Numerical Methods for Solid Mechanics

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COMPUTATIONAL MECHANICS

– FINITE ELEMENT METHOD –

* PHYLOSOPHY of NUMERICAL METHODS
* PROBLEMS IN SOLID MECHANICS
* FEM: GENERAL FORMULATION
* FINITE ELEMENT FOR TRUSS
* FINITE ELEMENT FOR BEAM
* INTRODUCTION TO MATLAB
* FINITE ELEMENT CODE

"Science is the study of what is, Engineering builds what will be. The scientist merely explores that which exists, while the engineer creates what has never existed before"

Theodore von Kármán

PHYSICAL MODEL (REAL) ⇝ REAL PROBLEM, WE NEED TO DEFINE THE PHYSICAL "ENTITIES" THAT PLAY A ROLE IN THE ANALYSIS OF THE PROBLEM

↓

MATHEMATICAL MODEL (FORMAL) ⇝ SET OF PARTIAL DIFFERENTIAL EQUATIONS, WITH BOUNDARY AND INITIAL CONDITIONS

$$\begin{cases} \underline{A}\left(\underline{x}, t, \underline{u}(\underline{x}), \frac{\partial u_i}{\partial x_j}, \dots, \frac{\partial^m u_i}{\partial x_i}, \dots \dot{u}_i\right) = 0 \quad \text{in } \Omega \times T \\ \underline{B}\left(\underline{x}, t, \underline{u}, \dots \frac{\partial u}{\partial x_j} \dots\right) = 0 \quad \text{in } \partial\Omega \quad (BC) \\ \underline{C}\left(\underline{x}, t, \underline{u}, \dot{u}\right) = 0 \quad \text{in } t=0 \quad (IC) \end{cases}$$

(A SOLUTION CAN RARELY BE FOUND !!!)

WE ARE HERE !!!

↓

COMPUTATIONAL MODEL (VIRTUAL) ⇝ SET OF ALGEBRAIC EQUATIONS

$$\boxed{\underline{\underline{K}}\, \underline{u} = \underline{f}}$$

→ THIS CAN ALWAYS BE SOLVED !!

COMPUTATIONAL MODEL → VIRTUAL MODEL THAT ALLOWS THE SOLUTION OF A PHYSICAL PROBLEM

↓

THE ALTERNATIVE WOULD BE A SCALED PHYSICAL MODEL...

$$\underset{=}{K}\,\underline{u} = \underline{f}$$

SYSTEM MATRIX

UNKNOWNS

DATA VECTOR

($\underset{=}{K} = \underset{=}{K}(\underline{u})$ IN NONLINEAR PROBLEMS)

COMPUTATIONAL MECHANICS → THE FOCUS IS ON THE PASSAGE FROM MATHEMATICAL MODEL TO COMPUTATIONAL MODEL

↳ INTRODUCE AN APPROXIMATION

APPROXIMATION ORIGINATES FROM:

1) "ACTIONS" ON THE DOMAIN
2) "ACTIONS" ON THE FUNCTIONS

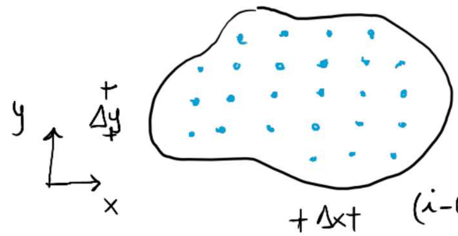
} ON THE PROBLEM

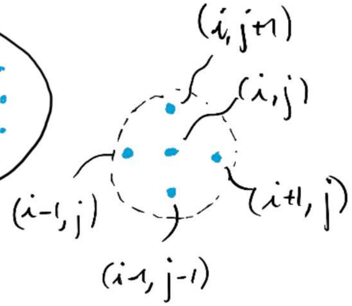
3) "APPROXIMATION" DUE TO NUMERICAL METHODS
4) "APPROXIMATION" DUE TO THE COMPUTER

} ON THE COMPUTER

POINTS (1) and (2) DEFINE THE NUMERICAL METHOD:

1) FINITE DIFFERENCE METHOD (late '800)



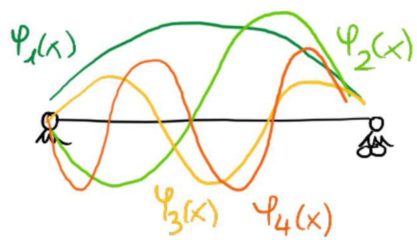


$$\frac{\partial u}{\partial x} = \frac{u_{i,j+1} - u_i}{\Delta x} \quad \forall i,j$$

$$\frac{\partial u}{\partial y} = \frac{u_{j+1,i} - u_j}{\Delta y}$$

STILL USED IN FLUID MECHANICS AND FOR TIME INTEGRATION

2) RAYLEIGH-RITZ METHOD (early '900)



APPROX. OF THE UNKNOWN FIELD AS:

$$u(x) = \underbrace{\alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + \alpha_3 \varphi_3(x) + \alpha_4 \varphi_4(x)}$$

WEIGHTS OF THE LINEAR COMBINATION (RITZ COEFFICIENTS)

SUBSTITUTED IN A VARIATIONAL FORM...

