Cosmology

and

Particle Physics

Relatività generale

spaziotempo curvo e in espansione

QFT in spazio curvo

quantum gravity etc...?

Teoria dei campi

è il framework in cui si lavora

Cosmologia

(infinitamente grande)

Termodinamica

universo "piccolo" e caldo

fisica delle alte energie ⟵ $p \Longleftrightarrow T$

Fisica delle particelle

(infinitamente piccolo)

studia l'universo "nel suo insieme" e "all'inizio" (ma anche in futuro)

- qual è l'origine dell'universo?
- di cosa è fatto?
- come è fatto? quanto è grande?
- più specifico: come si sono formate le prime stelle, galassie etc
- cosa accadrà?

Studia i componenti "ultimi" della materia e le loro interazioni

molecole ⟶ atomi ⟶ nuclei ed elettroni ⟶ particelle subnucleari

⇝ Modello Standard

⇝ BSM

ci dà informazioni sulla

cosmologia fisica delle alte energie

determina l'evoluzione della

osservando l'Universo su larga scala $\gtrsim 100\ \mathrm{Mpc}$ possiamo tentare di risalire alle sue condizioni iniziali per capire cosa c'era prima.

Facendo questo esercizio con la massima precisione possiamo fare delle ipotesi su cosa avverrà al nostro universo in futuro

"NATURAL" UNITS

Set some natural constant to 1 in order to

- simplify equations
- make symmetries and relations explicit (eg $\vec{x}$ and t)

$$c = \hbar = k_B = 1$$

Measure in powers of energy:

$[S] = [\hbar] = 0$

$[E] = [T] = [\phi] = [A_\mu] = [m] = 1$ $\qquad [\psi] = 3/2$

$[x] = [t] = -1$

We will measure temperature in GeV

$$\langle E \rangle \sim T$$

Conversion: $1\,eV = 1.60 \times 10^{-19}\,J = 1.16 \times 10^{4}\,K$

Planck mass: $M_P = (8\pi G_N)^{-1/2} = 2.4 \times 10^{18}\,GeV$

$m_P = G_N^{-1/2} = 1.22 \times 10^{19}\,GeV$

Another useful reference: $\dfrac{T}{1\,MeV} = \left(\dfrac{t}{1\,s}\right)^{-1/2}$ valid in radiation dominance

$T_\oplus \sim 300\,K \sim 0.03\,eV$ $\qquad (0.01\,eV \lesssim m_\nu \lesssim 0.1\,eV)$

$T_{\odot\,surf} \sim 6000\,K \sim 0.5\,eV$

$T_{\odot\,core} \sim 10^{7}\,K \sim 1\,keV$

Astronomical distances

pc = distance at which 1 AU subtends an angle of 1 arcsecond (1/3600 of a degree)

$1\,pc = 3.26\,lyr = 3\times10^{13}\,km = 2\times10^{5}\,AU$

* proxima centauri ~ 1.3 pc

- visible stars $\lesssim$ few $\times 10^3$ pc
- Andromeda 700 000 pc
- universe: $\gtrsim$ 100 Mpc

Planck mass

Why is $M_P = (8\pi G_N)^{-1/2}$ so important?

What does it mean that M_P is the scale of quantum gravity?

Two ways:

1) For a particle of mass M_P, the Compton wavelength and the Schwarzschild radius coincide

* Compton wavelength

$$\lambda_c = \frac{\hbar}{mc}$$ (reduced Compton wavelength)

"scale of quantum effects"

It controls the dynamics of QFT

eg KG eq: $\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = \left(\frac{mc}{\hbar}\right)^2\phi \leftrightarrow \lambda \sim \frac{\hbar}{mc}$

Dirac: $-i\not{\partial}\psi + \left(\frac{mc}{\hbar}\right)\psi = 0$

Eg Compton scattering (photon off electron)

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

Uncertainty principle: In order to call a particle "particle" it must be

$\Delta E < mc^2$ $\qquad E^2 = (pc)^2 + (mc^2)^2$

$E\Delta E = pc^2\Delta p \Rightarrow \Delta p < \frac{E}{pc^2}mc^2 \overset{\text{high energy}}{\approx} mc$

but $\Delta x \Delta p > \frac{\hbar}{2} \Rightarrow \Delta x > \frac{1}{2}\frac{\hbar}{mc} = \frac{1}{2}\lambda_c$

⇒ If I go closer than λ_c, I cannot consider the particle as a single, isolated one. Expect particle production to happen!

