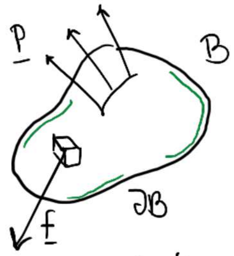
Finite Elements
in Structural Mechanics

Truss elements

Marco Rossi

STRUCTURAL MECHANICS

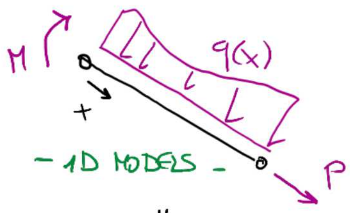


HOMOGENEOUS
LINEAR ELASTIC
ISOTROPIC SOLID

"COMPLEX" DIFF. EQUATIONS

$$\begin{cases} \operatorname{div} \underline{\underline{\sigma}} + f = 0 \\ \underline{\underline{\varepsilon}} = \dfrac{\nabla \underline{u} + \nabla \underline{u}^T}{2} \\ \underline{\underline{\sigma}} = \mathbb{C}\, \underline{\underline{\varepsilon}} \end{cases}$$

STRUCTURAL MODELS

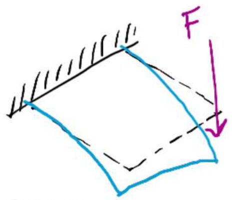


– 1D MODELS –

$$EA u'' = p(x)$$
$$EJ v^{IV} = q(x)$$

– 2D MODELS –

PLANE STRAIN,
PLANE STRESS

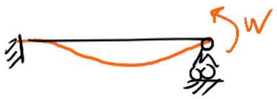


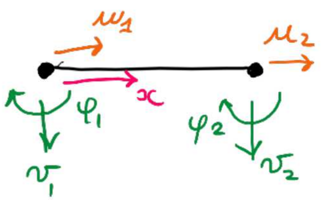
– PLATES –

FINITE ELEMENTS FOR STRUCTURAL MECHANICS

* 1D MODELS : i) TRUSS FINITE ELEMENT

ii) BEAM FINITE ELEMENT

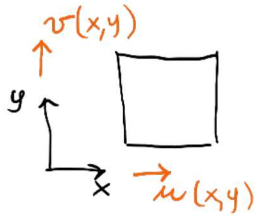




- THIN BEAMS (SLENDER) → EULER–BERNOULLI MODEL
- THICK BEAMS → TIMOSHENKO MODEL

* 2D MODELS : i) FINITE ELEMENT FOR 2D PLANAR PROBLEMS

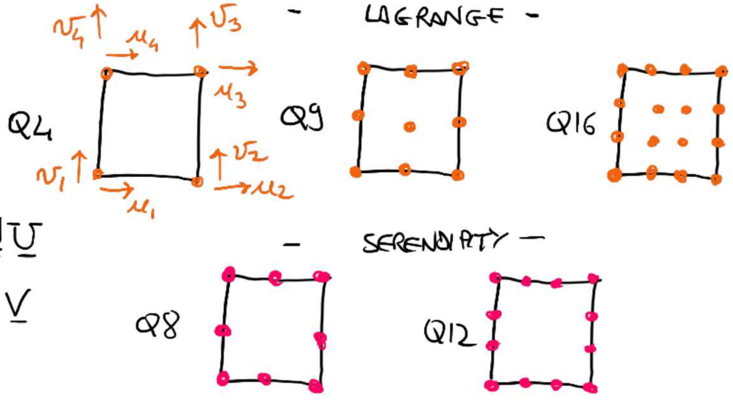
– RECTANGULAR –



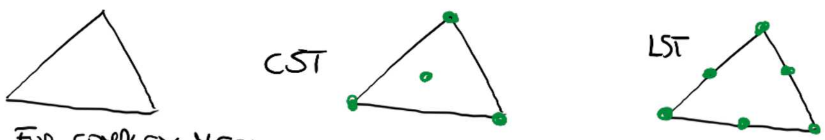
$$u = \underline{\underline{N}}\, \underline{U}$$
$$v = \underline{\underline{N}}\, \underline{V}$$

– LAGRANGE –

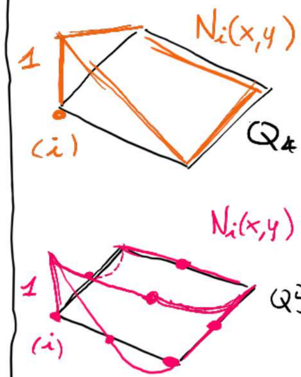
– SERENDIPITY –



– TRIANGULAR –



BETTER FOR COMPLEX MESH,
BUT "INTERNALLY NOT SYMMETRIC"



1 Q4

$x \quad y$

$x^2 \quad xy \quad y^2$ Q9

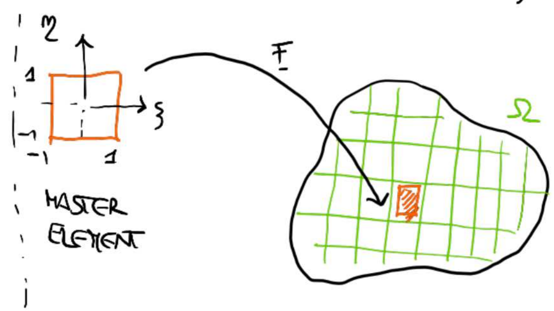
$x^3 \quad x^2y \quad xy^2 \quad y^3$

$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$

(LAGRANGE)

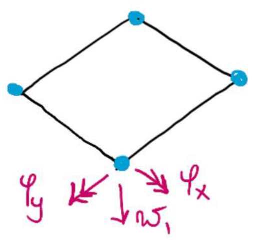
– ISOPARAMETRIC FE –

(HOW THE PROBLEM IS WRITTEN)



MASTER ELEMENT

ii) 2D MODELS FOR PLATES



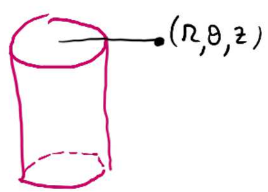
ACM (12 dofs)

BFS (16 dofs)

MORE COMPLICATED TO STUDY AND USE!

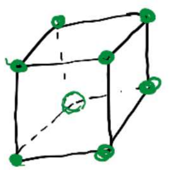
* 3D MODELS

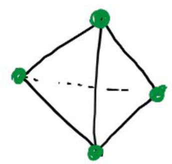
i) AXISYMMETRIC ELEMENTS



θ DOES NOT PLAY A ROLE!! → SIMPLIFIED PROBLEMS

ii) 3D GENERAL PROBLEMS

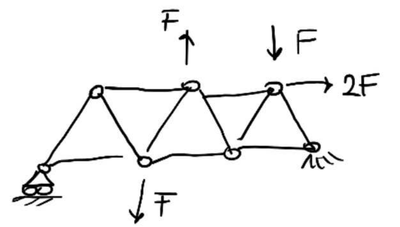




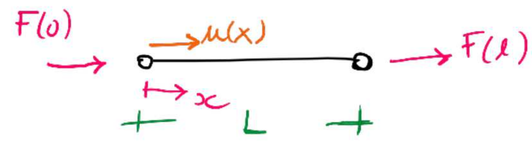
ELEMENTI FINITI TRUSS

TRUSS → SUBJECTED ONLY TO AXIAL LOADS

↳ THE INTERNAL FORCE IS ONLY AXIAL FORCE, NO FLEXURAL DISPLACEMENTS

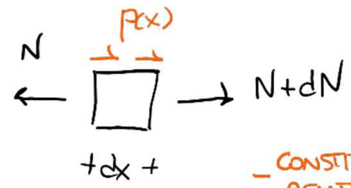


TRUSS RODS ARE THE ELEMENTS OF TRUSSES WHERE ONLY THE NODES ARE LOADED



- EQUILIBRIUM EQUATIONS

$$N + \frac{dN}{dx}dx - N + p\,dx = 0 \rightarrow \boxed{\frac{dN}{dx} = -p(x)}$$