THE UNKNOWNS OF THE PROBLEM ARE THE WEIGHTS α_i

TO HAVE BETTER APPROX. → INCREASE THE NUMBER OF FUNCTION IN THE LINEAR COMBO → FOR COMPLEX DOMAINS, HARD TO FIND THE THE GLOBAL $\varphi(x)$ THAT FULLFIL BOUNDARY CONDITIONS

3) FINITE ELEMENT METHOD (FEM)

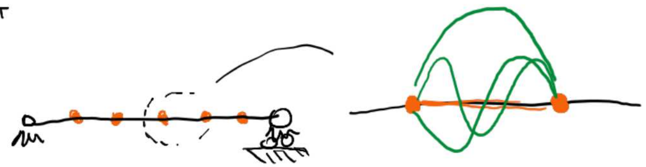
~1920 GALERKIN

~1940 HRENNIKOFF COURANT

~1950 (NASA, TURNER)

EXTENSION OF RR METHOD

$\varphi_i(x)$



THE DOMAIN IS DIVIDED INTO SMALLER PARTS (FINITE ELEMENT) AND ON EACH PART WE CONSIDER AN APPROXIMATION OF THE SOLUTION LIKE:

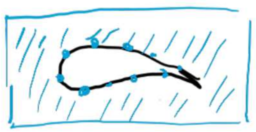
$$u(x) = \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + \dots$$

2 ORDERS OF APPROXIMATION
- i) GEOMETRIC ⇝ DOMAIN DECOMPOSITION (MESH)
- ii) ANALITICAL ⇝ SOL. APPROXIMATED BY LINEAR COMBO

UNKNOWNS → THE α_i FOR ALL FINITE ELEM. → THE α_i ARE LINKED TO PHYSICAL QUANTITIES (DISPLACEMENTS...)

OTHER METHODS:

4) BEM - BOUNDARY ELEMENT METHOD



5) FVM - FINITE VOLUME METHOD

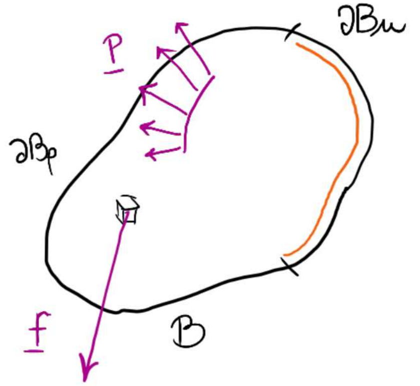
⋮

PROBLEM "TAXONOMY"

	LINEAR	NONLINEAR
STATICS	1) LINEAR STATIC $\underline{\underline{K}}\,\underline{u} = \underline{F}$ 2) INSTABILITY - BUCKLING $\left[\underline{\underline{K}}_M + K_G(P)\right]\underline{u} = \underline{0}$ (eigenvalues)	1) NONLINEAR STATIC $\underline{\underline{K}}(\underline{u})\,\underline{u} = f(\underline{u})$ $(\underline{\underline{K}} = \underline{\underline{K}}_M)$ 2) COMPLETE BUCKLING ANALYSIS $\left[\underline{\underline{K}}_M + \lambda \underline{\underline{K}}_G(\underline{u})\right]\underline{u} = \underline{0}$ (NL eigenvalues)
DYNAMICS	1) EIGENMODES $\left[\underline{\underline{K}} - \omega^2 \underline{\underline{M}}\right]\underline{u} = 0$ (eigenvalues) 2) LINEAR DYNAMIC $\underline{\underline{M}}\,\ddot{\underline{u}} + \underline{\underline{C}}\,\dot{\underline{u}} + \underline{\underline{K}}\,\underline{u} = \underline{F}$ DIRECT INTEGRATION OF THE FUNCTION $\underline{u}(t)$	NONLINEAR DYNAMIC ANALYSIS $\underline{\underline{M}}\,\ddot{\underline{u}} + \underline{\Phi}(\underline{u}, \dot{\underline{u}}, \dots) = 0$ (THE MOST GENERAL)

ALL THE PROBLEMS IN MECHANICS ARE HERE!!

