FORM:

$$\begin{cases} EJv^{IV}(x) - q(x) = 0, \quad x \in [0,\ell] \\ v(x), v'(x), v''(x), v'''(x), \quad x = 0 \vee x = \ell \end{cases}$$

$\longrightarrow$ LET'S FIND THE WEAK FORM USING WEIGHTED RESIDUAL METHOD

$$\int_0^\ell w(x)\left\{EJv^{IV}(x) - q(x)\right\}dx = 0, \quad \forall w(x) \in H_2^0([0,\ell])$$

$\longrightarrow$ THE TEST FUNCTION MUST BE "GOOD ENOUGH" SO THAT ALL THE DERIVATIVES ARE WELL-DEFINED

$\downarrow$

WE CAN "PASS" THE DERIVATIVE FROM THE UNKNOWN TO THE TEST FUNCTION $\longrightarrow$ WE CAN OBTAIN A SYMMETRIC WEAK FORM

$$\int_0^\ell w\left[EJv^{IV}-q\right]dx = EJ\int_0^\ell w v^{IV} - \int_0^\ell wq =$$

$$= EJ\int_0^\ell \left[(wv''')' - w'v'''\right] - \int_0^\ell wq =$$

$$= EJ\int_0^\ell \left[(wv''')' - (w'v'')' + w''v''\right] - \int_0^\ell wq =$$

$$= \int_0^\ell EJv''w'' + [\underbrace{EJv'''}_{-T(x)}w]_0^\ell - [\underbrace{EJv''}_{-M(x)}w']_0^\ell - \int_0^\ell wq =$$

$$= \underbrace{\int_0^\ell EJv''w''}_{a[v,w]} \underbrace{- \int_0^\ell wq + \left[M(x)w'(x)\right]_0^\ell - \left[T(x)w(x)\right]_0^\ell}_{-\ell[w]} = 0, \forall w$$

$a[v,w]$ BILINEAR FORM

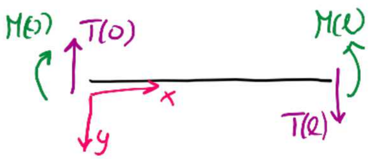
$-\ell[w]$ LINEAR FORM
DISTRIBUTED AND CONCENTRATED LOADS

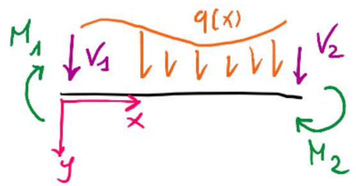
$$\left[Tw\right]_0^\ell = T(\ell)w(\ell) - T(0)w(0)$$

$$\left[Mw'\right]_0^\ell = M(\ell)w(\ell) - M(0)w(0)$$

→ WE CAN FIX THE VALUES ACCORDING TO THE BOUNDARY COND.

IF WE INTERPRET w AS A VIRTUAL DISPLACEMENT,
THESE TERMS ARE THE VIRTUAL WORK OF THE CONCENTRATED LOADS AT THE BOUNDARY





$$\begin{cases} T(0) = -V_1 \\ T(\ell) = V_2 \\ M(0) = M_1 \\ M(\ell) = -M_2 \end{cases}$$

EQUILIBRIUM OF EULER-BERNOULLI BEAM $\iff \boxed{a[v,w] = \ell[w]}, \forall w \in \mathcal{H}_0^2$

$$w \in \mathcal{H}_0^2 = \left\{ g:[0,\ell]\to\mathbb{R},\ g\in L_2,\ g'\in L_2,\ g''\in L_2,\ g(\bar{x})=0 \ \& \ v(\bar{x})=v_0,\ \bar{x}=0 \vee \bar{x}=\ell \right\}$$

$$L_2 = \left\{ f:[0,\ell]\to\mathbb{R},\ \int_0^\ell |f|^2 < +\infty \right\}$$

ex:

BOUNDARY CONDITIONS:

$$\begin{cases} v(0) = 0 \\ v'(0) = \varphi \end{cases} \qquad \begin{cases} v(\ell) = \delta \\ -EJv''(\ell) = W \end{cases}$$

$$\ell[w] = \int_0^\ell \underset{\overset{\parallel}{0}}{qw} + T(\ell)w(\ell) - T(0)w(0) - M(\ell)w'(\ell) + M(0)w'(0)$$

$$= \underbrace{V_2}_{\text{UNKNOWNS}} w(\ell) + \underbrace{V_1}_{\text{UNKNOWNS}} w(0) + \underbrace{M_2}_{\substack{\text{KNOWN} \\ M_2 = W}} w'(\ell) + \underbrace{M_1}_{\text{UNKNOWNS}} w'(0) = W w'(\ell)$$

SINCE WE CAN CHOOSE $w=0$ WHEN $v(x)$ IS KNOWN, WE CAN FIX w SUCH THAT THE UNKNOWNS DISAPPEAR

$$a[v,w] = \int_0^\ell EJv''w''$$

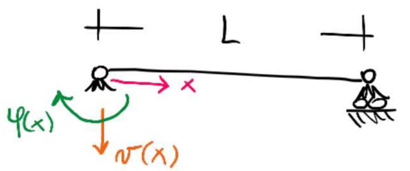
$$a[v,w] = \ell[w],\ \forall w$$

IF WE INTERPRET $w \equiv \delta v$

WE OBTAIN THE VARIATIONAL PRINCIPLE OF THE MINIMUM of TPE

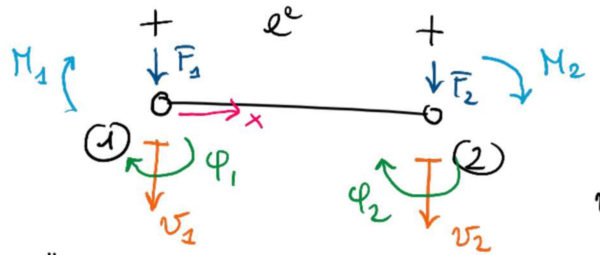
$$\Pi[v] = \underbrace{\frac{1}{2}\int_0^\ell EJ(v'')^2 dx}_{\text{ELASTIC ENERGY}} \underbrace{- \int_0^\ell qv - V_1 v(0) - M_1 v'(0) - V_2 v(\ell) - M_2 v'(\ell)}_{\text{WORK of EXTERNAL LOADS}}$$

FINITE ELEMENT MODEL for TRANSVERSAL BEHAVIOUR



$v(x) \rightarrow$ UNKNOWN PRINCIPAL FIELD

LET'S DIVIDE THE STRUCTURE AND CONSIDER A SINGLE ELEMENT



CLASS of THE FUNCTIONS:

$v(x) \in \mathcal{C}^2, \; x \in]0,\ell[$

$v(x) \in \mathcal{C}^1, \; x = 0 \vee x = \ell$

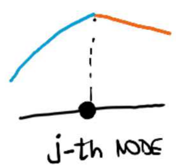
$v''(x)$ MUST BE DEFINED INSIDE FE $\rightarrow v \in \mathcal{C}^2$ AT LEAST IN THE DOMAIN

A DISCONTINUITY IS ADMISSIBLE AT THE BOUNDARY $\rightarrow \mathcal{C}^1$ AT THE BOUNDARY

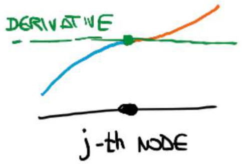
ASKING FOR $v \in \mathcal{C}^1$ AT BOUNDARY IMPLIES THAT BOTH $v(x)$ AND $v'(x)$ ARE CONTINUOUS FUNCTIONS AT EACH NODE

$\downarrow$

THIS BOUNDARY CONDITION IS A STRONG CONSTRAINT



NOT ADMISSIBLE (LIKE TRUSS)



ADMISSIBLE

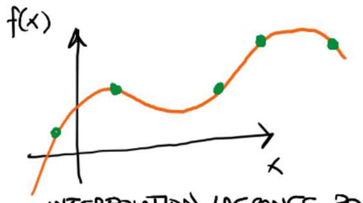
$\downarrow$

AT EACH NODE WE MUST CONTROL THE VALUE OF THE FUNCTION AND ITS FIRST DERIVATIVE $\rightarrow$ WE NEED AT LEAST 2 FREE PARAMETERS AT EACH NODE $\rightarrow$ QUADRATIC SHAPE FUNCTIONS ARE NOT ENOUGH

$\downarrow$

ONLY 3 DOFS, WE NEED AT LEAST 4 DOFS

OBSERVE: LINK BETWEEN SHAPE FUN. AND INTERPOLATION POLYNOMIAL



GIVEN A SET of POINTS $\{x_K, y_K\}$ $\rightarrow$ WE CAN FIND THE FUNCTION THAT INTERPOLATES THE POINTS $\rightarrow$ WE CAN USE LAGRANGE POLYNOMIAL

$\downarrow$

WE CONTROL THE FUNCTION IN A GIVEN SET of POINTS

INTERPOLATION LAGRANGE POLYNOMIAL, m+1 POINTS:

$$L_K = \prod_{i=0,\, i\neq K}^{m} \frac{x - x_i}{x_K - x_i} \quad , \quad f(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + \dots$$

$$m=1 \rightarrow (0,u_1),(L,u_2) \qquad L_0 = \frac{x-L}{0-L} = 1-\frac{x}{\ell} \quad , \quad L_1 = \frac{x-0}{L-0} = \frac{x}{L} \qquad f(x) = u_1\left(1-\frac{x}{L}\right) + u_2\frac{x}{L}$$

WHEN WE NEED TO CONTROL NOT ONLY THE FUNCTION, BUT ALSO ITS FIRST DERIVATIVE $\rightarrow$ INTERPOLATION POLYNOMIAL USING HERMITE FUNCTIONS $\rightarrow$ THE SHAPE FUNCTIONS FOR EULER–BERNOULLI FEM MODEL ARE CALLED HERMITIAN SHAPE FUNCTIONS

DEFINE THE SHAPE FUNCTIONS

WE NEED 2 DOFS FOR EACH NODE $\rightarrow (v_i, \varphi_i) \rightarrow$ 4 DOFS FOR EACH ELEMENT $\rightarrow$ WE NEED A CUBIC POLYNOMIAL

$$\boxed{v(x) = a + bx + cx^2 + dx^3}$$

a, b, c, d RITZ-GALERKIN PARAMETERS

LET'S PASS TO "PHYSICAL" DOFS

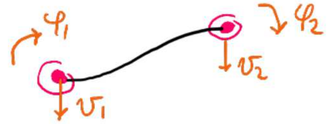
$$\begin{cases} v_1 = v(0) = a \\ \varphi_1 = v'(0) = b \\ v_2 = v(\ell) = a + b\ell + c\ell^2 + d\ell^3 \\ \varphi_2 = v'(\ell) = b + 2c\ell + 3d\ell^2 \end{cases} \longrightarrow \begin{cases} a = v_1 \\ b = \varphi_1 \\ c = \frac{3}{\ell^2}(v_2 - v_1) - \frac{2\varphi_1}{\ell} - \frac{\varphi_2}{\ell} \\ d = \frac{2}{\ell^3}(v_1 - v_2) + \frac{\varphi_1}{\ell^2} + \frac{\varphi_2}{\ell^2} \end{cases}$$

NOW, ISOLATING THE DOFS of THE STRUCTURE

$$v(x) = v_1\underbrace{\left(1-\frac{3x^2}{\ell^2}+\frac{2x^3}{\ell^3}\right)}_{N_1(x)} + \varphi_1\underbrace{\left(x-\frac{2x^2}{\ell}+\frac{x^3}{\ell^2}\right)}_{N_2(x)} + v_2\underbrace{\left(\frac{3x^2}{\ell^2}-\frac{2x^3}{\ell^3}\right)}_{N_3(x)} + \varphi_2\underbrace{\left(-\frac{x^2}{\ell}+\frac{x^3}{\ell^2}\right)}_{N_4(x)}$$

HERMITIAN SHAPE FUNCTIONS

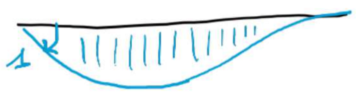
$$v(x) = N_1(x)v_1 + N_2(x)\varphi_1 + N_3(x)v_2 + N_4(x)\varphi_2$$



$$N_1(x) = 1 - 3\left(\frac{x}{\ell}\right)^2 + 2\left(\frac{x}{\ell}\right)^3$$

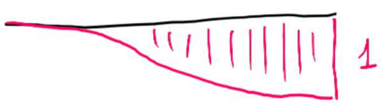
(v_1)

$$N_2(x) = x - \frac{2x^2}{\ell} + \frac{x^3}{\ell^2}$$



(φ_1)

$$N_3(x) = 3\left(\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3$$



(v_2)

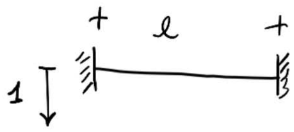
$$N_4(x) = -\frac{x^2}{\ell} + \frac{x^3}{\ell^2}$$

(φ_2)

SAME PROPERTIES of SHAPES FUNCTIONS ARE VALID

$$\text{"} N_i(x_j) = \delta_{ij} \text{"} \longrightarrow$$ NOW THE DISPLACEMENT AND ROTATION DOFS CANNOT BE SUMMED, NO MEANING...