- STATIC EQUIVALENCE $\sigma = \frac{N}{A} \rightarrow \boxed{N = A\sigma}$ ($\sigma = \sigma_x$ ONLY STRESS COMPONENT)

- CONSTITUTIVE RELATIONS $\boxed{\sigma = E\varepsilon}$ ε IS THE AXIAL STRAIN COMPONENT

- COMPATIBILITY of DISPLACEMENTS $\boxed{\varepsilon = \frac{du}{dx}}$

COMBINING ALL THE PREVIOUS EQUATIONS...

$$N = A\sigma = AE\varepsilon = EA\frac{du}{dx} \rightarrow$$

INDEFINITE EQUILIBRIUM EQUATIONS FOR A TRUSS (DISPLACEMENT FORMULATION)

$$\boxed{\left[EA\,u'(x)\right]' + p(x) = 0}$$

ON THE DOMAIN $x \in [0, L]$

IN CASE OF EA = const.

$$\boxed{EA\,u''(x) + p(x) = 0}, \; x \in [0, L]$$

BOUNDARY CONDITIONS

- ESSENTIAL BC: $u(0) = u^0$, $u(L) = u^\ell$

- NATURAL BC: on $u'(0)$ e $u'(L)$ → $N(0) = -F(0)$, $N(L) = F(L)$

$EA\,u'(0) = -F(0)$

$EA\,u'(L) = F(\ell)$

TOTAL POTENTIAL ENERGY

$$\Pi[u] = \mathcal{E}[u] - \mathcal{L} = \underbrace{\frac{1}{2}\int_0^L EA(u')^2}_{\substack{\mathcal{E}[u] \\ \text{ELASTIC POTENTIAL} \\ \text{ENERGY}}} - \underbrace{\int_0^L pu - F(0)u(0) - F(L)u(L)}_{\substack{\text{POTENTIAL of EXTERNAL} \\ \text{LOADS} \\ (-\text{ WORK})}}$$

($u(x)$ IS ASSUMED TO BE A COMPATIBLE DISPLACEMENT)

SOLUTION $u(x)$ FULFILLS EQUILIBRIUM $\Longleftrightarrow$ MINIMUM OF TPE

LET'S ASSUME THAT $\tilde{u}(x)$ FULFILLS BOTH COMPATIBILITY AND EQUILIBRIUM $\longrightarrow$ THE VARIATION FUNCTION $\tilde{u} + \delta u(x)$ IS DEFINED IN SUCH A WAY $\delta u(x)$ IS HOMOGENEOUS AT THE BOUNDARY

$$EA\tilde{u}''(x) + p(x) = 0$$

$$\begin{cases} \tilde{u}(0) = \tilde{u}_0 \\ \tilde{u}(L) = \tilde{u}_L \end{cases} \quad \text{OR} \quad \begin{cases} u'(0) = -\frac{F(0)}{EA} \\ u'(L) = \frac{F(L)}{EA} \end{cases}$$

$$\delta u(0) = 0 \quad \text{AND/OR} \quad \delta u(L) = 0$$

THE VARIATION OF THE TPE IS

$$\Pi[\tilde{u} + \delta u] = \frac{1}{2}\int_0^L EA(u' + \delta u')^2 - \int_0^L p(\tilde{u} + \delta u) - F(0)\left(\tilde{u}(0) + \delta u(0)\right) - F(L)\left(\tilde{u}(L) + \delta u(L)\right)$$

$$= \left.\frac{1}{2}\int_0^L EA\tilde{u}'^2 - \int_0^L p\tilde{u} - F(0)\tilde{u}(0) - F(L)\tilde{u}(L)\right\} \Pi[\tilde{u}] \; +$$

$$+ \int_0^L EA u'\delta u' + \frac{1}{2}\int_0^L EA\delta u'^2 - \int_0^L p\delta u - F(0)\delta u(0) - F(L)\delta u(L)$$

USING INTEGRATION BY PARTS:

$$\tilde{u}'\delta u' = (\tilde{u}'\delta u)' - \tilde{u}''\delta u$$

$$EA\int_0^L \tilde{u}'\delta u' = EA\left[\tilde{u}'(L)\delta u(L) - \tilde{u}'(0)\delta u(0)\right] - \int_0^L EA\tilde{u}''\delta u$$

$$= F(L)\delta u(L) + F(0)\delta u(0) - \int_0^L EA\tilde{u}''\delta u$$

HENCE

$$\Pi[\tilde{u} + \delta u] = \Pi[\tilde{u}] + F(L)\delta u(L) + F(0)\delta u(0) - \int_0^L (EA\tilde{u}'' + p)\delta u \; +$$

$$- F(L)\delta u(L) - F(0)\delta u(0) + \frac{1}{2}\int_0^L EA(\delta u')^2$$

$$\Pi[\tilde{u} + \delta u] = \Pi[\tilde{u}] - \underbrace{\int_0^L (EA\tilde{u}'' + p)\delta u}_{\substack{\delta\Pi = 0 \\ \text{FIRST VARIATION}}} + \underbrace{\frac{1}{2}\int_0^L EA\,\delta u'^2}_{\substack{\delta\Pi^2 \\ \text{SECOND VARIATION}}}$$

$\tilde{u}$ FULFILLS EQUILIBRIUM EQUATIONS ($\delta\Pi = 0$) $\longleftarrow$ FIRST VARIATION

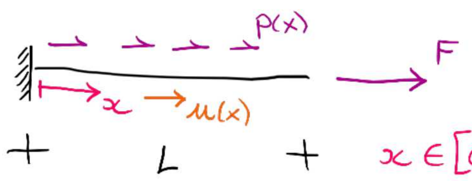
SECOND VARIATION $\longrightarrow$ ALWAYS POSITIVE

$$\Pi[\tilde{u} + \delta u] - \Pi[\tilde{u}] = \frac{1}{2}\int_0^L EA\,\delta u'^2 \begin{cases} \geq 0, \; \forall \delta u \text{ ADMISSIBLE} \\ = 0 \text{ iff } \delta u = 0 \end{cases}$$

THIS MEANS THAT $\longrightarrow$

$\Pi[\tilde{u}]$ IS A <u>MINIMUM</u> !!!

FINITE ELEMENT FORMULATION



$x \in [0, L]$

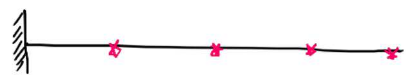
INTEGRAL FORMULATION $\longrightarrow$ LET'S USE TOTAL POTENTIAL ENERGY

$$\Pi[u] = \frac{1}{2}\int_0^L EA(u')^2 - \int_0^L pu - F(0)u(0) - F(L)u(L)$$

$u(x)$ t.c. $u \in L^2([0,L])$ S.T. THE INTEGRALS INTO TPE ARE WELL DEFINED

ASSUME $u(x) \in \mathcal{C}^1 \rightarrow$ CONTINUE AND DIFFERENTIABLE FUNCTIONS

1) DISCRETISATION : DIVIDE THE ROD INTO FINITE ELEMENTS AN STUDY THE SINGLE ONE



e-th FE

$x \in [0, \ell^e]$

THE TPE IS SEEN AS A SUM

$$\Pi[u(x)] = \sum_{e=1}^{n_e} \Pi^e[u^e(x)] \rightarrow$$ ADDITIVITY IS USEFUL !!