FINITE ELEMENTS IN SOLID MECHANICS



UNKNOWNS $u_i, \sigma_{ij}, \varepsilon_{ij}$ (15)

$$\begin{cases} \dfrac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 & (3) \\ \varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) & (6) \quad \text{in } \Omega \\ \sigma_{ij} = \mathbb{C}_{ijkl}\, \varepsilon_{kl} & (6) \end{cases}$$

$$\partial B \equiv \underbrace{\partial B_p}_{\text{NATURAL BC}} \cup \underbrace{\partial B_u}_{\text{ESSENTIAL BC}}$$

BOUNDARY CONDITIONS
$$\begin{cases} u_i = u_i^0 & \text{in } \partial B_u \\ \sigma_{ij} n_j = p_i^0 & \text{in } \partial B_p \end{cases}$$

N.B.: THE UNKOWNS ARE NOT ALWAYS 15, DIFFERENT FORMULATIONS CAN BE ADOPTED

i) "NAVIER" FORMULATION, THE UNKNOWNS ARE THE DISPLACEMENTS $(3, u_i)$

→ DISPLACEMENT FORMULATION FOR THE FINITE ELEMENTS (TPE)

ii) "BELTRAMI-MITCHELL" FORMULATION, THE UNKNOWNS ARE THE STRESS $(6, \sigma_{ij})$

→ STRESS FORMULATION FOR THE FINITE ELEMENTS (CPE)

iii) "MIXED" FORMULATION, THE UNKNOWNS ARE BOTH DISPLACEMENTS AND STRESS $(6\sigma_{ij}, 3u_i)$

→ MIXED FINITE ELEMENT FORMULATION (HELLINGER - REISSNER)

iv) "COMPLETE" FORMULATION, THE UNKNOWNS ARE $(15, u_i, \sigma_{ij}, \varepsilon_{ij})$

→ (HU - WASHIZU)

DISPLACEMENT FORMULATION → CLASSICAL FORMULATION FOR GENERIC PROBLEM

MIXED FORMULATION → GOOD FOR INCOMPRESSIBLE MATERIAL, CONTACT PROBLEM

FROM MATHEMATICAL TO COMPUTATIONAL MODEL

TARGET FEM → FIND "GLOBAL" DESCRIPTION OF THE PROBLEM

PDE (LOCAL) $\Longleftrightarrow$ INTEGRAL FORMULATION (GLOBAL)

3 METHODS TO FIND INTEGRAL FORMULATION

1) WEIGHTED RESIDUAL
2) VARIATIONAL PRINCIPLES
3) PRINCIPLE of VIRTUAL WORK

1) WEIGHTED RESIDUAL METHOD

– GALERKIN METHOD –

MOST GENERAL METHOD, BASED ON CALCULUS OF VARIATIONS

PASS FROM A SET OF PDEs (STRONG FORM)
TO INTEGRAL FORMULATION (WEAK FORM)

LET'S SEE A GENERAL EXAMPLE:

$u(x) : \underset{(\Omega)}{[a,b]} \to \mathbb{R}$

PDE: $\begin{cases} u''(x) + f(x) = 0 \text{ in } \Omega \\ u(a) = u_a \\ u(b) = u_b \end{cases}$] C.C.

STRONG FORMULATION OF THE DIFFERENTIAL PROBLEM

LET'S FIND A "MEAN" FORMULATION, VALID ON THE ENTIRE DOMAIN Ω

$$\int_a^b \left[u''(x) + f(x) \right] \cdot v(x) = 0 \quad , \forall v \in H_0^1(\Omega)$$

WEAK FORMULATION OF THE DIFFERENTIAL PROBLEM

$\equiv$ NEED TO ADD THIS TERM, FUNDAMENTAL FOR THE FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS

v: TEST FUNCTION

$v(x)$ MUST BE "GOOD" !!! $\longrightarrow v(x) \in H_0^1(\Omega)$ $\quad H(\Omega)$ IS A SOBOLEV SPACE!!

$$H_0^m(\Omega) = \{ v : \Omega \to \mathbb{R}, v \in L_2(\Omega), v' \in L_2(\Omega), \dots v^{(m)} \in L_2, v(\partial\Omega) = 0 \}$$

$$L_2 = \left\{ g : \int_\Omega |g|^2 d\Omega < +\infty \right\}$$

ACCORDING TO THE FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS THE TWO FORMULATIONS ARE EQUIVALENT

$$u''(x) + f(x) = 0 \quad + \text{B.C.} \quad \Longleftrightarrow \quad \int_a^b (u'' + f) v \, dx = 0, \ \forall v \in H_0^1$$

