10 Aprile

$$u(t) = e^{it\Delta} - i\lambda \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\,ds$$

Prop (buona positura locale in $L^2(\mathbb{R}^d)$)

Per $1 < p < 1 + \frac{4}{d}$ e per $u_0 \in L^2(\mathbb{R}^d)$

$\exists\ T > 0$ ed una unica soluzione con

$u \in C^0([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T], L^{p+1}(\mathbb{R}^d))$

dove $\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$ ($(q, p+1)$ è una coppia ammissibile per Strichartz)

$\exists$ una $T(\cdot) : [0,+\infty) \to (0,+\infty]$ decrescente

t.c. $T \geq T(|u_0|_{L^2(\mathbb{R}^d)})$

Inoltre $\forall\ T' \in (0,T)$ $\exists$ intorno V di u_0 in $L^2(\mathbb{R}^d)$ t.c.

$v_0 \in V \longrightarrow v(t)$

$u_0 \longrightarrow T_{u_0}$

spedisce

$$V \longrightarrow C^0([-T',T'], L^2(\mathbb{R}^d)) \cap L^q([-T',T'], L^{p+1}(\mathbb{R}^d))$$

ed è Lipschitziana.

Infine $u \in L^a([-T,T], L^b(\mathbb{R}^d))$

$\forall$ coppia ammissibile (a,b).

Dim

Sia $a > 0$ $\overline{\qquad T \qquad}$

$$E(T,a) = \{ v \in C^0([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T], L^{p+1}(\mathbb{R}^d))$$

$$|v|_T = |v|_{L^\infty([-T,T];L^2)} + |v|_{L^q([-T,T],L^{p+1}(\mathbb{R}^d))}$$

$$|v|_T \leq a$$

$$\Phi(u) = e^{it\Delta}u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\,ds$$

$$u = \Phi(u) \qquad u \in E(a,T)$$

$$\Phi : E(a,T) \circlearrowleft$$

$$|\Phi(u)|_T \leq C_0 |u_0|_{L^2} + C_0 \, \big| |u|^{p-1}u \big|_{L^{q'}([-T,T], L^{\frac{p+1}{p}})}$$

$$\leq C_0 |u_0|_{L^2} + C_0 \, |u|^p_{L^{q'p}([-T,T], L^{p+1})}$$

$$p \in \left(1, 1+\frac{4}{d}\right) \Longleftrightarrow p q' < q$$

$$\frac{1}{pq'} = \frac{1}{q} + \frac{1}{\tilde{q}}$$

$$\leq C_0 |u_0|_{L^2} + C_0 \quad |1|^p_{L^{\tilde{q}}[-T,T]} \quad |u|^p_{L^q([-T,T], L^{p+1})}$$

$$= C_0 |u_0|_{L^2} + C_0 C \; T^{\alpha} \quad \underbrace{|u|^p_{L^q([-T,T], L^{p+1})}}_{\leq |u|^p_T}$$

$$\alpha = \frac{p}{\tilde{q}}$$

$$|\phi(u)|_T \leq C_0 |u_0|_{L^2} + C_0 \; T^{\alpha} \quad |u|^p_T$$

$$|\phi(u)|_T \leq C_0 |u_0|_{L^2} + \underbrace{C_0 \; T^{\alpha} \quad a^{p-1}}_{< \frac{1}{2}} \quad |u|_T$$

$$|\phi|_T \leq C_0 |u_0|_{L^2} + \frac{1}{2} |u|_T \leq C_0 |u_0|_{L^2} + \frac{a}{2}$$

$$< \frac{a}{2} + \frac{a}{2} = a$$

$$a > 2 C_0 |u_0|_{L^2}$$

$$\Phi : E(a, T) \circlearrowleft$$

$$|\phi(u) - \phi(v)|_T = \left| \int_0^t ds \, e^{i(t-s)\Delta} \left(|u|^{p-1} u - |v|^{p-1} v \right) \right|_T$$

$$\leq C_0 \; \left| \, |u|^{p-1} u - |v|^{p-1} v \right|_{L^{q'}([-T,T], L^{\frac{p+1}{p}})}$$