OBS: $N_i(x)$ REPRESENTS THE SOLUTION OF A CLAMPED - CLAMPED BEAM WITH UNIT DISPLACEMENT OR ROTATION ALONG i-th COMPONENT



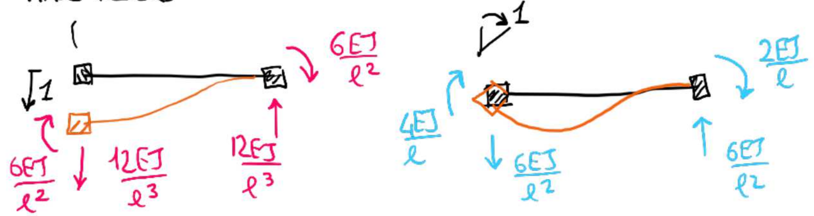
$$EJv^{IV} = 0 \longrightarrow v(x) = A + Bx + Cx^2 + Dx^3 \longrightarrow \text{CUBIC!!}$$

$$\begin{cases} v(0) = 1 \\ v'(0) = 0 \end{cases} \quad \begin{cases} v(\ell) = 0 \\ v'(\ell) = 0 \end{cases} \longrightarrow$$ SAME PREVIOUS SYSTEM TO BE SOLVED WHEN BC ARE APPLIED $\longrightarrow$ $v(x) = v_1 N_1 + \varphi_1 N_2 + v_2 N_3 + \varphi_2 N_4$

BUT $v_1 = 1,\ v_2 = \varphi_2 = \varphi_1 = 0 \implies v(x) = N_1(x)$

NOTE : STRONG RELATION BETWEEN CLASSIC METHOD of ANALYSIS BASED ON DISPLACEMENTS AND FINITE ELEMENT METHOD

FIXED BLOCKS



$\rightarrow$ THE ELASTIC FORCES DUE TO A UNIT DOF ARE THE COLUMNS of $\underline{\underline{K}}$

⋮

STIFFNESS MATRIX

$$v^e(x) = v_1^e N_1 + \varphi_1^e N_2 + v_2^e N_3 + \varphi_2^e N_4 = \underbrace{\begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}}_{\substack{\text{MATRIX of SHAPE FUNCTIONS} \\ \underline{\underline{N}}(x)}} \underbrace{\begin{Bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{Bmatrix}}_{\substack{\text{VECTOR of THE NODAL DEGREES OF FREEDOM} \\ \underline{U}(x)}} = \underline{\underline{N}}(x)\,\underline{U}^e$$

SINCE THE CURVATURE PLAY A ROLE IN THE MODEL, THE STRAIN DISPLACEMENT MATRIX CONTAINS THE 2^{nd} DERIVATIVE OF $v(x)$

$$-\chi(x) = v''(x) = \underbrace{\begin{bmatrix} N_1'' & N_2'' & N_3'' & N_4'' \end{bmatrix}}_{\underline{\underline{B}}(x)} \begin{Bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{Bmatrix}$$

LET'S USE WEIGHTED RESIDUAL METHOD

ON EACH FINITE ELEMENT

$$\int_0^{\ell^e} EJ^e\, v'' w'' - \int_0^{\ell^e} q^e(x)\, w - V_1^e w(0) - M_1^e w'(0) - V_2^e w(\ell^e) - M_2^e w'(\ell^e) = 0, \quad \forall w \in H_0^2$$

GALERKIN METHOD $\rightarrow$ THE SAME SHAPE FUNCTIONS BOTH FOR v AND w

$$v^e(x) = \underline{\underline{N}}\,\underline{U}^e \rightarrow v_e''(x) = \underline{\underline{B}}(x)\,\underline{U}_e$$

$$w_e(x) = \underline{\underline{N}}\,\underline{W}^e \rightarrow w_e''(x) = \underline{\underline{B}}(x)\,\underline{W}_e$$

HENCE

$$\int_0^{\ell} \underline{W}^T \underline{\underline{B}}^T EJ \underline{\underline{B}}\,\underline{U} - \int_0^{\ell} q\, \underline{W}^T \underline{\underline{N}}^T - \underline{W}^T \underline{\tilde{F}} = 0, \ \forall \underline{W} \qquad \underline{\tilde{F}} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix}, \ \underline{W} = \begin{Bmatrix} w(0) \\ w'(0) \\ w(\ell) \\ w'(\ell) \end{Bmatrix}$$

$$\cancel{\underline{W}^T}\left\{\left(\int_0^{\ell} \underline{\underline{B}}^T EJ \underline{\underline{B}}\right)\underline{U} - \int_0^{\ell} q\,\underline{\underline{N}}^T - \underline{\tilde{F}}\right\} = 0, \ \forall \underline{W}$$

$$\underbrace{\left(\int_0^{\ell} \underline{\underline{B}}^T EJ \underline{\underline{B}}\,dx\right)}_{\substack{\underline{\underline{K}} \\ \text{STIFFNESS MATRIX}}} \underline{U} - \underbrace{\left(\int_0^{\ell} q\,\underline{\underline{N}}^T - \underline{\tilde{F}}\right)}_{\substack{\underline{F} \\ \text{VECTOR of EQUIVALENT NODAL FORCES}}} = 0 \rightarrow \boxed{\underline{\underline{K}}^e \underline{U}^e = \underline{F}^e}$$

ALGEBRAIC PROBLEM ON EACH FINITE ELEMENT

THEN ASSEMBLY !!

$$\underline{\underline{B}} = \left[\frac{12x}{\ell^3} - \frac{6}{\ell^2},\ \frac{6x}{\ell^2} - \frac{4}{\ell},\ \frac{6}{\ell^2} - \frac{12x}{\ell^3},\ \frac{6x}{\ell^2} - \frac{2}{\ell}\right] \rightarrow \text{USED TO COMPUTE } \underline{\underline{K}}$$

STIFFNESS MATRIX

$$\underline{\underline{K}} = \frac{EJ}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ & 4\ell^2 & -6\ell & 2\ell^2 \\ & & 12 & -6\ell \\ \text{Sym} & & & 4\ell^2 \end{bmatrix}$$

STIFFNESS MATRIX of EACH FINITE ELEMENT ACCORDING TO THE EULER–BERNOULLI BEAM MODEL

THE NODAL FORCE DEPENDS ON THE SHAPE of DISTRIBUTED LOADS

$$\underline{F}^e = \underbrace{\int_0^{\ell} q(x)\begin{Bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \\ N_4(x) \end{Bmatrix}}_{\text{NODAL FORCE EQUIVALENT TO DISTRIBUTED LOADS}} + \underbrace{\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix}}_{\text{CONCENTRATED LOADS ON THE NODES of THE ELEMENTS}}$$