– REDUCTION OF THE ORDER OF DERIVATIVES

$$\int_a^b \underbrace{u''}_{\text{2ND DERIVATIVE}} \underbrace{v}_{\text{0-TH DERIVATIVE}} + f v =$$

INTEGRATION BY PARTS

$$\int_a^b u'' v = \underbrace{\int_a^b (u'v)'}_{\text{Th. of GAUSS } [u'v]_a^b} - \int_a^b u'v'$$

$$\int_a^b u''v + fv = \underbrace{-\int_a^b u'v' dx}_{\substack{\text{BILINEAR FORM} \\ a[u,v]}} + \underbrace{[u'v]_a^b + \int_a^b fv}_{\substack{\text{LINEAR FORM} \\ \ell[v]}} = 0$$

$$\boxed{a[u,v] = \ell[v]}$$

$a[u,v]$ IS BILINEAR

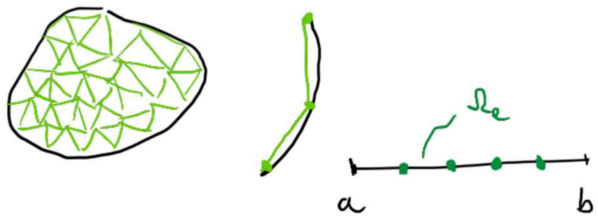
$a[\alpha u, \beta v] = \alpha a[u, \beta v] = \beta a[\alpha u, v]$

SYMMETRIC

$a[u,v] = a[v,u]$

- DISCRETISATION

$$\Omega \approx \sum_{e=1}^{m_e} \Omega_e$$



$$\int_a^b v\left[u'' + f\right]dx \approx \sum_{e=1}^{m_e} \int_{\Omega_e} v\left[u''+f\right]dx$$

STRONG FORM $\longrightarrow$ WEAK FORM $\longrightarrow$ DISCRETE FORM

DISCRETISED FORM

TEST FUNCTION IS HOMOGENEOUS AT THE BOUNDARY OF THE FE

$$\sum_{e=1}^{m_e} \left\{ -\int_0^{l_e} u_e'(x)\, v_e'(x) + \cancel{\left[u' v\right]_0^{l_e}} + \int_0^{l_e} f_e v_e \right\} = 0$$

ANALYTICAL APPROXIMATION

UNKNOWN FIELD $\quad u_e(x) = N_j(x)\, U_j^{(e)}$

TEST FUNCTION $\quad v_e(x) = H_i(x)\, V_i^{(e)}$

THE SAME SHAPE FUNCTIONS... $\boxed{H_i(x) = N_i(x)}$ GALERKIN METHOD

LOOK AT THE SINGLE FINITE ELEMENT :

$$\int_0^{l_e} u_e' v_e' - \int_0^{l_e} f_e v_e = 0, \ \forall v_e \longrightarrow \int_0^{l_e} N_i' N_j' U_j V_i - \int_0^{l_e} f N_i(x) V_i = 0, \ \forall \underline{\underline{V_i}}$$

$$V_i \left\{ \underbrace{\left(\int_0^{l_e} N_i'(x) N_j'(x)\, dx \right)}_{K_{ij}} U_j - \underbrace{\int_0^{l_e} f(x) N_i(x)\, dx}_{F_i} \right\} = 0 \quad \forall \underline{\underline{V_i}}$$

CAN BE DELETED

K_{ij} STIFFNESS MATRIX

F_i VECTOR OF GENERALISED NODAL FORCES

$$\boxed{\begin{aligned} K_{ij}^e U_j^e &= F_i^e \\ \underline{\underline{K}}^e \underline{U}^e &= \underline{F}^e \end{aligned}}$$

DISCRETISED FORM ON EACH FINITE ELEMENT

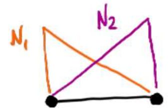
$$\underline{\underline{K}}^e = \begin{bmatrix} \int N_1'N_1' & \int N_1'N_2' & \cdots \\ \int N_2'N_1' & \int N_2'N_2' & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \qquad \underline{F}^e = \begin{Bmatrix} \int fN_1 \\ \int fN_2 \\ \vdots \\ \int fN_m \end{Bmatrix}$$