* Schwarzschild radius

$$r_s = \frac{2Gm}{c^2}$$

an object of mass m and radius $r < r_s$ is a BH

⇒ In order to neglect GR effects $\lambda_c \gg r_s$

$$\frac{r_s}{\lambda_s} = \frac{2Gm}{c^2}\,\frac{mc}{\hbar} = \frac{2G}{c\hbar}\,m^2$$

$$\Rightarrow m^2 \ll \frac{c\hbar}{G} = m_p^2 \qquad \left(M_p^2 = \frac{c\hbar}{8\pi G} \right)$$

2) The Planck mass controls the perturbative expansion

- photon scattering: expansion in $\alpha = \dfrac{e^2}{4\pi}$

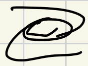
- graviton scattering:

(at low momentum) $\alpha = Gm^2 = \left(\dfrac{m}{m_p}\right)^2$

(at large momentum) $\left(\dfrac{p}{m_p}\right)^2$

⇒ At $E \sim m_p$ quantum effects in gravity become relevant and non-perturbative

OBSERVED UNIVERSE

- Stars : typical distance ~ few pc
- Galaxies : MW :  $R \simeq 12.5\,\mathrm{kpc}$ $h = 0.3\,\mathrm{kpc}$
  $r_\odot = 8\,\mathrm{kpc}$ $v_\odot = 200\,\mathrm{km/s}$
  $\sim 10^{11}$ stars
- Local group : distance: 50 - 1000 kpc , size Mpc
- Cosmology : typical size $\gtrsim 100$ Mpc (today)
  (distance between galaxy clusters)

highly inhomogeneous due to gravitational collapse

Temperature $T_{CMB} \simeq 2.73\,\mathrm{K} \simeq 2.4 \times 10^{-4}\,\mathrm{eV}$

with very small fluctuations $\dfrac{\delta T_{CMB}}{T_{CMB}} \simeq 10^{-5}$

homogeneity on large scales $100\,\mathrm{Mpc} \lesssim \lambda \lesssim 3000\,\mathrm{Mpc}$

COSMOLOGICAL PRINCIPLE

homogeneous & isotropic (on large enough scales)

not equivalent : - $\vec{E} \neq 0$ uniform is homogeneous but not isotropic
- $\rho = \rho(r)$ is isotropic when seen from the centre, but not homogeneous
- it selects a preferred reference frame!

FLRW metric the metric of a homogeneous and isotropic universe can be written as

$$ds^2 = -\,dt^2 + a^2(t)\left[\frac{dr^2}{1 - k r^2/R_0^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right]$$

curvature of 3-space : ${}^{3}K(t) = \dfrac{k}{a^2(t)}$

3D curvature

$$k = \begin{cases} +1 & \text{positive curvature, closed universe (finite)} \\ 0 & \text{flat universe (infinite)} \\ -1 & \text{open universe (infinite)} \end{cases}$$

$t, \vec{x}$ "comoving coordinates": an observer at rest lives at $\vec{x} = \text{constant}$ (ie $\vec{x} = \text{const}$ is a geodesic)

t: proper time of a observer at rest

Conformal time $dt = a \, d\tau$

Cosmic reference frame (privileged): the reference frame in which $g_{\mu\nu}$ is diagonal $\Rightarrow$ the frame in which the universe is indeed isotropic

Scale factor: $a(t) > 0$

$a = 1$: Minkowsky

in general: $a(t)$ depends on the amount and on the type of matter in the Universe ("matter content")

Rescaling symmetry

$r \to \lambda r \qquad R_0 \to \lambda R_0 \qquad a \to a/\lambda$

I get the same metric. $\Rightarrow$ I can define $a_0 = 1$ today

Peculiar velocity and Hubble flow

position of galaxy : $\vec{r}_{phys} = a\,\vec{r}$

$\Rightarrow$ physical velocity : $\vec{v}_{phys} = \dfrac{d\vec{r}_{phys}}{dt} = \dot{a}\,\vec{r} + a\,\dot{\vec{r}}$

$$= \left(\frac{\dot{a}}{a}\right) a\,\vec{r} + a\,\dot{\vec{r}}$$

$$= H\,\vec{r}_{phys} + \vec{v}_{peculiar}$$

peculiar velocity : $\vec{v}_{pec} = a\,\dot{\vec{r}}$ velocity measured by a comoving (i.e. free-falling) observer at position $\vec{r}$.