2) ANALYTICAL APPROXIMATION

THE UNKNOWN FIELD $u(x)$ FULFILLS

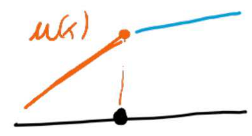
- $u(x) \in \mathcal{C}^1 \rightarrow$ INSIDE THE DOMAIN of THE FUNCTION
- $u(x) \in \mathcal{C}^0 \rightarrow$ ON THE BOUNDARY $(x=0, x=\ell^e)$

$u(x) \in \mathcal{C}^1$ if $x \in]0, L_e[$

$u(x) \in \mathcal{C}^0$ if $x = 0 \vee x = L$

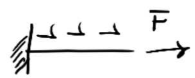
$\downarrow$

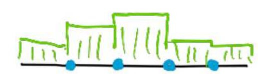
THIS CHOICE GUARANTEES THE CONTINUITY OF THE PRINCIPAL UNKNOWN FIELD $u(x)$ BUT NOT OF ITS FIRST DERIVATIVE $\rightarrow$ APPROXIMATION ALLOWED FOR THIS KIND OF PROBLEMS



j-th NODE

$\hookrightarrow$ HENCE, THE DISPLACEMENTS ARE CONTINUOUS, BUT THE STRESSES ARE NOT





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NOTE: IN THE FINITE ELEMENT METHOD, THE APPROXIMATION OF DERIVED FIELDS (STRESS, STRAIN) IS ALWAYS WORSE THAN THE APPROXIMATION OF THE PRINCIPAL UNKOWN FIELD

$\hookrightarrow$ TO INCREASE THE QUALITY OF THE SOLUTION WE CAN :

i) REFINE THE MESH, BUT THE SOLUTION STILL REMAINS AN APPROXIMATION

ii) INCREASE THE ORDER OF FE, i.e. CONSIDER $u(x) \in \mathcal{C}^m$ $(m>1)$, TO ENRICH THE FORMULATION AND HAVE CONTINUOUS DERIVATIVE

THE SHAPE FUNCTIONS DEFINE THE ORDER OF APPROXIMATION

NOTE: CHOOSE A COMPLETE FUNCTIONAL BASIS

$$\lim_{K \to \infty} \left[u(x) - \sum_{i=1}^{K} \alpha_i \varphi_i(x)\right] = 0 \rightarrow$$ CHOOSE ALL THE TERMS OF THE POLYNOMIAL UNDER THE GIVEN DEGREE

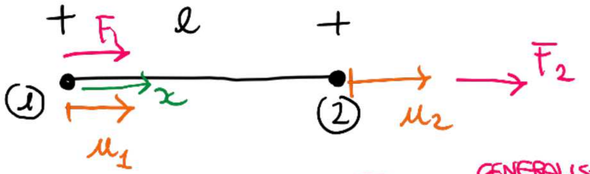
LINEAR TRUSS FINITE ELEMENT

$$u(x) = \alpha_i \varphi_i(x) = \alpha_1 + \alpha_2 x \rightarrow$$ THE DISPLACEMENT IS ASSUMED TO BE LINEAR ON EACH EL.

RITZ-GALERKIN APPROACH

$u(x)$ DEPENDS ON GENERAL COORDINATES α_i

LET'S PASS TO A "PHYSICAL" REPRESENTATION, WHERE THE DOFS ARE THE NODAL DISPLACEMENTS



u_1, u_2 : NODAL DISPLACEMENTS DOFS $\quad$ F_1, F_2 : GENERALISED NODAL FORCES

PASSAGE FROM RITZ-GALERKIN TO NODAL DOFS

$$\begin{cases} u(0) = \alpha_1 + \alpha_2 \cdot 0 = u_1 \\ u(L) = \alpha_1 + \alpha_2 \cdot L = u_2 \end{cases} \rightarrow \quad \alpha_1 = u_1 \qquad \alpha_2 = \frac{u_2 - u_1}{\ell}$$

HENCE

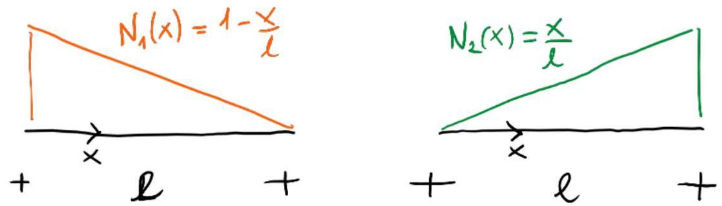
$$u(x) = u_1 + \frac{u_2 - u_1}{\ell} x = u_1 \underbrace{\left(1 - \frac{x}{\ell}\right)}_{N_1(x)} + u_2 \underbrace{\left(\frac{x}{\ell}\right)}_{N_2(x)}$$

SHAPE FUNCTIONS

$$u(x) = N_i(x) u_i \qquad i = 1,2$$

NODAL 2 DOFS

LINEAR MODEL : $N_1(x) = 1 - \frac{x}{\ell}$, $N_2(x) = \frac{x}{\ell}$



THE VALUE OF THE SHAPE FUNCTION IS 1 IF COMPUTED IN THE NODE, 0 OTHERWISE.

$$N_i(x_j) = \delta_{ij}$$

$$N_1(x) + N_2(x) = 1 - \frac{x}{\ell} + \frac{x}{\ell} = 1 \longrightarrow$$ SINCE ALSO A RIGID MOTION MUST BE REPRESENTED

RIGID MOTION $\quad u(x) = \bar{u} \qquad u_1 = u_2 = \bar{u}$

$$u(x) = \bar{u} N_1 + \bar{u} N_2 = \bar{u}(N_1 + N_2) \rightarrow$$ IT IMPLIES $N_1 + N_2 = 1$

LET'S USE MATRIX NOTATION

$$u(x) = N_1(x) u_1 + N_2(x) u_2 = \underbrace{\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}}_{\text{SHAPE FUNCTION MATRIX}} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{\text{VECTOR OF NODAL DOFS}} = \underline{\underline{N}}\, \underline{U}$$

THIS MODEL IS VALID $\forall$ FE

$$u^e(x) = \underline{\underline{N}}^e(x^e)\, \underline{U}_e = \underline{\underline{N}}(x^e)\, \underline{U}_e \quad (\underline{\underline{N}} \text{ ARE ALWAYS THE SAME, } \forall e)$$

DEFORMATION :

$$\varepsilon(x) = u'(x) = \frac{d}{dx}\left[N_1(x) u_1 + N_2(x) u_2 \right] = N_1'(x) u_1 + N_2'(x) u_2 =$$

$$= \underbrace{\begin{bmatrix} N_1'(x) & N_2'(x) \end{bmatrix}}_{\substack{\underline{\underline{B}} \\ \text{STRAIN-DISPLACEMENTS MATRIX}}} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{\ell} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underline{\underline{B}}\, \underline{U}_e$$

$$B_1 + B_2 = 0 \rightarrow$$ DUE TO RIGID MOTION...

$$u(x) = \bar{u} \rightarrow \varepsilon = 0 = \bar{u}(B_1 + B_2)$$

THE APPROXIMATION of THE UNKNOWN FIELD CAN ALWAY BE DONE USING MATRIX NOTATION

$$u(x) = \underline{\underline{N}}(x)\,\underline{U} \qquad \varepsilon(x) = \underline{\underline{B}}(x)\,\underline{U}$$

$\underline{\underline{N}}, \underline{\underline{B}}$ CHANGE ON THE BASIS OF THE FORMULATION