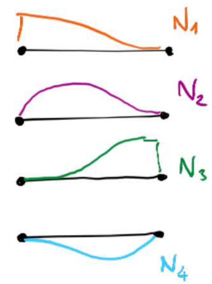
$\underline{\underline{K}} \in Sym \rightarrow$ IT COMES FROM THE BILINEAR SIMMETRIC FORM

PROPERTIES of THE SHAPE FUNCTIONS

" $N_i(x_j) = \delta_{ij}$ "

VALUE IS 1 ON THE CONSIDERED DOF., 0 OTHERWISE





N.B. : DIFFERENT KIND of APPROXIMATIONS :

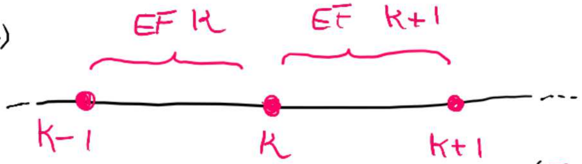
$u(x) = \varphi_j(x)\, \alpha_j$ — ONLY ALGEBRAIC MEANING — COEFF. di RITZ-GALERKIN

$u(x) = N_j(x)\, U_j$ — MECHANICAL MEANING, MOST USEFUL, U_j IS A DISPLACEMENT — DEGREES OF FREEDOM (dof)

ASSEMBLY

THE WEAK FORM IS FOR EACH SINGLE FE

$$K_{ij}^{(e)} U_j^{(e)} = F_i^{(e)}$$



$\underline{e=k} \rightarrow \underline{\underline{K}}^{(k)} \underline{U}^{(k)} = \underline{F}^{(k)}, \quad \underline{U}^{(k)} = \begin{Bmatrix} U_{k-1} \\ U_k \end{Bmatrix}, \quad \underline{F}^{k} = \begin{Bmatrix} F_{k-1} \\ F_k \end{Bmatrix}$

$\underline{e=k+1} \rightarrow \underline{\underline{K}}^{(k+1)} \underline{U}^{(k+1)} = \underline{F}^{(k+1)}, \quad \underline{U}^{(k+1)} = \begin{Bmatrix} U_k \\ U_{k+1} \end{Bmatrix}, \quad \underline{F}^{(k+1)} = \begin{Bmatrix} F_k \\ F_{k+1} \end{Bmatrix}$

SOME DOFS BELONG TO MORE THAN ONE FE
THE COMMON DOFS CAN BE CONSIDERED ONCE

↳ THIS OPERATION IS CALLED ASSEMBLY

VECTOR OF GLOBAL DOF $\quad \underline{U} = \{ U_1, U_2, \ldots\ldots, U_N \}^T \quad$ (N NODES)

$$\underline{\underline{K}} = \begin{bmatrix} K^{(1)} & & & 0 \\ & K^{(2)} & & \\ & & K^{(3)} & \\ 0 & & & \ddots \end{bmatrix}, \quad \underline{F} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{Bmatrix}$$

GLOBAL STIFFNESS MATRIX

FINAL PROBLEM

$$\boxed{\underline{\underline{K}}\,\underline{U} = \underline{F}}$$

WHAT ABOUT BOUNDARY CONDITIONS ?

THE DETERMINANT OF THE COMPLETE STIFFNESS MATRIX IS NULL $\quad \det(\underline{\underline{K}}) = 0 \rightarrow$ THE SOLUTION IS NOT UNIQUE!! THERE ARE RIGID MOTIONS

"DELETE" THE DOFS RELATIVE TO RIGID MOTION THROUGH BOUNDARY CONDITION

- ESSENTIAL BC : KNOWN DISPLACEMENTS u_U — UNKNOWN FORCES (REACTIONS) F_U
- NATURAL BC : UNKNOWN DISPLACEMENTS u_F — KNOWN FORCES F_F

$\underline{U}$ e $\underline{F}$ CAN BE PARTITIONED

$$\underline{U} = \{ \underline{u}_U^T, \underline{u}_F^T \}^T, \quad \underline{F} = \{ F_U^T, F_F^T \}^T \quad \text{AND ALSO } \underline{\underline{K}} \ldots$$

$$\begin{bmatrix} \underline{\underline{K}}_{UU} & \underline{\underline{K}}_{UF} \\ \underline{\underline{K}}_{FU} & \underline{\underline{K}}_{FF} \end{bmatrix} \begin{Bmatrix} U_U \\ U_F \end{Bmatrix} = \begin{Bmatrix} F_U \\ F_F \end{Bmatrix} \rightarrow \begin{aligned} \underline{F}_U &= \underline{\underline{K}}_{UU} \underline{U}_U + \underline{\underline{K}}_{UF} \overbrace{\underline{U}_F}^{\text{UNKNOWNS}} \\ \underline{F}_F &= \underline{\underline{K}}_{FU} \underline{U}_U + \underline{\underline{K}}_{FF} \underbrace{\underline{U}_F}_{\text{UNKNOWNS}} \end{aligned}$$

$$\boxed{\underline{\underline{K}}_{FF} \underbrace{\underline{U}_F}_{\text{UNKNOWNS}} = \underline{F}_F - \underbrace{\underline{\underline{K}}_{FU} U_U}_{\text{FORCE DUE TO FIXED DISPLACEMENT}}}$$