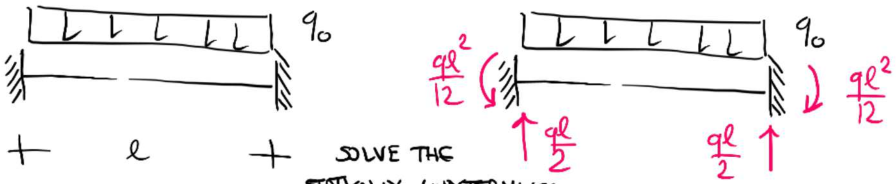
NOTE: THE SIGN IS GIVEN BY FORCE CONVENTION AND NOT BY THE SIGN of N, T AND M

EXAMPLE: $q(x) = q_0$, CONSTANT DISTRIBUTED LOAD

$$\underline{F} = \int_0^{\ell} q_0 \begin{Bmatrix} 1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3} \\ x - \frac{2}{\ell}x^2 + \frac{x^3}{\ell^2} \\ 3\left(\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3 \\ -\frac{x^2}{\ell} + \frac{x^3}{\ell^2} \end{Bmatrix} = q_0 \begin{Bmatrix} \ell\,(1 - 1 + 1/2) \\ \ell^2\,(1/2 - 2/3 + 1/4) \\ \ell\,(1 - 1/2) \\ \ell^2\,(1/3 + 1/4) \end{Bmatrix} = \boxed{\begin{Bmatrix} \frac{q_0 \ell}{2} \\ \frac{q_0 \ell^2}{12} \\ \frac{q_0 \ell}{2} \\ -\frac{q \ell^2}{12} \end{Bmatrix}}$$

MECHANICAL INTERPRETATION $\rightarrow$ NODAL LOAD EQUIVALENT TO A DISTRIBUTED LOAD FROM THE POINT of VIEW OF THE ENERGY

NOTE : RECALLING THE ANALOGY WITH THE CLAMPED - CLAMPED BEAM, THE EQUIVALENT NODAL LOAD SET IS THE OPPOSITE OF THE SET OF THE REACTIONS PRODUCED BY THE DISTRIBUTED LOADS



SOLVE THE STATICALLY - UNDETERMINED STRUCTURE

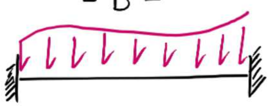
(EQUIVALENT LOADS = − LOADS GIVING EQUILIBRIUM)

PROOF USING BETTI'S THEOREM

- A -

$v(x) = N_1(x) \cdot 1$

- B -



$\uparrow R_1$: REACTION DUE TO LOAD $q(x)$

$\mathcal{L}_{AB} = \mathcal{L}_{BA}$

R_1 IS THE ONLY FORCE DUAL TO NOT-VANISHING BOUNDARY DISPLACEMENTS

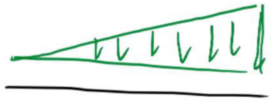
$$\mathcal{L}_{AB} = \int_0^\ell v(x)\, q(x) + \overbrace{1 \cdot R_1}$$

$$\rightarrow R_1 = -\int_0^\ell q N_1(x)$$

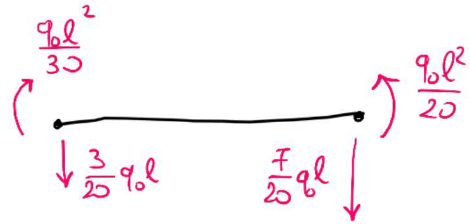
$\mathcal{L}_{BA} = 0$ (VANISHING DISPL. IN B WHERE THERE IS A FORCE IN A)

$\underbrace{}$ 1^{st} COMPONENT of $\underline{F}$

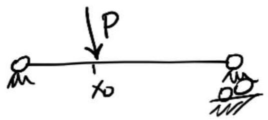
LINEAR DISTRIBUTED LOAD



$q(x) = \frac{q_0 x}{\ell}$

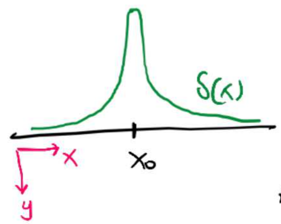


CONCENTRATED LOAD



↳ IT'S A PROBLEM IN THE FIELD OF CONTINUUM MECHANICS → DISTRIBUTION ANALYSIS → EXTENSION of THE CONCEPT of FUNCTIONS

METHOD 1] DIRAC'S DELTA FUNCTION



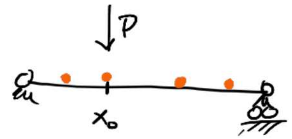
$$\delta(x) \text{ t.c. } \begin{cases} \delta(x) = 0, & x \in \mathbb{R} \setminus \{x_0\} \\ \delta(x) = \infty, & x = x_0 \end{cases}$$

IT CAN BE INTEGRATED

$$\rightarrow \int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$

WE CAN SEE A CONCENTRATED FORCE AS A DISTRIBUTED LOAD $q(x) = P\delta(x)$

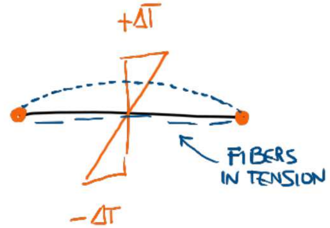
METHOD 2] DISCRETIZATION IN SUCH A WAY POINT x_0 IS ON A NODE of AN ELEMENT → CONCENTRATED LOADS ARE ADMITTED ON FE NODES



BUT BE CAREFUL !! THE LENGTHS of THE ELEMENTS SHOULD NOT BE TO MUCH DIFFERENT ONE ANOTHER

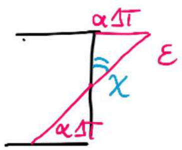
THERMAL LOADS

ENRICH THE FE FORMULATION TO CONSIDER ALSO THERMAL DISTORSIONS



↳ IT ENTERS IN THE DEFINITION OF EXTERNAL LOAD

$$\chi^{TOT} = \underbrace{\chi_E}_{\text{ELASTIC CURVATURE}} + \underbrace{\chi_T}_{\text{THERMAL CURVATURE}}$$



$$|\chi^T| = \frac{\alpha\Delta T}{\frac{h}{2}} = \frac{2\alpha\Delta T}{h} \longrightarrow \boxed{\chi^T = -\frac{2\alpha\Delta T}{h}}$$

1) IN GENERAL, THE TOTAL CURVATURE CAN BE SEEN AS THE SUM OF AN ELASTIC and ANELASTIC (THERMAL, VISCOUS, ...) COMPONENT

$$\boxed{\chi^{TOT}(x) = \underbrace{\chi_E(x)}_{} + \chi_T(x)}$$

$$\chi_E(x) = \frac{M}{EJ}$$

UNTIL NOW, WE HAVE ONLY CONSIDERED THIS COMPONENT !!