3) WEAK FORM / STATIONARY POINT of FUNCTIONAL

FOR EACH FE, THE TOTAL POTENTIAL ENERGY IS

$$\Pi^e[u] = \frac{1}{2}\int_0^\ell EA^e (u')^2 - \int_0^\ell p u - \tilde{\tilde{F}}_1 u_1 - \tilde{\tilde{F}}_2 u_2 =$$

$$= \frac{1}{2}\int_0^\ell (u')^T EA (u') - \int_0^\ell (u')^T p - \tilde{\tilde{F}}_1 u_1 - \tilde{\tilde{F}}_2 u_2 =$$

$$= \frac{1}{2}\int_0^\ell \underline{U}^T \underline{\underline{B}}^T(x) EA\, \underline{\underline{B}}(x)\, \underline{U} - \int_0^\ell \underline{U}^T \underline{\underline{N}}^T(x)\, p - u_1\tilde{\tilde{F}}_1 - u_2\tilde{\tilde{F}}_2 =$$

$$= \frac{1}{2}\underline{U}^T \underbrace{\left(\int_0^\ell \underline{\underline{B}}^T(x) EA\, \underline{\underline{B}}(x)\right)}_{\underline{\underline{K}}\ \text{STIFFNESS MATRIX}} \underline{U} - \underbrace{\underline{U}^T\left(\int_0^\ell \underline{\underline{N}}^T(x) p(x)\right) - U_1\tilde{\tilde{F}}_1 - U_2\tilde{\tilde{F}}_2}$$

$$-\underline{U}^T\underline{F} = -\underline{U}^T \underbrace{\left[\int_0^\ell \underline{\underline{N}}^T p + \begin{Bmatrix}\tilde{\tilde{F}}_1 \\ \tilde{\tilde{F}}_2\end{Bmatrix}\right]}_{\underline{F}\ \text{NODAL FORCE VECTOR}}$$

HENCE

$$\Pi^e[u] = \frac{1}{2}\underline{U}^{e^T} \underline{\underline{K}}^e \underline{U}^e - \underline{U}^{e^T}\underline{F}^e$$

← IT'S A FUNCTION of THE NODAL UNKNOWNS $\underline{U}^e$

STIFFNESS MATRIX

$$\underline{\underline{B}} = \frac{1}{\ell}\{-1, 1\}$$

$$\underline{\underline{K}}^e = \int_0^\ell \underline{\underline{B}}^T(x) EA\, \underline{\underline{B}} = \int_0^\ell \frac{EA}{\ell^2}\begin{Bmatrix}-1 \\ 1\end{Bmatrix}\{-1, 1\} = \frac{EA}{\ell^2}\int_0^\ell \begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix} dx = \frac{EA}{\ell^2}\begin{bmatrix}\ell & -\ell \\ -\ell & \ell\end{bmatrix}$$

FINITE ELEMENT WITH LINEAR FORMULATION

$$\boxed{\underline{\underline{K}}^e = \frac{EA^e}{\ell_e}\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}}$$

← FUNDAMENTAL RESULT! IT DEPENDS ON THE EA^e OF EACH SINGLE ELEMENT

NODAL FORCE VECTOR

$$\int_0^\ell \underline{\underline{N}}^T p(x) = \int_0^\ell \begin{Bmatrix}\left(1 - \frac{x}{\ell}\right) p(x) \\ \frac{x}{\ell} p(x)\end{Bmatrix} dx \longrightarrow$$

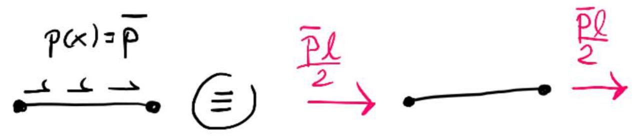
IT REPRESENTS A NODAL LOAD WHICH IS EQUIVALENT TO A DISTRIBUTED LOAD, FROM THE POINT OF VIEW OF THE ENERGY

(i.e. THE SAME CONTRIBUTION IN THE TPE)

- CONSTANT LOAD

$$p(x) = \bar{p} = \text{cst}$$

$$\underline{F} = \bar{p}\int_0^\ell \begin{Bmatrix}1 - \frac{x}{\ell} \\ \frac{x}{\ell}\end{Bmatrix} dx = \begin{Bmatrix}\frac{\bar{p}\ell}{2} \\ \frac{\bar{p}\ell}{2}\end{Bmatrix}$$



- LINEAR LOAD

$$p(x) = p_1 + \frac{p_2 - p_1}{\ell}x, \quad \underline{F} = \int_0^{\ell_e} \begin{Bmatrix}1 - \frac{x}{\ell} \\ \frac{x}{\ell}\end{Bmatrix}\left(p_1 - \frac{p_2 - p_1}{\ell}x\right) dx = \begin{Bmatrix}\frac{p_1\ell}{3} + \frac{p_2\ell}{6} \\ \frac{p_1\ell}{6} + \frac{p_2\ell}{3}\end{Bmatrix}$$

MINIMUM OF THE TOTAL POTENTIAL ENERGY

$$\Pi[u(x)] \approx \Pi(\underline{U}) \approx \sum_e \Pi^e(\underline{U}^e)$$

FUNCTIONAL — FUNCTION of THE UNKNOWNS $\underline{U}$ — DISCRETISATION

$$\delta\Pi[u] = 0, \quad \forall \delta u$$

IMPOSE THE CONDITION THAT THE FUNCTIONAL IS STATIONARY

VALID FOR EACH VARIATION δu,
HENCE VALID FOR EACH FINITE ELEMENT $\rightarrow$ $\boxed{\text{MINIMUM of } \Pi^e(\underline{U}^e) \ \forall \text{ EF}}$

$$\frac{\partial \Pi^e(\underline{U})}{\partial \underline{U}} = 0, \quad \Pi^e(\underline{U}) = \frac{1}{2}\underline{U}_e^T \underline{\underline{K}}_e \underline{U}_e - \underline{U}_e^T \underline{F}_e$$

INDEX NOTATION :

$$\Pi = \frac{1}{2} K_{ij} U_i U_j - U_i F_i$$

$$\frac{\partial \Pi}{\partial U_k} = \frac{1}{2} K_{ij} \left\{ \frac{\partial U_i}{\partial U_k} U_j + \frac{\partial U_j}{\partial U_k} U_i \right\} - F_i \frac{\partial U_i}{\partial U_k}$$

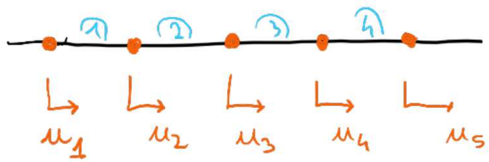
$$= \frac{1}{2} K_{ij} \left(2 \frac{\partial U_i}{\partial U_k} U_j \right) - F_i \delta_{ik}$$

$$= K_{ij} \delta_{ik} U_j - F_i \delta_{ik} = K_{kj} U_j - F_k = 0$$

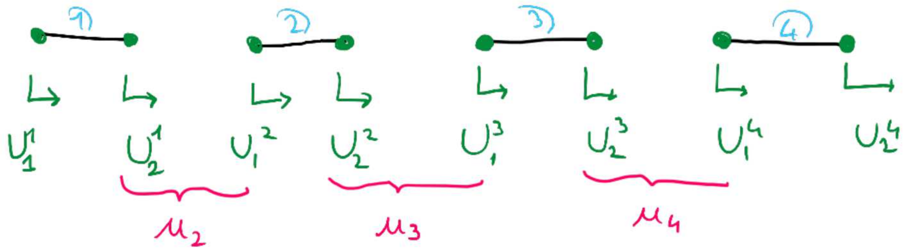
$$\frac{\partial \Pi^e}{\partial \underline{U}} = \underline{\underline{K}}_e \underline{U}_e - \underline{F}_e = 0 \rightarrow \boxed{\underline{\underline{K}}_e \underline{U}_e = \underline{F}_e}$$