THIS IS THE FINAL PROBLEM !! → U_F IS THE UNKNOWN

WHEN U_F IS KNOWN, WE CAN COMPUTE THE REACTIONS A POSTERIORI

$$\rightarrow \boxed{\underline{F}_U = \underline{\underline{K}}_{UU} \underline{U}_U + \underline{\underline{K}}_{UF} U_F}$$

2) VARIATIONAL METHODS

WE CAN OBTAIN AN INTEGRAL FORMULATION (WEAK FORM) THROUGH THE STATIONARY CONDITION OF A FUNCTIONAL

FOR THE SPECIFIC PHYSICAL PROBLEM, WE CAN DEFINE SPECIAL FUNCTIONALS THAT CAN BE MINIMISED TO HAVE

EXAMPLE: THE MINIMUM OF THE TOTAL POTENTIAL ENERGY FOR MECHANICAL SYSTEMS

CALCULUS of VARIATIONS

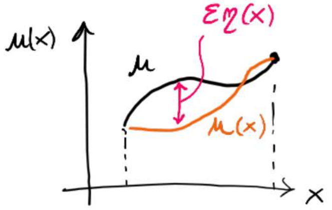
$u : [a,b] \to \mathbb{R}$

$$I[u] = \int_a^b F(x, u, u', u'', \dots)\,dx$$

SCALAR

FUNCTIONAL

"FUNCTION of FUNCTIONS"



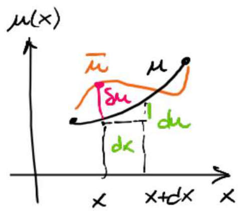
$$\bar{u}(x) = u(x) + \varepsilon\,\eta(x)$$

VARIED FUNCTION $\quad (\varepsilon \in \mathbb{R})$

ADMISSIBLE VARIATION

$$\delta u(x) = \bar{u}(x) - u(x) = \varepsilon\,\eta(x)$$

IT MUST FULFILL THE BC OF THE "BASE" FUNCTION



DERIVATIVE OF A FUNCTIONAL

$$I'[u]\eta = \lim_{\varepsilon \to 0} \frac{I[\bar{u}] - I[u]}{\varepsilon}$$

($\eta(x)$ IS A SORT OF DIRECTION) DIRECTIONAL DERIVATIVE

$$\Delta I = I[\bar{u}] - I[u] \qquad \Delta I = \delta I[u] + O(\varepsilon^2) \qquad \Delta F = \underbrace{\varepsilon\left(\frac{\partial F}{\partial u}\eta + \frac{\partial F}{\partial u'}\eta'\right)}_{\delta F} + O(\varepsilon^2)$$

$$\delta I[u] = \int_a^b \delta F = \int_a^b \left(\frac{\partial F}{\partial u}\varepsilon\eta + \frac{\partial F}{\partial u'}\varepsilon\eta'\right)dx$$

EULER-LAGRANGE EQUATIONS

STATIONARY FUNCTIONAL		CAUCHY'S PROBLEM
$\delta I = 0, \ \forall \eta$	$\Longleftrightarrow$	$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0, \quad \frac{\partial F}{\partial u'}\varepsilon\eta\Big\vert_a^b = 0$

A DIFFERENTIAL PROBLEM IN STRONG FORM CAN BE REWRITTEN AS A PROBLEM OF MINIMUM OF A GIVEN FUNCTIONAL

WE CAN INTRODUCE AN APPROXIMATION:

$$u(x) = N_j(x)\,U_j \;\rightarrow\; I[u] \approx \tilde{I}[U_j]$$

(FUNCTION!!)

$$\delta I \approx \delta\tilde{I} = \frac{\partial\tilde{I}}{\partial U_1}\delta U_1 + \frac{\partial\tilde{I}}{\partial U_2}\delta U_2 + \dots + \frac{\partial\tilde{I}}{\partial U_m}\delta U_m = 0,$$