2) THE TOTAL CURVATURE χ^{TOT} IS LINKED TO THE CURVATURE OF THE BEAM AXIS

$$\boxed{\chi^{TOT} = -v''(x)}$$

3) LET'S START FROM THE TPE, KNOWING THAT $\mathcal{U}(v) = \frac{1}{2}\int_0^\ell EJ\chi_E^2(x)$ (CLAPEYRON)

$$\Pi[v] = \frac{1}{2}\int_0^\ell EJ\chi_E^2(x)dx - \int_0^\ell q(x)v(x) - V_1 v(0) - V_2 v(\ell) - M_1 v'(0) - M_2 v'(\ell) =$$

$$= \frac{1}{2}\int_0^\ell EJ(\chi^{TOT} - \chi_T)^2 - \int_0^\ell qv - V_1 v_1 - V_2 v_2 - M_1\varphi_1 - M_2\varphi_2 =$$

$$= \frac{1}{2}\int_0^\ell EJv''^2 + \underbrace{\frac{1}{2}\int_0^\ell EJ\chi_T^2}_{\text{CAN BE DELETED}} - \int_0^\ell EJv''\chi_T - \int_0^\ell qv - \underline{\tilde{F}}\cdot\underline{U}$$

→ THE TERM $v''(x)$ IS NOT PRESENT, PERFORMING DIFFERENTIATION IT'S A CONSTANT AND VANISHES

LET'S INTRODUCE THE APPROXIMATION ON EACH FINITE ELEMENT $\qquad v(x) = \underline{\underline{N}}\,\underline{U} \qquad v''(x) = \underline{\underline{B}}(x)\,\underline{U}$

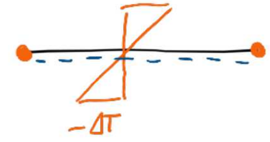
$$\Pi^\ell(\underline{U}) = \frac{1}{2}\int_0^\ell \underline{U}^T\underline{\underline{B}}^T EJ\,\underline{\underline{B}}\,\underline{U} - \int_0^\ell EJ\,\underline{U}^T\underline{\underline{B}}^T\chi_T - \int_0^\ell \underline{U}^T\underline{\underline{N}}^T q - \underline{U}^T\underline{\tilde{F}} =$$

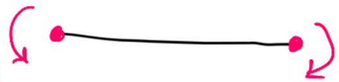
$$= \frac{1}{2}\underline{U}^T\underbrace{\left(\int_0^\ell \underline{\underline{B}}^T EJ\,\underline{\underline{B}}\right)}_{\text{"CLASSICAL" STIFFNESS MATRIX}}\underline{U} - \underline{U}^T\left(\underbrace{\int_0^\ell EJ\chi_T\underline{\underline{B}}^T}_{\text{NODAL FORCE VECTOR EQUIVALENT TO THERMAL LOADS}} + \underbrace{\int_0^\ell q\underline{\underline{N}}^T + \underline{\tilde{F}}}_{\text{"CLASSICAL" NODAL FORCE VECTOR}}\right)$$

THERMAL CURVATURE ENTERS IN THE PROBLEM AS A NODAL FORCE

$$\underline{F}_T = \int_0^\ell EJ\chi_T\underline{\underline{B}}^T = \int_0^\ell \frac{2\alpha\Delta T}{h}EJ\,\underline{\underline{B}}^T = \frac{2\alpha\Delta T}{h}EJ\int_0^\ell \underline{\underline{B}}^T$$

$$\underline{F}_T = \frac{2\alpha\Delta T}{h}EJ\int_0^\ell \begin{Bmatrix} \frac{12x}{\ell^3} - \frac{6}{\ell^2} \\ \frac{6x}{\ell^2} - \frac{4}{\ell} \\ \frac{6}{\ell^2} - \frac{12x}{\ell^3} \\ -\frac{2}{\ell} + \frac{6x}{\ell^2} \end{Bmatrix} = \frac{2\alpha\Delta T}{h}EJ\begin{Bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{Bmatrix}$$





AND... ON TRUSS !?

$\varepsilon_T = \alpha\Delta T$

$$\mathcal{U} = \frac{1}{2}\int_0^\ell EA\varepsilon_E^2 = \frac{1}{2}\int_0^\ell EA(\varepsilon^{TOT} - \varepsilon_T)^2 =$$

$$= \frac{1}{2}\int_0^\ell EAu'^2 - \int_0^\ell EAu'\varepsilon_T$$

$$\underline{F}_T = \int_0^\ell \varepsilon_T EA\,\underline{\underline{N}}'^T(x) = \alpha\Delta TEA\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

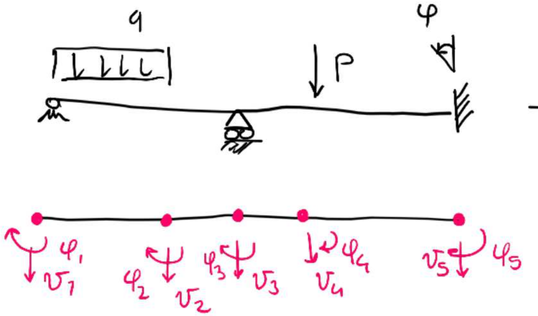
FROM THE FINITE ELEMENT TO THE BEAM / FRAME

WITH THE ASSEMBLY PROCEDURE WE OBTAIN THE FINAL ALGEBRIC SYSTEM

$$\underline{\underline{K}}\,\underline{u} = \underline{F}$$

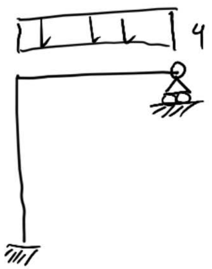
$\downarrow$

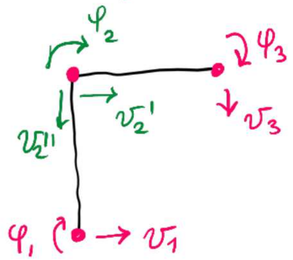
FOR COMPLEX STRUCTURES, ALSO ASSEMBLY IS COMPLICATED



$\rightarrow$ DISCRETISATION IS SIMPLE (and ALSO ASSEMBLY)

DISCRETISATION of FRAME STRUCTURES IS MORE COMPLEX !