EQUILIBRIUM EQUATION FOR EACH FINITE ELEMENT

ASSEMBLY



THE SAME CONCEPT IS VALID ALSO CONSIDERING THE FORCES



TWO KINEMATIC UNKOWNS FOR EACH FINITE ELEMENT $\rightarrow$ BUT FROM A GLOBAL POINT OF VIEW THE NUMBER OF UNKNOWNS DECREASES $\rightarrow$ SOME UNKNOWNS ARE SHARED BY TWO OR MORE ELEMENTS

$$U_2^{i+1} = U_1^i \qquad F_2^{i-1} = F_1^i$$

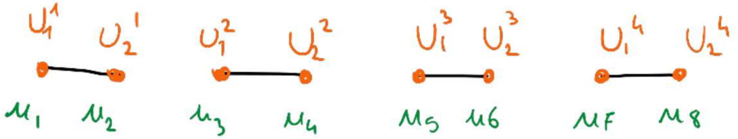
THROUGH ASSEMBLY, WE PASS FROM THE PROBLEM ON THE SINGLE ELEMENT TO THE GLOBAL PROBLEM

$$\boxed{\underline{\underline{K}}\,\underline{U} = F}$$

ASSEMBLY METHODS
- o) "COMPLETE" METHOD
- i) EXPANDED MATRIX METHOD
- ii) DIRECT METHOD

0) COMPLETE METHOD $\longrightarrow$ DO NOT USE IT!! APPLIED WITHOUT THINKING...

THE GLOBAL UNKNOWNS ARE ALL THE UNKNOWNS FOR EACH FINITE ELEMENT $\longrightarrow$ WE NEED OTHER CONSTRAINT EQUATIONS



VECTOR OF GLOBAL UNKNOWN $\quad \underline{u} = \{u_1, u_2, u_3, u_4, u_5, \dots\dots\}^T$

ADD OTHER CONSTRAINT EQUATION TO DEFINE WHICH DOFS ARE EQUAL

$\underline{\underline{A}}\,\underline{u} = 0$

CONSTRAINTS

$u_2 = u_3$

$u_4 = u_5$

$u_6 = u_7$

$\vdots$

THIS METHOD IS NOT "SMART"...

i) EXPANDED MATRIX METHOD

$\forall e$, WE HAVE THESE RELATIONS $\quad \underline{\underline{K}}^e \underline{U}^e = \underline{F}^e$

$\underline{\underline{K}}^e$: 2×2

$\underline{U}^e, \underline{F}^e$: 2×1

NOW, REWRITE THESE RELATIONS IN THIS WAY:

$$\hat{\underline{\underline{K}}}^e \underline{u} = \hat{\underline{F}}^e$$

$\underline{u} = \{u_1, u_2, \dots\}$

GLOBAL VECTOR

THEY ARE STILL WRITTEN FOR EACH FINITE ELEMENT, BUT THE DIMENSION OF $\hat{K}$ AND $\hat{F}$ IS COMPATIBLE WITH VECTOR $\underline{u}$

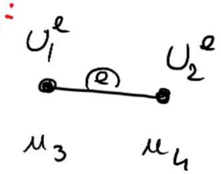
IF $\underline{u}$ HAS N COMPONENTS $\longrightarrow$ $\hat{K}^e$: $N\times N$, $\hat{F}^e$: $N\times 1$ $\longrightarrow$ THE REASON WHY IT'S CALLED "EXPANDED MATRIX"...

LET'S INTRODUCE THE CONNECTIVITY MATRIX

$$\underline{U}^e = \underline{\underline{C}}^e \underline{u}$$

$\underline{\underline{C}}^e$ IS A $\underline{2\times N}$ BOOLEAN MATRIX

es:



$$\underline{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{Bmatrix} \qquad \underline{\underline{C}}^e = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \end{bmatrix}$$

$$\underline{\underline{C}}^e \underline{u} = \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} U_1^e \\ U_2^e \end{Bmatrix}$$

THE TPE FOR EACH ELEMENT IS:

$$\Pi^e(\underline{U}) = \frac{1}{2}\underline{U}_e^T \underline{\underline{K}}_e \underline{U}_e - \underline{U}_e^T \underline{F}_e = \frac{1}{2}(\underline{\underline{C}}^e \underline{u})^T \underline{\underline{K}}\, \underline{\underline{C}}^e \underline{u} - (\underline{\underline{C}}^e \underline{u})^T \underline{F} =$$

$$= \frac{1}{2}\underline{u}^T \underbrace{\underline{\underline{C}}_e^T \underline{\underline{K}}_e \underline{\underline{C}}_e}_{\hat{\underline{\underline{K}}}_e} \underline{u} - \underline{u}^T \underbrace{\underline{\underline{C}}_e^T F}_{\hat{\underline{F}}_e} \equiv \frac{1}{2}\underline{u}^T \hat{\underline{\underline{K}}}^e \underline{u} - \underline{u}^T \hat{\underline{F}}$$

$$\hat{\underline{\underline{K}}}_e = \underline{\underline{C}}_e^T \underline{\underline{K}}_e \underline{\underline{C}}_e \qquad \hat{\underline{F}}_e = \underline{\underline{C}}_e^T \underline{F}$$

$\hat{\underline{\underline{K}}}^e$ ed $\hat{\underline{F}}^e$ ARE SPARSE MATRICES

THEY ARE NULL ALMOST EVERYWHERE, FEW ELEMENTS ARE NOT VANISHING

$$\Pi(\underline{u}) = \sum_{e=1}^{Me} \Pi^e = \sum_e \frac{1}{2}\underline{U}_e^T \underline{K}_e \underline{U}_e - \underline{U}_e^T F = \sum_e \frac{1}{2}\underline{u}^T \hat{\underline{\underline{K}}}^e \underline{u} - \underline{u}^T \hat{F}^e =$$

$$= \frac{1}{2}\underline{u}^T\left(\sum_e \hat{\underline{\underline{K}}}^e\right)\underline{u} - \underline{u}^T\left(\sum_e \hat{\underline{F}}^e\right) \equiv \frac{1}{2}\underline{u}^T \underline{\underline{K}}\,\underline{u} - \underline{u}^T \underline{F} \Rightarrow \quad \underline{\underline{K}} = \sum_e \hat{\underline{\underline{K}}}^e, \quad \underline{F} = \sum_e \hat{\underline{F}}_e$$

HENCE

$$\underline{\underline{K}} = \sum_e \hat{\underline{\underline{K}}}^e \ , \ \underline{F} = \sum_e \hat{\underline{F}}^e \rightarrow$$

THEN SOLVE THE GLOBAL PROBLEM

$$\underline{\underline{K}}\,\underline{u} = \underline{F}$$

THE PROBLEM OF THIS METHOD IS THAT WE NEED TO STORE THE EXPANDED MATRICES, WHICH ARE COMPOSED OF ZERO ELEMENTS ...