ALONG EACH DIRECTION $\longrightarrow \forall\,\delta\underline{U}$

$$\delta\tilde{I} = \left\{\begin{matrix} \frac{\partial\tilde{I}}{\partial U_1} \\ \frac{\partial\tilde{I}}{\partial U_2} \\ \vdots \\ \frac{\partial\tilde{I}}{\partial U_m} \end{matrix}\right\} \cdot \left\{\begin{matrix} \delta U_1 \\ \delta U_2 \\ \vdots \\ \delta U_m \end{matrix}\right\} = 0, \ \forall\,\delta\underline{U}$$

MUST VANISH

ARBITRARY

$$\boxed{\delta I = 0 \;\Updownarrow\; \frac{\partial\tilde{I}}{\partial U_i} = 0}$$

MINIMUM of the TOTAL POTENTIAL ENERGY $\longrightarrow$ FINITE ELEMENTS <u>DISPLACEMENT</u> FORMULATION

$$\Pi[\underline{u}] = \frac{1}{8}\underbrace{\int_\Omega (u_{i,j}+u_{j,i})\,\mathbb{C}_{ijkh}\,(u_{k,h}+u_{h,k})}_{\text{ELASTIC POTENTIAL ENERGY}} \underbrace{- \int_\Omega f_i u_i - \int_{\partial\Omega} p_i u_i}_{\text{POTENTIAL of EXTERNAL LOADS (- WORK)}}$$

$$= \frac{1}{2}\int_\Omega \varepsilon_{ij}\,\mathbb{C}_{ijkh}\,\varepsilon_{kh} - \int_\Omega f_i u_i - \int_{\partial\Omega} p_i u_i \quad \left(\text{with } \varepsilon_{ij} = \varepsilon_{ij}(\underline{u})\right)$$

GEOMETRIC APPROXIMATION

$$\Pi[u] \approx \sum_e \Pi^e[u] \longrightarrow$$

ADDITIVITY IS VERY USEFUL!!
WE CAN STUDY THE ENERGY OF THE SINGLE FE AND THEN SUM UP THE CONTRIBUTION...

ANALYTICAL APPROXIMATION

$\forall e$, $u_i^e(\underline{x}) = N_{ip} U_p^e \rightarrow \varepsilon_{ij}^e = B_{ijp} U_p^e$ STRAIN-DISPLACEMENT TENSOR

THEN

$$\tilde{\Pi}_e[U] = \frac{1}{2}\int_{\Omega_e} B_{ijp} U_p^e\, \mathbb{C}_{ijkh}\, B_{khq} U_q^e - \int_{\Omega_e} f_i N_{ip} U_p^e - \int_{\partial\Omega_e} p_i N_{ip} U_p^e =$$

$$= \frac{1}{2} U_p^e \underbrace{\left(\int_{\Omega_e} B_{ijp}\, \mathbb{C}_{ijkh}\, B_{khq}\right)}_{\underline{\underline{K}}^e_{pq}} U_q^e - U_p^e \underbrace{\left(\int_{\Omega_e} f_i N_{ip} + \int_{\partial\Omega_e} p_i N_{ip}\right)}_{F_p^e}$$

$$\tilde{\Pi}^e[\underline{U}] = \frac{1}{2}\underline{U}^e \cdot \underline{\underline{K}}^e \underline{U}^e - \underline{U}^e \cdot \underline{F}^e = \frac{1}{2}\underline{U}^{e\,T} \underline{\underline{K}}^e \underline{U}^e - \underline{U}^{e\,T} \underline{F}^e$$

$$\frac{\partial \tilde{\Pi}^e}{\partial \underline{U}^e} = \underline{\underline{K}}^e \underline{U}^e - \underline{F}^e = 0 \longrightarrow \boxed{\underline{\underline{K}}^e \underline{U}^e = \underline{F}^e}$$

– CONTINUE AS IN CASE OF WEIGHTED RESIDUAL –

[. . . . .]

3) PRINCIPLE of VIRTUAL WORK

IT IS A WEAK FORM of THE PROBLEM, BASED on A PHYSICAL PRINCIPLE !!
(MIX of PREVIOUS METHODS)

WEAK FORM $\longleftrightarrow$ PRINCIPLE of VIRTUAL WORK

WEAK FORM $\longleftrightarrow$ MINIMUM of FUNCTIONAL

MINIMUM of FUNCTIONAL $\longleftrightarrow$ PRINCIPLE of VIRTUAL WORK

PVW IS A WEAK FORMULATION

$$\begin{cases} \operatorname{div} \underline{\underline{\sigma}} + \underline{f} = 0 & \text{in } \Omega \\ \underline{\underline{\sigma}}\,\underline{n} = 0 & \text{in } \partial\Omega \end{cases} \rightarrow$$