Hubble flow : $H\,\vec{r}_{phys}$

Hubble constant : ('20s)

- Cepheid stars in distant galaxies $\longrightarrow$ measure distance (intrinsic luminosity I known from the period of luminosity variation, measure $\Phi = \dfrac{I}{4\pi d^2}$ $\rightarrow$ determine d)

- velocity from redshift $z = \dfrac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} \propto v$

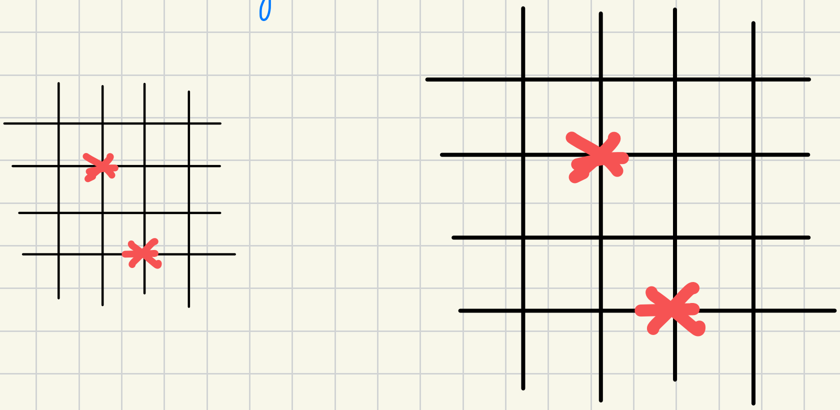
$\Rightarrow$ $v \propto d$ $\quad v = H_0\, d$

$\left(\begin{array}{l} H_0 = 100\, h\ \mathrm{Mpc}^{-1}\ \mathrm{km\,s}^{-1} \\ \text{w/ } h \simeq 0.7 \end{array} \right)$

interpretation:

Things do not move away from us, we being at the center.
The universe expands, ie distances grow

Galaxies live at fixed coordinates $\vec{x}$



$a(t_0)$ $\qquad$ $a(t>t_0)$

If the distance is $\Delta\vec{r} = a\,\Delta\vec{x}$

$$\vec{v} = \dot{\vec{r}} = \dot{a}\,\Delta\vec{x} = \frac{\dot{a}}{a}\,\Delta\vec{r}$$

thus $\quad H = \dfrac{\dot{a}}{a}$

Hubble today: $\boxed{H_0 = H(t_0) \simeq 100\, h \ \mathrm{km\ s^{-1}\ Mpc^{-1}} \qquad h \simeq 0.7}$

$$\left(\begin{aligned} 1\mathrm{pc} &= 3.26\ \mathrm{lyr} = 2\times 10^5\ \mathrm{AU} \\ &= 3.09\times 10^{13}\ \mathrm{km} \end{aligned} \right)$$

$$\simeq 2.2\, h \times 10^{-42}\ \mathrm{GeV}$$

(*) Distances are tricky in cosmology.

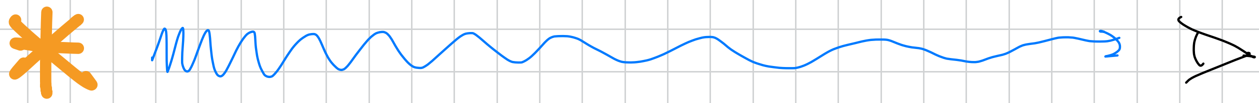
$\left\{ \begin{array}{l} \text{comoving distance } \Delta\vec{x} \\ \text{physical distance } \Delta\vec{r} = a(t)\,\Delta\vec{x} \end{array} \right.$

are not measurable because defined from two separate events at fixed time

Redshift

Everything we know about the Universe is inferred from light received from distant objects. How does the expansion affect the light?