(BUT ACTUALLY IS NOT A GREAT PROBLEM !!)

ii) DIRECT METHOD

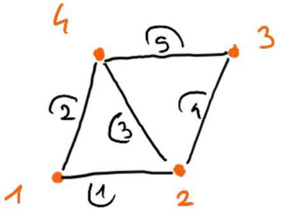
ACCORDING TO THIS METHOD, THE ELEMENTS of $\underline{\underline{K}}^e$ ARE PUT DIRECTLY IN THE $\underline{\underline{K}}$ GLOBAL MATRIX, IN THE RIGHT POSITION

↳ LET'S CONSIDER THE PREVIOUS RESULT $\quad \underline{\underline{K}} = \sum_e \underline{\underline{K}}^e \ , \ \underline{F} = \sum_e \underline{F}^e$

↳ HENCE, THE ELEMENTS OF $\underline{\underline{K}}^e$ ENTER IN THE GLOBAL MATRIX $\underline{\underline{K}}$ AS A <u>"SUM"</u>

$$\underline{\underline{K}}\,\underline{u} = \underline{F} \qquad \underline{u} = \{u_1, u_2, \dots, u_N\}^T$$

LET'S DEFINE A CONNECTIVITY MATRIX :

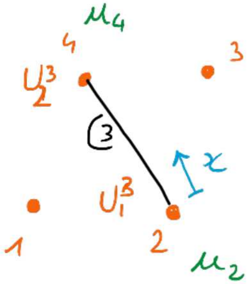


$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 4 \\ 2 & 3 \\ 4 & 3 \end{bmatrix} \leftarrow$$

IN THE j-th ROW THERE ARE THE LABELS OF THE NODES OF THE j-th ELEMENT

CONNECTIVITY MATRIX GIVES US INFORMATIONS ALSO ON THE ORIENTATION of THE REF. SYSTEM ON THE ELEMENTS

ex : ELEMENT 3 $\rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ \boxed{2 \quad 4} \\ 2 & 3 \\ \vdots \end{bmatrix}$



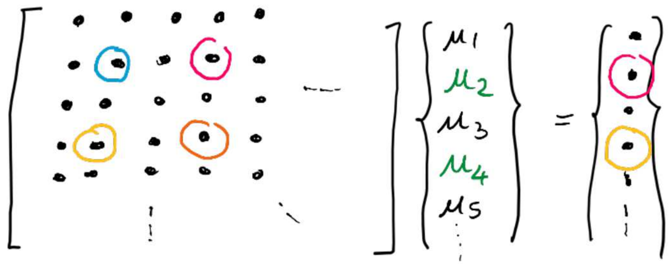
$$\begin{bmatrix} K_{11}^3 & K_{12}^3 \\ K_{21}^3 & K_{22}^3 \end{bmatrix} \underbrace{\begin{Bmatrix} U_1^3 \\ U_2^3 \end{Bmatrix}}_{} = \begin{Bmatrix} F_1^3 \\ F_2^3 \end{Bmatrix}$$

WE NEED THOSE OF THE GLOBAL REFERENCE →

THE CONNECTIVITY MATRIX TELLS US THE LINK FROM LOCAL TO GLOBAL COORDINATES

$$\begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} U_1^3 \\ U_2^3 \end{Bmatrix}$$

$$\underline{\underline{K}}\,\underline{u} = \underline{F}$$



→ THE COMPONENTS OF $\underline{\underline{K}}^{(3)}$ AND $\underline{F}^{(3)}$ ENTER IN THE GLOBAL MATRIX/VECTOR IN THIS WAY

THEY ARE SUMMED TO THE ELEMENTS ALREADY PRESENT IN THE SAME POSITION

<u>ITERATIVE PROCEDURE</u> :

<u>STEP 1</u> : INITIALISE <u>NULL</u> MATRIX $\underline{\underline{K}}$ AND VECTOR $\underline{F}$

for $e = 1 : M_e$

<u>STEP 2</u> : THROUGH CONNECTIVITY MATRIX, UNDERSTAND THE POSITION OF $\underline{\underline{K}}^{(e)}$ AND $\underline{F}^{(e)}$ IN THE GLOBAL MATRIX / VECTOR

<u>STEP 3</u> : <u>SUM</u> THE ELEMENTS of $\underline{\underline{K}}^{(e)}$ AND $\underline{F}^{(e)}$ TO THOSE ALREADY PRESENT IN THE CONSIDERED POSITIONS

end

PROPERTY of THE GLOBAL SYSTEM

$$\underline{\underline{K}}\,\underline{u} = \underline{F} \longrightarrow \Pi[\underline{u}] = \frac{1}{2}\underline{u}^T \underline{\underline{K}}\,\underline{u} - \underline{u}^T \underline{F}$$

TPE of THE GLOBAL SYSTEM (FUNCTION of $\underline{u}$)

$\downarrow$ MECHANICAL INTERPRETATION

$\underline{F}$: EXTERNAL FORCES APPLIED TO THE NODES OF THE STRUCTURE

$\underline{\underline{K}}\,\underline{u}$: INTERNAL FORCES DUE TO THE ELASTIC STRUCTURE (RESPONSE of EXTERNAL LOADS)

THROUGH FEM, THE STRUCTURE BECOMES AN ELASTIC SPRING $\rightarrow$ THE STIFFNESS OF THE SPRING IS DESCRIBED BY $\underline{\underline{K}}$, IT DEPENDS ON THE STRUCTURE

NOTE : THE EQUATIONS $\underline{\underline{K}}\,\underline{u} = \underline{F}$ CAN BE INTERPRETED AS THE EQUILIBRIUM EQUATIONS of THE STRUCTURE

PROPERTY of $\underline{\underline{K}}$:

i) THE ROWS OF $\underline{\underline{K}}$ REPRESENT EQUILIBRIUM EQUATION OF THE NODES OF THE TRUSS STRUCTURE

ii) THE COLUMNS of $\underline{\underline{K}}$ ARE THE NODAL FORCES CAUSED BY A UNIT DISPLACEMENT ASSOCIATED TO THE DOF OF THE CONSIDERED COLUMN

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots \\ K_{21} & K_{22} & K_{23} & \cdots \\ \vdots & \vdots & \vdots & \\ K_{N1} & K_{N2} & K_{N3} & \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{Bmatrix} \rightarrow \underbrace{\begin{Bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{N1} \end{Bmatrix} u_1}_{\text{FORCE DUE ONLY TO DISPL. } u_1} + \begin{Bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{N2} \end{Bmatrix} u_2 + \cdots = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{Bmatrix}$$

iii) $\underline{\underline{K}}$ IS SINGULAR FOR THE UNCONSTRAINED STRUCTURE

$\hookrightarrow$ THE ELASTIC SOLUTION IS UNIQUE, UNLESS THE PRESENCE OF A RIGID MOTION $\rightarrow$ THE ELASTIC FORCES DUE TO A RIGID MOTION ARE NULL !!!

$$\underline{\underline{K}}\,\underline{u}^R = 0 \,, \quad \forall\, \underline{u}^R \text{ ADMISSIBLE} \quad \Rightarrow \quad \det(\underline{\underline{K}}) = 0$$

iv) $\underline{\underline{K}}$ IS POSITIVE SEMIDEFINITE $\quad \mathcal{E} = \frac{1}{2}\underline{u}^T \underline{\underline{K}}\,\underline{u} \geq 0 \,, \forall \underline{u} \quad (\mathcal{E} = 0 \text{ iff } \underline{u} = 0)$

v) $\underline{\underline{K}} \in Sym \longrightarrow$ LET'S APPLY BETTI'S THEOREM

$$\mathcal{L}_{ab} = \mathcal{L}_{ba}$$

CONDITION a) $\underline{u}^a, \underline{F}^a = \underline{\underline{K}}\,\underline{u}^a$

CONDITION b) $\underline{u}^b, \underline{F}^b = \underline{\underline{K}}\,\underline{u}^b$

$$\underline{u}^b \cdot \underline{F}^a = \underline{u}^a \cdot \underline{F}^b \rightarrow \underline{u}^b \cdot \underline{\underline{K}}\,\underline{u}^a = \underline{u}^a \cdot \underline{\underline{K}}\,\underline{u}^b$$

$$\underline{u}^b \cdot \underline{\underline{K}}\,\underline{u}^a = \underline{u}^b \cdot \underline{\underline{K}}^T \underline{u}^a \Rightarrow \boxed{\underline{\underline{K}} = \underline{\underline{K}}^T}$$