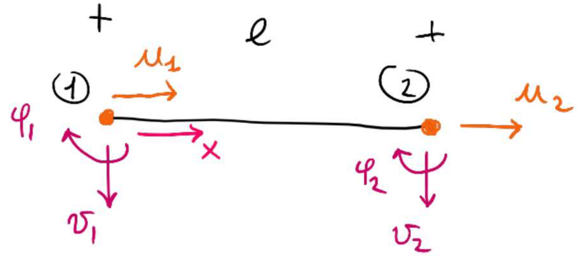
IN THIS NODE, SHEAR AND NORMAL FORCES of THE CONVERGING BEAMS INTERACT

$\downarrow$

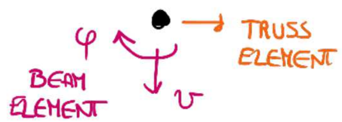
2 DOFS ARE GENERALLY NOT ENOUGH

WE CAN COMBINE THE TRUSS ELEMENT (AXIAL BEHAVIOUR) AND THE BEAM ELEMENT (FLEXURAL BEHAVIOUR)

COMPLETE EULER-BERNOULLI FINITE ELEMENT



THERE ARE 3 DOFS FOR EACH NODE



$$\underline{U} = \begin{Bmatrix} u_1 \\ v_1 \\ \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \end{Bmatrix}$$ VECTOR of NODAL DOFS

THERE ARE 2 UNKNOWN FIELDS

$$\boxed{u(x),\ v(x)}$$

$$\begin{cases} EAu'' + p = 0 \\ + \text{B.C.} \end{cases}$$

- AXIAL PROBLEM -

$$\begin{cases} EJv^{IV} - q = 0 \\ + \text{B.C.} \end{cases}$$

- FLEXURAL PROBLEM -

HENCE THE TPE BECOMES

$$\Pi[u,v] = \underbrace{\frac{1}{2}\int_0^\ell EAu'^2}_{\text{ELASTIC ENERGY (AXIAL)}} + \underbrace{\frac{1}{2}\int_0^\ell EJ(-v'')^2}_{\text{ELASTIC ENERGY (FLEXURAL)}} - \underbrace{\mathcal{L}[u,v]}_{\text{EXTERNAL WORK}}$$

$\leftarrow$ THE FUNCTIONAL IS VERY USEFUL, WE CAN SUM UP ALL THE CONTRIBUTIONS !!

APPROXIMATION of THE FIELDS :

$$u(x) = \begin{bmatrix} N_1^u(x) & N_2^u(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$v(x) = \begin{bmatrix} N_1^v & N_2^v & N_3^v & N_4^v \end{bmatrix} \begin{Bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{Bmatrix}$$

$$\leadsto \begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} = \underbrace{\begin{bmatrix} N_1^u & 0 & 0 & N_2^u & 0 & 0 \\ 0 & N_1^v & N_2^v & 0 & N_3^v & N_4^v \end{bmatrix}}_{\text{COMPLETE MATRIX of SHAPE FUNCTIONS}} \begin{Bmatrix} u_1 \\ v_1 \\ \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \end{Bmatrix}$$

$$\underline{u}(x) = \underline{\underline{N}}\,\underline{U}$$

$$\Pi[u,v] = \frac{1}{2}\int_0^\ell (u')^T EA(u') + \frac{1}{2}\int_0^\ell v''^T EJ v'' - \mathcal{L}[u,v]$$

$$u'(x) = N_{1j}'(x)\, U_j$$

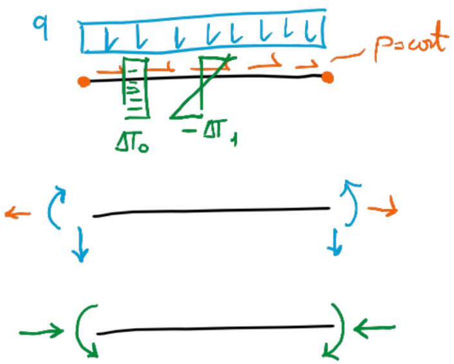
$$v''(x) = N_{2j}''(x)\, U_j$$

THE FINAL STIFFNESS MATRIX IS 6×6,
"INSERT" IN THE CORRECT PLACE THE ELEMENTS of AXIAL AND FLEXURAL STIFFNESS MATRICES

$$\underline{\underline{K}} = \begin{array}{c} \begin{bmatrix} \frac{EA}{\ell} & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 \\ 0 & \frac{12EJ}{\ell^3} & \frac{6EJ}{\ell^2} & 0 & -\frac{12EJ}{\ell^3} & \frac{6EJ}{\ell^2} \\ 0 & \frac{6EJ}{\ell^3} & \frac{4EJ}{\ell} & 0 & -\frac{6EJ}{\ell^2} & \frac{2EJ}{\ell} \\ -\frac{EA}{\ell} & 0 & 0 & \frac{EA}{\ell} & 0 & 0 \\ 0 & -\frac{12EJ}{\ell^3} & -\frac{6EJ}{\ell^2} & 0 & \frac{12EJ}{\ell^3} & -\frac{6EJ}{\ell^2} \\ 0 & \frac{6EJ}{\ell^2} & \frac{2EJ}{\ell} & 0 & -\frac{6EJ}{\ell^2} & \frac{4EJ}{\ell} \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \end{array}$$

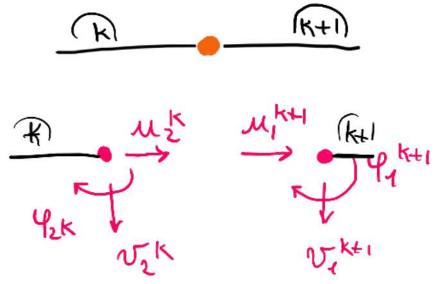
THE SAME APPROACH CAN BE USED FOR THE NODES

$$\underline{F} = \underline{\tilde{F}} + \int_0^\ell \underline{\underline{N}}^T \begin{Bmatrix} p(x) \\ q(x) \end{Bmatrix}$$



$$\underline{F} = \begin{Bmatrix} \frac{p\ell}{2} + \alpha\Delta T_0 EA \\ \frac{q\ell}{2} \\ \frac{q\ell^2}{12} - \frac{2\alpha\Delta T_1 EJ}{h} \\ \frac{p\ell}{2} - \alpha\Delta T_0 EA \\ \frac{q\ell}{2} \\ -\frac{q\ell^2}{12} + \frac{2\alpha\Delta T_1 EJ}{h} \end{Bmatrix}$$

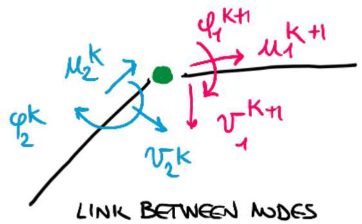
WE MUST PAY ATTENTION IN THE ASSEMBLY PROCESS $\rightarrow$ THERE MAY BE PROBLEMS IN CASE of INCLINED BEAMS