Look at geodesics w/ $ds^2 = 0$ in FLRW:



emitted $\longrightarrow$ $\lambda_* = \dfrac{a(t_*)}{\underbrace{a(t_0)}_{=1}} \lambda_0$ $\longleftarrow$ measured today

Define redshift as

$$z = \frac{\lambda_0 - \lambda_*}{\lambda_*} \quad \longleftrightarrow \quad 1 + z = \frac{1}{a(t)}$$

Redshift is a measure of time (closer to actual observations: redshift is measured, while time is derived from z w/ a model)

DYNAMICS

Einstein equation $$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\frac{8\pi G}{c^4} \to = \frac{1}{M_p^2} = \frac{8\pi}{m_p^2}$$

$G_{\mu\nu}$ computed for the metric

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho} \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\rho}$$

$$R_{tt} = 3\frac{\ddot{a}}{a} \qquad R_{ti} = R_{it} = 0 \qquad R_{ij} = -\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\right) g_{ij}$$

In order to respect the symmetries (homogeneous & isotropic) the energy-momentum tensor must be that of a perfect fluid

$$T_{\mu\nu} = \left(\begin{array}{c|c} T_{00} & T_{0j} \\ \hline T_{i0} & T_{ij} \end{array}\right)$$

T_{00} : scalar under 3-rotations $SO(3)$, depends on t

$T_{i0} = T_{0j} = 0$ in the comoving frame

$T_{ij} \propto \delta_{ij} \propto g_{ij}$ (isotropy again)

$\Rightarrow$ in the comoving frame

ρ : energy density

P : pressure

$$T_{00} = \rho(t) \qquad T_{i0} = T_{0j} = 0 \qquad T_{ij} = p(t)\, g_{ij}$$

which I can rewrite as

$$T_{\mu\nu} = (\rho + P) U_\mu U_\nu + P g_{\mu\nu}$$

Now this is a tensor in the GR sense

U is the 4-velocity of a fluid at rest in the comoving frame

ie in the comoving frame $U_t = 1 \quad U_i = 0$

Continuity equation

GR version of energy conservation, derived from the conservation of $T_{\mu\nu}$

$$\nabla_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$$

Exercise: obtain the continuity eq from thermodynamics

$$dU = -P\,dV \quad \text{w/} \quad U = \rho V \quad \text{and} \quad V \propto a^3$$

solution:

$$\frac{1}{V}\frac{dU}{dt} = \dot{\rho} + \rho\frac{\dot{V}}{V} = \dot{\rho} + \rho\frac{3a^2\dot{a}}{a^3} = \dot{\rho} + 3\frac{\dot{a}}{a}\rho$$

$$= -\frac{1}{V}P\frac{dV}{dt} = -3P\frac{\dot{a}}{a}$$

$$\Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$$

Equation of state

Some physics enters and describes any fluid

$$P = w\rho \qquad w = \text{constant}$$

For constant w one finds exact solutions

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho \propto a^{-3(1+w)}$$

(will discuss w later)

First Friedmann equation

$$G^0{}_0 = 8\pi G T^0{}_0 \quad : \quad H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\left(\begin{array}{ll} G^i{}_j = 8\pi G T^i{}_j \; ! & \dfrac{\ddot{a}}{a} = -\dfrac{4\pi G}{3}(\rho + 3P) \\ (\text{or } {}^0_0 + \text{continuity}) & \end{array} \right)$$

An instructive Newtonian derivation is also possible

Take a test particle at rest at a distance $\vec{r}$ from the origin.
The particle is attracted towards the origin with a force

$$F = -G\frac{Mm}{r^2} \qquad w/ \; M = \frac{4}{3}\pi r^3 \rho$$

and I can define an energy $\quad U = -G\dfrac{Mm}{r}$

The energy of the test particle is $\quad E = \dfrac{1}{2} m \dot{r}^2 - \dfrac{G M(r) m}{r}$

