7 aprile

$$\begin{cases} i\partial_t u = -\Delta u + \lambda |u|^{p-1}u \\ u_{|t=0} = u_0 \in H^1(\mathbb{R}^d, \cancel{\mathbb{C}}) \end{cases}$$

$$\lambda = 1, -1$$

$$1 < p < d^* = \begin{cases} +\infty & \text{se } d = 1, 2 \\ \frac{d+2}{d-2} \end{cases}$$

$$\partial_t u - i\Delta u = -i\lambda |u|^{p-1}u$$

$$\partial_t\left(e^{-it\Delta}u\right) = -i\lambda\, e^{-it\Delta}|u|^{p-1}u$$

$$u(t) = e^{it\Delta}u_0 - i\lambda\int_0^t e^{i(t-s)\Delta}|u|^{p-1}u\,ds$$

Lemm

1) $1 < p < d^*$

$$|u|_{L^{p+1}(\mathbb{R}^d)} \leq C_p |\nabla u|^{\alpha}_{L^2(\mathbb{R}^d)} |u|^{1-\alpha}_{L^2(\mathbb{R}^d)}$$

$$\frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}$$

2) $u \to |u|^{p-1}u$ è loc Lipch

$$H^1(\mathbb{R}^d) \to H^{-1}(\mathbb{R}^d)$$

$$f \in \mathcal{S}'(\mathbb{R}^d)$$

$$\langle \xi \rangle^{-1} \hat{f} \in L^2(\mathbb{R}^d)$$

3) $u \in W^{1,p+1}(\mathbb{R}^d)$

$$\nabla\left(|u|^{p-1}u\right) \in L^{\frac{p+1}{p}}(\mathbb{R}^d)$$

$$= p|u|^{p-1}\nabla u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2 \nabla \bar{u}$$

Dim

(2) $u, v \in \mathbb{C}$ $\qquad w = u - v$

$$|u|^{p-1}u - |v|^{p-1}v =$$

$$= \int_0^1 \frac{d}{dt} |v+tw|^{p-1}(v+tw)\, dt$$

$$= \int_0^1 |v+tw|^{p-1} w \, dt$$

$$+ \int_0^1 (v+tw) \frac{d}{dt} \left((v_1+tw_1)^2 + (v_2+tw_2)^2 \right)^{\frac{p-1}{2}} dt$$

$$= \int_0^1 |v+tw|^{p-1} w \, dt$$

$$+ \frac{p-1}{\cancel{2}} \int_0^1 (v+tw)\, |v+tw|^{p-3}\, \cancel{2} \,\cdot$$

$$\left((v_1+tw_1)w_1 + (v_2+tw_2)w_2 \right) dt$$

$$\left| (v+tw)\, |v+tw|^{p-3} (v_j+tw_j) w_j \right| \leq$$

$$\leq |v+tw|^{p-1} \, |w|$$

$w = u - v$

$$\left| |u|^{p-1}u - |v|^{p-1}v \right| \le \int_0^1 |v+tw|^{p-1} |w|\,dt$$

$$(2(p-1)+1)$$

$$\le C_p \int_0^1 |tu+(1-t)v|^{p-1} |w|\,dt$$

$$\le C_p \int_0^1 (|u|+|v|)^{p-1} |w|\,dt$$

$$\le C_p (|u|+|v|)^{p-1} |w|\,dt$$

$$\le C_p\, 2^{p-1}(|u|^{p-1}+|v|^{p-1}) |w|$$

$$|u| \le |v|$$

$$(|u|+|v|)^{p-1} \le (2|v|)^{p-1} = 2^{p-1}|v|^{p-1}$$

$$\le 2^{p-1}(|u|^{p-1}+|v|^{p-1})$$

$$\left| |u|^{p-1}u - |v|^{p-1}v \right| \le C(|u|^{p-1}+|v|^{p-1})\,|u-v|$$

$u, v \in L^{p+1}(\mathbb{R}^d)$

$$\| |u|^{p-1}u - |v|^{p-1}v \|_{L^{\frac{p+1}{p}}(\mathbb{R}^d)}$$

$$\leq C \| (|u|+|v|)^{p-1} (u-v) \|_{L^{\frac{p+1}{p}}(\mathbb{R}^d)}$$

$$\frac{p}{p+1} = \frac{1}{p+1} + \frac{p-1}{p+1}$$

$$\leq C \| u-v \|_{L^{p+1}(\mathbb{R}^d)} \| (|u|+|v|)^{p-1} \|_{L^{\frac{p+1}{p-1}}}$$

$$\leq C \| |u|+|v| \|^{p-1}_{L^{p+1}(\mathbb{R}^d)} \| u-v \|_{L^{p+1}(\mathbb{R}^d)}$$

$$\leq 2^{p-1} C \left(\|u\|^{p-1}_{L^{p+1}} + \|v\|^{p-1}_{L^{p+1}} \right) \|u-v\|_{L^{p+1}(\mathbb{R}^d)}$$

$*$ $u \longrightarrow |u|^{p-1}u$

$L^{p+1}(\mathbb{R}^d) \longrightarrow L^{\frac{p+1}{p}}(\mathbb{R}^d)$

è loc. Lipch.