$$\leq C_0 C_p \left| \; (|u| + |v|)^{p-1} (u - v) \right|_{L^{q'}([-T,T], L^{\frac{p+1}{p}})}$$

$$\leq c_0 C_p \left\| \left\| (|u|+|v|)^{p-1}(u-v) \right\|_{L_x^{\frac{p+1}{p}}} \right\|_{L^{q'}[-T,T]}$$

$$\leq c_0 C_p \left\| \left\| (|u|+|v|)^{p-1} \right\|_{L_x^{\frac{p+1}{p-1}}} \|u-v\|_{L_x^{p+1}} \right\|_{L^{q'}[-T,T]}$$

$$\frac{p}{p+1} = \frac{p-1}{p+1} + \frac{1}{p+1}$$

$$\leq c_0 C_p \left\| \left(\|u\|_{L^{p+1}} + \|v\|_{L^{p+1}} \right)^{p-1} \|u-v\|_{L_x^{p+1}} \right\|_{L^{q'}[-T,T]}$$

$$\frac{1}{q'} = \frac{p-1}{q} + \frac{1}{\rho}$$

$$p q' < q \qquad \frac{p}{q} < \frac{1}{q'}$$

$$\frac{1}{q'} > \frac{p}{q} = \frac{p-1}{q} + \frac{1}{q} \qquad \frac{1}{\rho} > \frac{1}{q} \qquad \rho < q$$

$$\leq c_0 C_p \left\| \left(\|u\|_{L^{p+1}} + \|v\|_{L^{p+1}} \right)^{p-1} \right\|_{L^{\frac{q}{p-1}}([-T,T])} \|u-v\|_{L^{\rho}([-T,T], L_x^{p+1})}$$

$$\leq c_0 C_p \left\| \|u\|_{L_x^{p+1}} + \|v\|_{L_x^{p+1}} \right\|_{L^{q}([-T,T])}^{p-1} \|u-v\|_{L^{\rho}([-T,T], L_x^{p+1})}$$

$$\leq c_0 \left(\|u\|_{L^q([-T,T], L^{p+1})} + \|v\|_{L^q([-T,T], L^{p+1})} \right)^{p-1} \|u-v\|_{L^q([-T,T], L_x^{p+1})} (2T)^{\frac{1}{\alpha}}$$

$$\frac{1}{\rho} = \frac{1}{q} + \frac{1}{\alpha}$$

$$\leq C_0\, 2^{\frac{1}{\alpha}}\, 2^{p-1}\, a^{p-1}\, T^{\frac{1}{\alpha}}\, |u-v|_{\underbrace{L^q([-T,T],L^{p+1}_x)}_{|u-v|_T}}$$

$$|\phi(u)-\phi(v)|_T \leq \underbrace{C_1\, a^{p-1}\, T^{\frac{1}{\alpha}}}_{<\frac{1}{2}}\, |u-v|_T$$