COMPUTE THE SOLUTION

$\det(\underline{\underline{K}}) = 0 \qquad rg(\underline{\underline{K}}) = 2m - \underbrace{3}$

WE NEED AT LEAST 3 DOFS TO HAVE AN INVERTIBLE SUB-MATRIX

$$\underline{u} = \begin{Bmatrix} \underline{u}_U \\ \hline \underline{u}_F \end{Bmatrix}, \quad \underline{F} = \begin{Bmatrix} \underline{F}_U \\ \underline{F}_F \end{Bmatrix}$$

– PARTITION –

$\underline{u}_U, \underline{F}_U \longrightarrow$ DISPLACEMENTS AND FORCES WHERE THE DISPLACEMENTS ARE KNOWN

$\underline{u}_F, \underline{F}_F \longrightarrow$ DISPLACEMENTS AND FORCES WHERE THE FORCES ARE KNOWN

THE PROBLEM CAN BE PARTITIONED IN THIS WAY:

$$\left[\begin{array}{c|c} \underline{\underline{K}}_{UU} & \underline{\underline{K}}_{UF} \\ \hline \underline{\underline{K}}_{FU} & \underline{\underline{K}}_{FF} \end{array}\right] \begin{Bmatrix} \underline{u}_U \\ -- \\ \underline{u}_F \end{Bmatrix} = \begin{Bmatrix} \underline{F}_U \\ -- \\ \underline{F}_F \end{Bmatrix}$$

UNKNOWNS OF THE PROBLEM

2^{nd} ROW:

$$\underline{\underline{K}}_{FF} \underbrace{\underline{u}_F}_{\text{UNKNOWNS}} = \underbrace{\underline{F}_F}_{\text{EXTERNAL FORCES}} - \underbrace{\underline{\underline{K}}_{FU} \underline{u}_U}_{\text{FORCES DUE TO PRESCRIBED DISPLACEMENT}}$$

ONE CAN SOLVE THE SYSTEM IN $\underline{u}_F \longrightarrow$ USE THE SOLUTION IN THE 1^{st} ROW

$$\underbrace{\underline{F}_U}_{\text{UNKNOWNS}} = \underline{\underline{K}}_{UU} \underline{u}_U + \underline{\underline{K}}_{UF} \underline{u}_F \longrightarrow$$ $\underline{F}_U$ ARE THE REACTIONS

(IN FACT, IN THE CONSTRAINTS THE DISPL. ARE KNOWN)

QUADRATIC FINITE ELEMENT FORMULATION

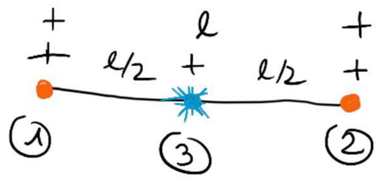
UNTIL NOW WE USED A LINEAR APPROXIMATION OF THE FINITE ELEMENT $\longrightarrow u(x) = \left(1 - \frac{x}{l}\right) u_1 + \frac{x}{l} u_2 \longrightarrow$ GOOD APPROXIMATION FOR DISPLACEMENTS, WORSE APPROX. FOR INTERNAL STRESS AND AXIAL FORCES

$N(x) = EA u'(x) \longrightarrow$ THE APPROXIMATION of $\underline{N}(x)$ IS PIECEWISE-LINEAR $\longrightarrow$ REFINING THE MESH, THE SOLUTION QUALITY of $u(x)$ IMPROVES, BUT $\underline{N}(x)$ IS STILL PIECEWISE LINEAR

TO IMPROVE THE SOLUTION, WE CAN INCREASE THE ORDER OF THE POLYNOMIAL OF THE SHAPE FUNCTIONS

$$\boxed{u(x) = a + bx + cx^2} \longrightarrow$$ WE NEED DOFS IN 3 NODES

(RITZ - GALERKIN)



$$\begin{cases} u(0) = u_1 = a \\ u(l/2) = u_3 = a + b\frac{l}{2} + c\frac{l^2}{4} \\ u(l) = u_2 = a + bl + cl^2 \end{cases}$$

HENCE

$$\begin{cases} a = u_1 \\ b\frac{l}{2} + c\frac{l^2}{4} = u_3 - u_1 \\ bl + cl^2 = u_2 - u_1 \end{cases} \qquad \begin{cases} a = u_1 \\ 2bl + cl^2 = 4u_3 - 4u_1 \\ bl + cl^2 = u_2 - u_1 \end{cases} \qquad \begin{cases} a = u_1 \\ b = \frac{1}{l}(3u_1 - u_2 + 4u_3) \\ c = \frac{1}{l^2}(2u_1 + 2u_2 - 4u_3) \end{cases}$$

THE QUADRATIC FORMULATION BECOMES:

$$\boxed{u(x) = \underbrace{\left[1-\frac{3x}{\ell}+\frac{2x^2}{\ell^2}\right]}_{N_1(x)}u_1 + \underbrace{\left[-\frac{x}{\ell}+\frac{2x^2}{\ell^2}\right]}_{N_2(x)}u_2 + \underbrace{\left[\frac{4x}{\ell}-\frac{4x^2}{\ell^2}\right]}_{N_3(x)}u_3}$$

$$u^e(x) = N_i(x)U_i^e = \left[N_1, N_2, N_3\right]\underline{U}^e = \underline{\underline{N}}(x)\,\underline{U}^e$$

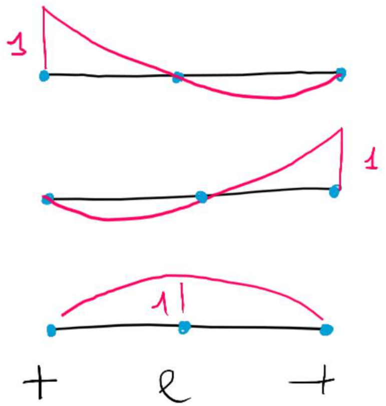
$$\varepsilon^e(x) = \frac{du^e}{dx} = \left[\frac{dN_1}{dx}, \frac{dN_2}{dx}, \frac{dN_3}{dx}\right]\underline{U}^e = \underline{\underline{B}}(x)\,\underline{U}^e$$

THEN $N_i(x_j) = \delta_{ij}$, $\sum N_i = 1$, $\sum B_i = 0$

$$N_1(x) = 1 - \frac{3x}{\ell} + \frac{2x^2}{\ell^2}$$

$$N_2(x) = \frac{x}{\ell}\left(-1 + \frac{2x}{\ell}\right)$$

$$N_3(x) = 4\frac{x}{\ell}\left(1-\frac{x}{\ell}\right)$$



MATRIX NOTATION:

$$\underline{\underline{N}}(x) = \left\{\left(1-\frac{3x}{\ell}+\frac{2x^2}{\ell^2}\right), \frac{x}{\ell}\left(-1+\frac{2x}{\ell}\right), \frac{4x}{\ell}\left(1-\frac{x}{\ell}\right)\right\}$$

$$\underline{\underline{B}}(x) = \left\{\left(-\frac{3}{\ell}+\frac{4x}{\ell^2}\right), \left(-\frac{1}{\ell}+\frac{4x}{\ell^2}\right), \left(\frac{4}{\ell}-\frac{8x}{\ell^2}\right)\right\}$$

TOTAL POTENTIAL ENERGY

$$\Pi[\underline{U}] = \frac{1}{2}\underbrace{\underline{U}^{e^T}}_{\text{NOW } \underline{U} \text{ HAS 3 COMPONENTS}}\underbrace{\left(\int_0^\ell \underline{\underline{B}}^T(x)\,EA\,\underline{\underline{B}}(x)\right)}_{\underline{\underline{K}}\ \text{STIFFNESS MATRIX}\ (3\times3)}\underline{U}^e - \underline{U}^{e^T}\underbrace{\left(\int_0^\ell \underline{\underline{N}}^T(x)p(x) - \begin{Bmatrix}\tilde{F}_1^e\\ \tilde{F}_2^e\\ \tilde{F}_3^e\end{Bmatrix}\right)}_{\underline{F}\ \text{VECTOR OF NODAL FORCES}}$$