$u \in H^1(\mathbb{R}^d) \longrightarrow H^{-1}(\mathbb{R}^d)$

$$u \in L^{p+1}(\mathbb{R}^d) \longrightarrow L^{\frac{p+1}{p}}(\mathbb{R}^d)$$

$$u \longrightarrow |u|^{p-1}u$$

$$L^{\frac{p+1}{p}}(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d)$$

$$\begin{tikzcd}
H^1(\mathbb{R}^d) \arrow[r, hook] \arrow[dr] & L^{p+1}(\mathbb{R}^d) \\
& L^\alpha(\mathbb{R}^d)
\end{tikzcd}$$

$$d^* < +\infty$$

$$2 \le \alpha \le d^*$$

$$\nabla\left(|u|^{p-1}u\right) \in L^{\frac{p+1}{p}}(\mathbb{R}^d)$$

$$\cancel{u \in H^1 f}$$

$$u \in W^{1,p+1}(\mathbb{R}^d)$$

$$G \in C^1(\mathbb{C},\mathbb{C})$$

$$G(0)=0, \qquad |\nabla G| \le M < +\infty$$

$$\nabla(G(u)) = \partial_u G(u) \nabla u + \partial_{\bar u} G(u) \nabla \bar u$$ è vero nel senso delle distribuzioni per $u \in W^{1,p+1}(\mathbb{R}^d)$

$z = x + iy$

$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$

$\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$

$u_n \in C_c^\infty(\mathbb{R}^d)$ $\qquad u_n \longrightarrow u$

$$\nabla(G(u_n)) = \partial_u G(u_n) \nabla u_n + \partial_{\bar u} G(u_n) \nabla \bar u_n$$

$$\downarrow \qquad\qquad \downarrow$$

$$\nabla(G(u)) = \partial_u G(u) \nabla u + \partial_u G(u) \nabla \bar u$$

$$G(u_n) \longrightarrow G(u) \text{ in } \mathcal{D}'(\mathbb{R}^d)$$

$$|G(u)-G(u_n)| \le \sup_{z\in\mathbb{C}} |\nabla G(z)|\, |u-u_n|$$

$$\le M\, |u-u_n|$$

$$u_n \to u \quad \text{in} \quad W^{1,p+1}(\mathbb{R}^d)$$

$$\Rightarrow \quad |G(u)-G(u_n)| \xrightarrow{n\to+\infty} 0 \quad \text{in} \quad L^{p+1}(\mathbb{R}^d)$$

$$\nabla G(u_n) \longrightarrow \nabla G(u) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d)$$

$$|u|^{p-1}u \qquad\qquad g(|u|^2)\, u$$

$$g(s) = \begin{cases} s^{\frac{p-1}{2}} & 0 \le s \le 1 \\ 2^{\frac{p-1}{2}} & s \ge 2 \end{cases}$$

$$g \in C^\infty(\mathbb{R}_+, \mathbb{R})$$

$$G_m(u) = m^{p-1}\, g\left(\frac{|u|^2}{m^2}\right) u \qquad m \in \mathbb{N}.$$

$$G_m(u) = |u|^{p-1}u \quad \text{se} \quad |u| \le m$$

$$\nabla G_m(u) = \partial_u G_m(u)\, \nabla u + \partial_{\bar{u}} G_m(u)\, \nabla \bar{u}$$

$$\downarrow$$

$$\nabla G(u) = \partial_u G(u)\, \nabla u + \partial_{\bar{u}} G(u)\, \nabla \bar{u}$$

$$G = |u|^{p-1} u$$

$$G_m(u) \longrightarrow G(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^d)$$

$$\int G_m(u)\, \varphi \, dx =$$

$$= \int_{|u| \le m} G_m(u)\, \varphi \, dx + \int_{|u|>m} G_m(u)\, \varphi \, dx$$

$$= \int_{\mathbb{R}^d} |u|^{p-1} u\, \varphi \, dx - \int_{|u| \ge m} |u|^{p-1} u\, \varphi \, dx$$

$$+ \int_{|u| \ge m} G_m(u)\, \varphi \, dx \xrightarrow{m \to +\infty}$$

$$\to \int_{\mathbb{R}^d} |u|^{p-1} u\, \varphi \, dx$$

$$\left| \int_{|u| \ge m} |u|^{p-1} u\, \varphi \, dx \right| \le$$

$$\leq \left| \int |u|^{p-1} u \; 1_{\underbrace{\{|u| \geq m\}}_{X_m}} \varphi \right|$$