ASSEMBLE MEANS ASSUME THAT

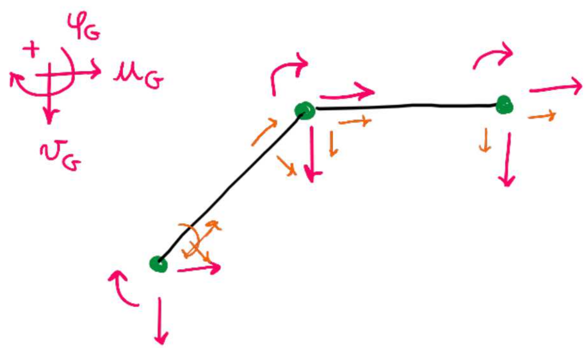
$$\begin{cases} u_2^k = u_1^{k+1} \\ v_2^k = v_1^{k+1} \\ \varphi_2^k = \varphi_1^{k+1} \end{cases}$$

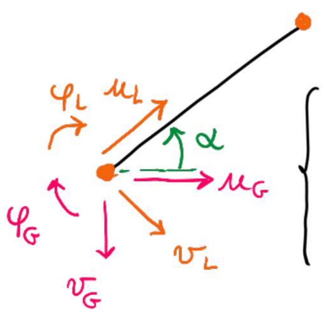
AND FOR INCLINED RODS ??



LINK BETWEEN NODES CANNOT BE IMPOSED DIRECTLY...

THE SOLUTION IS DEFINING GLOBAL DOFS RELATIVE TO A GLOBAL REFERENCE FRAME USED FOR ALL THE NODES of THE STRUCTURE $\rightarrow$ WE NEED TO INTRODUCE A ROTATION FROM THE LOCAL TO THE GLOBAL REFERENCE FRAME





THE LINK BETWEEN LOCAL AND GLOBAL REF. FRAME IS :

$$\begin{cases} u^L = u_G\cos\alpha - v_G\sin\alpha \\ v^L = u_G\sin\alpha + v_G\cos\alpha \\ \varphi^L = \varphi^G \end{cases}$$

HENCE

$$\begin{Bmatrix} u^L \\ v^L \\ \varphi^L \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{ROTATION MATRIX}} \begin{Bmatrix} u_G \\ v_G \\ \varphi_G \end{Bmatrix}$$

$$\underline{\underline{T}}_0 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

DOF VECTOR of EACH SINGLE NODE

$$\underline{u}^L = \underline{\underline{T}}_0 \underline{u}^G$$

THE VECTOR of DOFS of EACH FINITE ELEMENT IS

$$\underline{U}^L = \begin{Bmatrix} u_1^L \\ v_1^L \\ \varphi_1^L \\ u_2^L \\ v_2^L \\ \varphi_2^L \end{Bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{T}}_0 & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{T}}_0 \end{bmatrix}}_{\underline{\underline{T}}} \underbrace{\begin{Bmatrix} u_1^G \\ v_1^G \\ \varphi_1^G \\ u_2^G \\ v_2^G \\ \varphi_2^G \end{Bmatrix}}_{\underline{U}^G}$$

ROTATION MATRIX of THE COMPLETE FINITE ELEMENT

HENCE

$$\underline{U}^L = \underline{\underline{T}}\,\underline{U}^G$$ → LET'S USE IT IN THE TPE

$$\Pi[\underline{u}] = \frac{1}{2}\underline{U}^L \cdot \underline{\underline{K}}^L \underline{U}^L - \underline{U}^L \cdot \underline{F}^L =$$

$$= \frac{1}{2}\underline{\underline{T}}\,\underline{U}^G \cdot \underline{\underline{K}}\,\underline{\underline{T}}\,\underline{U}^G - \underline{\underline{T}}\,\underline{U}^G \cdot \underline{F}^L =$$

$$= \frac{1}{2}\underline{U}^G \cdot \underbrace{\underline{\underline{T}}^T \underline{\underline{K}}^L \underline{\underline{T}}}_{\underset{\text{GLOBAL}}{\underline{\underline{K}}}} \underline{U}^G - \underline{U}^G \cdot \underbrace{\underline{\underline{T}}^T \underline{F}^L}_{\underset{\text{GLOBAL}}{\underline{F}}}$$

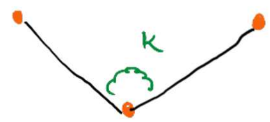
IN GENERAL

$$\underline{\underline{K}} = \underline{\underline{T}}^T \underline{\underline{K}}^L \underline{\underline{T}} \,, \quad \underline{F} = \underline{\underline{T}}^T \underline{F}^L$$ ← CORRECT ASSEMBLY

$\underline{\underline{K}}^L_e$ $\underline{F}^L$ KNOWN FROM FORMULATION

SPRINGS AND ELASTIC CONCENTRATED ELEMENTS

– GROUND SPRINGS –



– INTERNAL SPRINGS –

TWO KINDS of ELASTIC CONSTRAINTS → WE NEED TO ADD TO THE FEM FORMULATION THE ENERGY COMPONENT of THE SPRING

GROUND SPRINGS

ELASTIC ENERGY of EACH FE

DOFS RELATED TO THE SPRINGS

$$\mathcal{U}[u] = \frac{1}{2}\underline{U}_e^T \underline{\underline{K}}_e \underline{U}_e + \frac{1}{2}K_1 U_2^{e\,2} + \frac{1}{2}K_2 U_3^{e\,2}$$

THE ADDITIONAL QUANTITIES MUST BE WRITTEN IN TERMS OF THE GLOBAL VECTOR $\underline{U}^e$

HENCE

$$\frac{1}{2}K_1 U_2^{e\,2} = \underline{U}^{e\,T} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\tilde{\underline{\underline{K}}}_1} \underline{U}^e + \underline{U}^{e\,T} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\tilde{\underline{\underline{K}}}_2} \underline{U}^e$$

$$\frac{1}{2}\underline{U}_e^T \tilde{\underline{\underline{K}}}_1 \tilde{\underline{U}}_e$$

STIFFNESS MATRICES OF THE CONCENTRATED ELASTIC SPRINGS

→ ADDITIVITY!! THEY ARE SUMMED UP TO THE STIFFNESS MATRIX OF THE FE

THE FINAL STIFFNESS MATRIX IS $\underline{\underline{K}} + \tilde{\underline{\underline{K}}}_1 + \tilde{\underline{\underline{K}}}_2$

$$K = \begin{bmatrix} \frac{12EJ}{\ell^3} & \frac{6EJ}{\ell^2} & -\frac{12EJ}{\ell^3} & \frac{6EJ}{\ell^2} \\ \frac{6EJ}{\ell^2} & \frac{4EJ}{\ell} + K_1 & -\frac{6EJ}{\ell^2} & \frac{2EJ}{\ell} \\ -\frac{12EJ}{\ell^3} & -\frac{6EJ}{\ell^2} & \frac{12EJ}{\ell^3} + K_2 & -\frac{6EJ}{\ell^2} \\ \frac{6EJ}{\ell^2} & \frac{2EJ}{\ell} & -\frac{6EJ}{\ell^2} & \frac{4EJ}{\ell} \end{bmatrix}$$