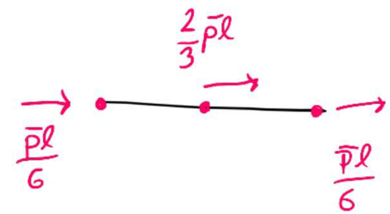
STIFFNESS MATRIX:

$$\underline{\underline{K}} = EA\int_0^L \begin{bmatrix} B_1^2 & B_1B_2 & B_{13} \\ & B_2^2 & B_{23} \\ \text{Sym} & & B_3^2\end{bmatrix}dx = \boxed{\frac{EA}{3\ell}\begin{bmatrix}7 & 1 & -8\\ 1 & 7 & -8 \\ -8 & -8 & 16\end{bmatrix}}$$

NODAL FORCE VECTOR:

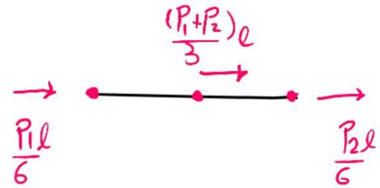
- CONSTANT LOAD

$$p(x) = \bar{p} \qquad \int_0^\ell \underline{\underline{N}}^T(x)\,\bar{p} = \bar{p}\int_0^\ell \underline{\underline{N}}^T(x) = \frac{\bar{p}\ell}{3}\begin{Bmatrix}1/2\\1/2\\2\end{Bmatrix}$$



- LINEAR LOAD

$$p(x) = P_1 + \frac{P_2-P_1}{\ell}x \qquad \int_0^\ell \underline{\underline{N}}^T(x)p(x) = \frac{\ell}{3}\begin{Bmatrix}P_1/2\\ P_2/2 \\ P_1+P_2\end{Bmatrix}$$



ALTERNATIVE METHOD FOR SHAPE FUNCTIONS

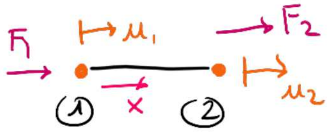
i) $N_1(x)$ IS QUADRATIC

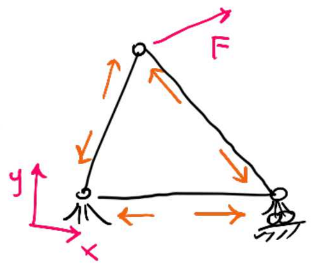
ii) $\begin{cases} N_1(0) = 1 \\ N_1(\ell/2) = 0 \\ N_1(\ell) = 0 \end{cases} \rightarrow N_1(x) = A_1\left(x-\frac{\ell}{2}\right)(x-\ell)$

$$N_1(0) = A_1\frac{\ell^2}{2} = 1 \rightarrow A_1 = \frac{2}{\ell^2}$$

$$\boxed{N_1(x) = \frac{2}{\ell^2}\left(x-\frac{\ell}{2}\right)(x-\ell)}$$

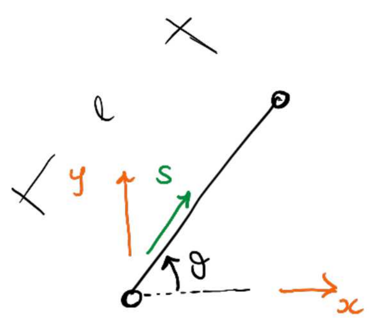
INCLINED ELEMENTS

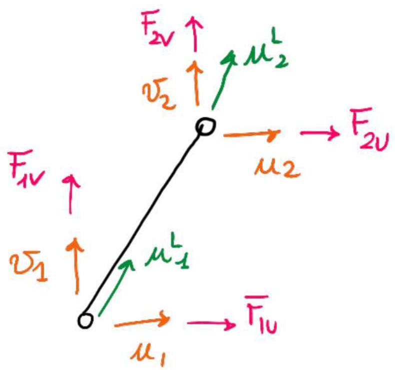




FOR "INTRINSIC 1D" STRUCTURE THE TRUSS FORMULATION IS ENOUGH TO SOLVE THE PROBLEM DIRECTLY

→ FOR 2D STRUCTURES, WE NEED A LINK BETWEEN DISPLACEMENTS $u(x,y)$ AND $v(x,y)$ WITH THE AXIAL DISPLACEMENT of EACH TRUSS ELEMENT



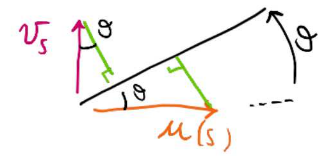


$$\underline{U}^A = \begin{Bmatrix} u_1^L \\ u_2^L \end{Bmatrix} \qquad \begin{cases} u(s) = N_1(s)\,u_1 + N_2(s)\,u_2 \\ v(s) = N_1(s)\,v_1 + N_2(s)\,v_2 \end{cases}$$

$$\underline{U} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad \begin{Bmatrix} u(s) \\ v(s) \end{Bmatrix} = \begin{bmatrix} N_1(s) & 0 & N_2(s) & 0 \\ 0 & N_1(s) & 0 & N_2(s) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

THE LINK BETWEEN $u(s)$, $v(s)$ AND THE LOCAL DISPLACEMENT $u^L(s)$ IS

$$u^L(s) = u(s)\cos\theta + v(s)\,\mathrm{sen}\,\theta$$



COMPUTED IN THE NODES

$$\begin{Bmatrix} u_1^L \\ u_2^L \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \mathrm{sen}\,\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \mathrm{sen}\,\theta \end{bmatrix}}_{\underline{\underline{T}} \to 2\times 4} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

THE TPE ON THE SINGLE FE IS:

$$\Pi = \frac{1}{2}\underline{u}_L^T \underline{\underline{K}}_L \underline{u}_L - \underline{u}_L^T \underline{F}_L = \frac{1}{2}\underline{u}^T \underbrace{\underline{\underline{T}}^T \underline{\underline{K}}_L \underline{\underline{T}}}_{\underline{\underline{K}}} \underline{u} - \underline{u}^T \underbrace{\underline{\underline{T}}^T \underline{F}_L}_{\underline{F}}$$

$$= \frac{1}{2}\underline{u}^T \underline{\underline{K}}\,\underline{u} - \underline{u}^T \underline{F}$$

$$\underline{\underline{T}}^T \underline{\underline{K}}_L \underline{\underline{T}} \to \begin{matrix} 4\times 2 \\ 2\times 2 \\ 2\times 4 \end{matrix} \to \underline{\underline{K}}_e \ 4\times 4!$$

$c = \cos\theta$

$s = \mathrm{sen}\,\theta$

$$\underline{\underline{T}}^T \underline{\underline{K}}_L \underline{\underline{T}} = \frac{EA}{l}\begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} = \frac{EA}{l}\begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix}\begin{bmatrix} c & s & -c & -s \\ -c & -s & c & s \end{bmatrix} =$$

$$= \frac{EA}{l}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} = \begin{bmatrix} \tilde{\underline{\underline{K}}} & -\tilde{\underline{\underline{K}}} \\ -\tilde{\underline{\underline{K}}} & \tilde{\underline{\underline{K}}} \end{bmatrix}, \quad \tilde{\underline{\underline{K}}} = \frac{EA}{l}\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}, \quad \underline{F} = \begin{Bmatrix} cF_1 \\ sF_1 \\ cF_2 \\ sF_2 \end{Bmatrix}$$