$$\leq_{\Omega} \qquad 1 = \frac{p}{p+1} + \frac{1}{p+1}$$

$$\leq \quad | u^p |_{L^{\frac{p+1}{p}}} \quad | 1_{X_m} \varphi |_{L^{p+1}}$$

$$\leq |u|^p_{L^{p+1}} \quad |\varphi|_{L^\infty} \quad |X_m|^{\frac{1}{p+1}}$$

$$\downarrow$$

$$0$$

$$X_m = \{ x : |u(x)|^{p+1} \geq m^{p+1} \}$$

$$|X_m| \leq \frac{1}{m^{p+1}} \quad |u|^{p+1}_{L^{p+1}}$$

$$\partial_u G_m(u) \partial_j u \longrightarrow \partial_u G(u)\, u$$

$$G = |u|^{p-1} u$$

$$\int \qquad \nabla \left(|u|^{p-1} u \right) = \partial_u \left(|u|^{p-1} u \right) \nabla u$$
$$+ \partial_{\bar{u}} \left(|u|^{p-1} u \right) \nabla \bar{u}$$

$$|u|^2 = u\bar{u}$$

$$\nabla\left(|u|^{p-1}u\right) = \nabla\left(\left(|u|^2\right)^{\frac{p-1}{2}} u\right) =$$

$$= \nabla\left(u^{\frac{p-1}{2}} \bar{u}^{\frac{p-1}{2}} u\right) =$$

$$= \nabla\left(u^{\frac{p+1}{2}} \bar{u}^{\frac{p-1}{2}}\right)$$

$$= \frac{p+1}{2} u^{\frac{p-1}{2}} \bar{u}^{\frac{p-1}{2}} \nabla u$$

$$+ \frac{p-1}{2} u^{\frac{p+1}{2}} \bar{u}^{\frac{p-3}{2}} \nabla \bar{u}$$

$$= \frac{p+1}{2} |u|^{p-1} \nabla u$$

$$+ \frac{p-1}{2} |u|^{p-3} u^2 \nabla \bar{u}$$

$$= \frac{p+1}{2} |u|^{p-1} \nabla u + \frac{p-1}{2} |u|^{p-1} \left(\frac{u}{|u|}\right)^2 \nabla \bar{u}$$

$$\begin{cases} i\partial_t u = -\Delta u + \lambda |u|^{p-1} u \\ u|_{t=0} = u_0 \end{cases} \qquad \dot{u} = i\Delta u - i\lambda |u|^{p-1} u$$

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx$$

$$Q(u) = \frac{1}{2}\int_{\mathbb{R}^d} |u|^2 dx = \frac{1}{2}\langle u, u\rangle$$

$$P_j(u) = \frac{1}{2} \operatorname{Im} \int_{\mathbb{R}^d} \partial_j u \ \bar{u} \ dx$$

$$j = 1, \dots, d$$

$$\frac{d}{dt} Q(u(t)) = \frac{1}{2}\frac{d}{dt}\langle u(t), u(t)\rangle =$$

$$= \langle \dot{u}(t), u(t)\rangle$$

$$= \langle i\Delta u - i\lambda |u|^{p-1} u, u\rangle =$$

$$= \langle i\Delta u, u\rangle - \lambda \langle i |u|^{p-1} u, u\rangle$$

$$= -\langle i\, \partial_j u, \partial_j u\rangle - \lambda \langle i |u|^{p-1} u, u\rangle$$

$$= 0 \qquad \langle f, g\rangle = \operatorname{Re}\int_{\mathbb{R}^d} f \ \bar{g}$$

$$Q(u) = \frac{1}{2} |u|_{L^2}^2$$

$$P_j, Q \in C^\infty(H^1(\mathbb{R}^d), \mathbb{R})$$

$$E \in C^1(H^1(\mathbb{R}^d), \mathbb{R})$$

$$E \in C^0(H^1(\mathbb{R}^d), \mathbb{R})$$

$$E = \underbrace{\frac{1}{2} |\nabla u|_{L^2}^2}_{E_K} + \underbrace{\frac{\lambda}{p+1} \int |u|^{p+1}\, dx}_{E_p}$$

$$E_K \in C^\infty(H^1(\mathbb{R}^d), \mathbb{R})$$

$$E_p \in C^0(H^1(\mathbb{R}^d), \mathbb{R})$$

$$\begin{tikzcd}
u \in H^1 \arrow[r] \arrow[d, hook] & |u|_{L^{p+1}}^{p+1} \\
L^{p+1} \arrow[r, "|\cdot|_{L^{p+1}}"'] & \mathbb{R}
\end{tikzcd}$$

$$dE_p \in C^0(H^1(\mathbb{R}^d), \underbrace{\mathcal{L}(H^1(\mathbb{R}^d), \mathbb{R})}_{H^{-1}(\mathbb{R}^d)})$$

$E \in C^1(H^1(\mathbb{R}^d), \mathbb{R})$

Ω su $H^1(\mathbb{R}^d)$

$\Omega(f, g) = \langle if, g \rangle$

E è la Hamiltoniana

$\dot{u} = +i\Delta u - i\lambda |u|^{p-1} u = X_E$

$S^1 \times \mathbb{R}^d \times H^1(\mathbb{R}^d, \mathbb{C}) \longrightarrow H^1(\mathbb{R}^d, \mathbb{C})$

$(e^{i\vartheta}, x_0, u) \longrightarrow e^{i\vartheta} u(\cdot - x_0)$

$E(e^{i\vartheta} u(\cdot - x_0)) = E(u)$