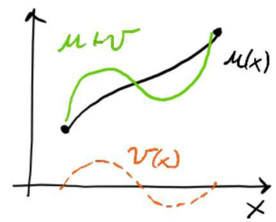
EQUILIBRIUM EQUATIONS of A SOLID (STRONG FORM)

LET'S ASSUME

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}[\underline{u}]$

$\underline{u}$ ADMISSIBLE

$$\int_\Omega \underline{v} \cdot \left[\operatorname{div}(\underline{\underline{\sigma}}) + \underline{f}\right] = 0, \quad \forall \underline{v} \text{ ADMISSIBLE}$$



LET'S ASSUME THE TEST FUNCTION $v = \delta u$

THE TEST FUNCTION CAN BE INTERPRETED AS A VIRTUAL DISPLACEMENT

INDEX NOTATION

$$\int_\Omega \delta u_i \left[\frac{\partial \sigma_{ij}}{\partial x_j} + f_i \right] = 0 \,, \quad \forall \delta u_i$$

TEST FUNCTION v CAN BE INTERPRETED AS A DISPLACEMENT

$(\underline{\underline{\sigma}} \in Sym)$

INTEGRATION BY PARTS

$$\int_\Omega \delta u_i \frac{\partial \sigma_{ij}}{\partial x_j} = \underbrace{\int_\Omega \frac{\partial}{\partial x_j}\left(\sigma_{ij}\, \delta u_i\right)}_{1} - \underbrace{\int_\Omega \sigma_{ij} \frac{\partial}{\partial x_j}(\delta u_i)}_{2}$$

1 Th Gauss: $\int_\Omega div(\underline{v}) = \int_{\partial\Omega} \underline{v} \cdot \underline{n}$

$$\int_\Omega \frac{\partial}{\partial x_j}\left(\sigma_{ij}\, \delta u_i\right) = \int_{\partial\Omega} \sigma_{ij}\, \delta u_i\, n_j = \int_{\partial\Omega} \delta u_i\, \sigma_{ij} n_j =$$

$$= \int_{\partial\Omega} \delta \underline{u} \cdot \underline{t}$$

2 $$\underbrace{\sigma_{ij} \frac{\partial}{\partial x_j}(\delta u_i)}_{2^{nd} \text{ ORDER TENSOR}} \overset{\underline{\underline{\sigma}} \in Sym}{=} \sigma_{ij} \left[\underbrace{\frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right)}_{\delta \varepsilon_{ij}} \right]$$

VIRTUAL STRAIN

HENCE THE WEAK FORM OF THE EQUILIBRIUM

$$\underbrace{-\int_\Omega \sigma_{ij}\, \delta \varepsilon_{ij}}_{-\delta L_{VI}} + \underbrace{\int_\Omega f_i\, \delta u_i + \int_{\partial\Omega} t_i\, \delta u_i}_{\delta L_{VE}} = 0 \,, \quad \forall \delta \underline{u}$$

$\forall$ DISPLACEMENT FULLFILLING THE BOUNDARY CONDITION

NOT LINKED TO $\underline{\underline{\sigma}} \rightarrow$ VIRTUAL DISPL.

$\boxed{\delta L_{VE} = \delta L_{VI}}$ PLV !!

WEAK FORM AND VARIATIONAL PRINCIPLES

WEAK FORM $\quad a[u, v] = l[v] \rightarrow$ INTERPRETATION OF v (TEST FUN.) AS AN ADMISSIBLE VARIATION OF u $\rightarrow$ $v = \delta u$

$$a[u, \delta u] - l[\delta u] = 0, \xrightarrow{\forall \delta u}$$

FOR a SYMMETRIC

$$a[u, \delta u] = \frac{1}{2}\left(a[u, \delta u] + a[u, \delta u] \right)$$

$$= \frac{1}{2}\left(a[u, \delta u] + a[\delta u, u] \right) = \frac{1}{2}\, \delta a[u, u]$$

(BILINEAR FORM)

l IS A LINEAR FORM

$$l[\delta u] = \delta l[u]$$

$\Downarrow$

$$\frac{1}{2}\, \delta a[u, u] - \delta l[u] = \delta \Big\{ \underbrace{\frac{1}{2} a[u, u] - l[u]}_{\text{IT'S A FUNCTIONAL! } I[u]} \Big\} = 0 \quad \forall \delta u$$

$$\boxed{\delta I[u] = 0, \ \forall \delta u}$$