INTERNAL SPRINGS

THE FORM IS LIKE $$\mathcal{U} = \frac{1}{2}K_3\Big(\underbrace{U_i^k - U_j^{k+1}}_{\text{RELATION AMONG DOFS OF DIFFERENT ELEMENTS!}}\Big)^2$$

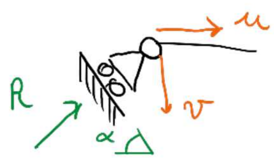
SAME PROCEDURE AS BEFORE, BUT:

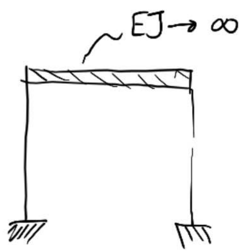
i) WE MUST "WORK" ON THE ASSEMBLED MATRIX

ii) THE STIFFNESS MATRIX of THE SPRING IS IN THE FORM

$$\tilde{\underline{\underline{K}}} = \begin{bmatrix} K_3 & -K_3 \\ -K_3 & K_3 \end{bmatrix} \begin{matrix} U_i^k \\ U_j^{k+1} \end{matrix}$$

ADDITIONAL CONSTRAINTS FOR A STRUCTURE





SOMETIMES WE NEED TO ADD SOME ADDITIONAL CONSTRAINTS AMONG THE DOFS

↓

HOW CAN WE DO THAT !?

MULTIFREEDOM CONSTRAINTS $\underline{\underline{A}}\,\underline{u} = \underline{g}$ LET'S CONSIDER ONLY LINEAR CONSTRAINTS AMONG THE DOFS

3 METHODS TO CONSIDER THAT :

i) PENALTY METHOD : ADD ELASTIC CONSTRAINTS DIRECTLY IN THE STIFFNESS MATRIX, WITH THE PROPER RIGIDITY

ex : $\underline{\underline{A}}\,\underline{u} = \underline{g}$ → IT CAN BE SEEN (FOR SOME $\underline{\underline{A}}$) AS A RIGID MOTION → IT CAN BE OBTAINED USING A TRUSS ELEMENT WITH $EA \to \infty$

ii) LAGRANGE MULTIPLIER METHOD

THE WEAK FORM CAN BE SEEN AS A MINIMUM PROBLEM → WE TRANSFORM IT INTO A CONSTRAINED MINIMUM PROBLEM

$\min \Pi \Rightarrow \min \mathcal{L}$ $\quad\mathcal{L}$: LAGRANGIAN

$$\mathcal{L} = \Pi - \underbrace{\underline{\lambda}^T}_{\text{LAGRANGE MULTIPLIERS}} \underbrace{(\underline{\underline{A}}\,\underline{u} - \underline{g})}_{\text{CONSTRAINTS}} \leftarrow$$

MINIMIZE THIS FUNCTION, CONSIDERING ALSO THE PARAMETERS $\underline{\lambda}$

$\dfrac{\partial \mathcal{L}}{\partial \underline{u}} = 0$ BUT ALSO $\dfrac{\partial \mathcal{L}}{\partial \underline{\lambda}} = 0$ ← ADDITIONAL CONDITION $\quad \mathcal{L} = \mathcal{L}(\underline{u}, \underline{\lambda})$

THE MINIMUM OF THE LAGRANGIAN LEAD TO THE FOLLOWING PROBLEM

$$\begin{bmatrix} \underline{\underline{K}} & \underline{\underline{A}}^T \\ \underline{\underline{A}} & \underline{\underline{0}} \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{\lambda} \end{Bmatrix} = \begin{Bmatrix} \underline{f} \\ \underline{g} \end{Bmatrix}$$

FROM A PHYSICAL POINT OF VIEW, $\underline{\lambda}$ ARE THE FORCES DUE TO THE CONSTRAINTS

iii) MASTER-SLAVE METHOD

$\underline{\underline{A}}\,\underline{u} = \underline{g}$ → $\underline{\underline{A}}$ IS NOT A SQUARE MATRIX, GENERIC DIMENSION (NUMBER OF ROWS = NUMBER OF EQUATIONS) → WE CAN "PARTIALLY" SOLVE A SYSTEM AND WRITE SOME DOFS AS A FUNCTION OF THE OTHER DOFS

THE VECTOR $\underline{u}$ OF DOFS CAN BE PARTITIONED INTO 2 SETS :

MASTER NODES $\underline{q}$, THE REAL UNKNOWNS OF THE PROBLEM

SLAVE NODES, LINKED TO THE MASTER FROM $\underline{\underline{A}}\,\underline{u} = \underline{g}$

$$\boxed{\underline{u} = \underline{\underline{L}}\,\underline{q}}$$

MASTER NODES $\underline{q}$ → VECTOR OF LINEARLY INDEPENDENT DOFS

THE TPE CAN BE WRITTEN AS

$$\Pi[\underline{u}] = \frac{1}{2}\underline{u}^T \underline{\underline{K}}\,\underline{u} - \underline{u}^T \underline{F} = \frac{1}{2}\underline{q}^T \underbrace{\underline{\underline{L}}^T \underline{\underline{K}}\,\underline{\underline{L}}}_{\tilde{\underline{\underline{K}}}}\,\underline{q} - \underline{q}^T \underbrace{\underline{\underline{L}}^T \underline{F}}_{\tilde{\underline{F}}} \to$$

THE PROBLEM TO SOLVE IS $\boxed{\tilde{\underline{\underline{K}}}\,\underline{q} = \tilde{\underline{F}}}$

ONCE $\underline{q}$ IS KNOWN, WE CAN COMPUTE $\underline{u} = \underline{\underline{L}}\,\underline{q}$

DIRECT LINK BETWEEN $\underline{\underline{A}}$ AND $\underline{\underline{L}}$

$$\boxed{\underline{\underline{L}} = \mathrm{nullspace}(\underline{\underline{A}})} \to$$

EASY TO COMPUTE DIRECTLY

NULLSPACE OF $\underline{\underline{A}}$:

ALL VECTORS $\underline{y}$ s.t. $\underline{\underline{A}}\,\underline{y} = 0$ $\quad L = [\underline{y}_1 | \underline{y}_2 \cdots]$

($\underline{y}$ LINEARLY INDEPENDENTS)