$$p q' < q \qquad \frac{p}{q} < \frac{1}{q'} = 1-\frac{1}{p}$$

$$\frac{p}{q} < 1 - \frac{1}{q} \qquad \frac{p+1}{q} < 1$$

$$\frac{1}{q} < \frac{1}{p+1}$$

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$$

$$\frac{1}{q} = \frac{d}{4} - \frac{d}{2p+2} < \frac{1}{p+1}$$

$$\frac{d}{4} < \left(\frac{d}{2}+1\right)\frac{1}{p+1}$$

$$p+1 < \frac{\frac{d}{2}+1}{\frac{d}{4}} = \frac{2d+4}{d} = 2+\frac{4}{d}$$

$$p+1 < 2+\frac{4}{d}$$

$$p < 1+\frac{4}{d}$$

$T = T(|u_0|_{L^2})$ $\qquad E(T, a)$

$a \geq 4 c_0 |u_0|_{L^2}$

u_0

$f(T(|u_0|_{L^2}), a) < 1$

$T' < T(|u_0|_{L^2})$

$f(T', \tilde{a}) < 1$

per $\tilde{a} > a$

scelto un $\tilde{a} > a$ posso prendere V intorno di u_0 in modo tale che

$\tilde{a} > 4 c_0 |v_0|_{L^2} \quad \forall v_0 \in V$

$\forall v_0 \in V \quad \exists \quad v \in \underline{E(\tilde{a}, T')}$

$$|v - u|_{T'} \leq \left| e^{it\Delta} v_0 - e^{it\Delta} u_0 + i\lambda \int_0^t ds\, e^{i(t-s)\Delta} \left(|v|^{p-1} v - |u|^{p-1} u \right) \right|_{T'}$$

$$\leq c_0 |u_0 - v_0|_{L^2} + C\, T'^{\frac{1}{\alpha}} \left(|u|_{T'} + |v|_{T'} \right)^{p-1} |u - v|_{T'}$$

$|u|_{T'} \leq 2 c_0 |u_0|_{L^2}$

$|v|_{T'} \leq 2 c_0 |v_0|_{L^2}$

$$|v - u|_{T'} \leq c_0 |u_0 - v_0|_{L^2} + \underbrace{C\, T'^{\frac{1}{\alpha}} \left(|u_0|_{L^2} + |v_0|_{L^2} \right)^{p-1}}_{\gamma \leq \frac{1}{2}} |u - v|_{T'}$$

$$\frac{1}{2} |v-u|_{T'} \leq c_0 |u_0 - v_0|_{L^2}$$

$$\gamma < \frac{1}{2}$$

$$\cancel{|v-u|_{T'} \leq 2c_0 |u_0-v_0|_{L^2}} \implies 1-\gamma > \frac{1}{2}$$

$$\leq (1-\gamma) |v-u|_{T'} \leq c_0 |u_0 - v_0|_{L^2}$$

$$\frac{1}{2} |v-u|_{T'}$$

$$\frac{1}{2} |v-u|_{T'} \leq c_0 |u_0 - v_0|_{L^2}$$

$$0 < T' < T$$

$$u : [-T, T] \to \cancel{\Delta}$$

$$v \in V \longrightarrow v$$

$$V \longrightarrow C^0([-T', T'], L^2) \cap L^q([-T', T'], L^{p+1})$$

$$u(t) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u|^{p-1} u \, ds$$

$$v(t) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |v|^{p-1} v \, ds$$

$$[0, T]$$

$$|u-v|_T = \left| \int_0^t e^{i(t-s)\Delta} \left(|u|^{p-1} u - |v|^{p-1} v \right) ds \right|_T$$

$$\leq \underbrace{C \, T^{\vartheta} \left(|u|_{L^q([0,T], L^{p+1})} + |v|_{L^q([0,T], L^{p+1})} \right)^{p-1}}_{1/2} |u-v|_{L^q([0,T], L^{p+1})}$$

$$|u-v|_T \le \frac{1}{2} |u-v|_T \Rightarrow |u-v|_T = 0$$

<u>Prop</u> Sia $p \in (1, d^*)$ $\quad d^* = \begin{cases} \infty & \text{se } d = 1,2 \\ \frac{d+2}{d-2} & \text{se } d \ge 3 \end{cases}$

Allora $\forall\, u_0 \in H^1(\mathbb{R}^d)$ $\exists$ ~~una~~ $T > 0$ ed una unica soluzione

$$u \in C^0([-T,T], H^1(\mathbb{R}^d)) \cap L^q([-T,T], W^{1,p+1}(\mathbb{R}^d))$$

q come prima $\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$

$$T = T(|u_0|_{H^1}) > 0$$

$$T_{u_0} \ge T(|u_0|_{H^1})$$

$\forall\ 0 < T' < T_{u_0}$ $\exists$ V intorno di u_0 in H^1

t.c. $v_0 \in V \to$

$$v \in C^0([0,T'], H^1(\mathbb{R}^d)) \cap L^q([0,T'], W^{1,p+1}(\mathbb{R}^d))$$

è Lipschitziana.

$$p < 1 + \frac{4}{d}$$