Use comoving coordinates $\quad \vec{r}(t) = a(t)\,\vec{x}$

$$\rightsquigarrow \quad E = \frac{1}{2} m \dot{a}^2 x^2 - Gm \frac{1}{ax}\frac{4}{3}\pi\rho a^3 x^2$$

$$\frac{2E}{mx^2} = \underbrace{\dot{a}^2 - \frac{\rho}{3M_p^2} a^2}_{x \text{ indep}} \qquad \Rightarrow \frac{2E}{mx^2} \text{ is } x \text{ independent}$$

call it $\quad k = -\dfrac{2E}{mx^2}$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_p^2} - \frac{k}{a^2}$$

Second Friedmann eq (acceleration eq)

$$G^i{}_j = 8\pi G T^i{}_j \quad ! \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) = -\frac{\rho + 3P}{6M_p^2}$$

(or $^0_0$ + continuity)

Derive the Friedmann eq :

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_p^2} - \frac{k}{a^2} \quad \leftarrow \quad \frac{d}{dt}$$

$$2\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2}\right) = \frac{\dot{\rho}}{3M_p^2} + 2k\frac{\dot{a}}{a^3}$$

$$2H\left(\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2}\right) = -\frac{1}{3M_p^2}3H(\rho + P) + 2\frac{k}{a^2}H$$

$$\frac{\ddot{a}}{a} = H^2 - \frac{\rho + P}{2M_p^2} + \frac{k}{a^2} = -\frac{\rho + P}{2M_p^2} + \frac{\rho}{3M_p^2} = -\frac{\rho + 3P}{6M_p^2}$$

For all familiar fluids :

$$\rho + 3P > 0 \quad \Rightarrow \quad \ddot{a} < 0$$

ie : gravity pulls the Universe together

but observations tell us $\ddot{a} > 0 \Rightarrow$ [our universe accelerates]

Matter, radiation, kinetic energy, dark energy

Continuity: $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$

EoS: $P = w\rho$

$\Rightarrow \rho \propto a^{-3(1+w)}$

Expansion: $\frac{\dot{a}}{a} = \frac{\sqrt{\rho}}{\sqrt{3}M_p} = \left(\frac{\rho_0}{3M_p^2}\right)^{1/2} \frac{a^{-\frac{3}{2}(1+w)}}{a_0^{-\frac{3}{2}(1+w)}}$

$$\int_0^a \underbrace{a^{\frac{3}{2}(1+w)-1}}_{a^{\frac{1+3w}{2}}} \, da = \left(\frac{\rho_0}{3M_p^2}\right)^{1/2} a_0^{\frac{3}{2}(1+w)} \int_0^t dt$$

$$\rightsquigarrow \frac{2}{1+3w} a^{\frac{3}{2}(1+w)} + K = \left(\frac{\rho_0}{3M_p^2}\right)^{1/2} a_0^{\frac{3}{2}(1+w)} t$$

Big bang: $t = 0$, $a = 0$ $\Rightarrow K = 0$, Rescaling: $a_0 = 1$

$\rightarrow \exists$ singularity theorem: $\forall$ matter w/ $\rho + 3P > 0$, $a = 0$ at some point in the past

$$\Rightarrow a = \left(\frac{\rho_0}{3M_p}\right)^{\frac{1}{3(1+w)}} \left(\frac{1+3w}{2}\right)^{\frac{2}{3(1+w)}} t^{\frac{2}{3(1+w)}}$$

$$a \propto t^{\frac{2}{3(1+w)}}$$

Matter (dust) non relativistic particles have $p \ll m$

$\Rightarrow P \ll \rho \Rightarrow$ eos: $P = 0$ or $w = 0$

$\Rightarrow \rho \propto a^{-3} \qquad a \propto t^{2/3} \qquad H = \frac{2}{3}\frac{1}{t} \propto a^{-3/2}$

- all SM particles (except γs)
- "dark matter"
- "many" new particles predicted by BSM if $\underline{m > T}$

Radiation relativistic particles have $p \gg m$ $\quad \left(p = \gamma m v \gg mc \right)$

$\Rightarrow$ eos: $P = \frac{1}{3}\rho \Rightarrow w = 1/3$ $\quad$ comes from $T_\mu{}^\mu = 0$ as for Maxwell theory

$\Rightarrow \rho \propto a^{-4} \qquad a \propto t^{1/2} \qquad H = \frac{1}{2t} \propto a^{-2}$

- photons
- for long time also neutrinos $\quad \left(T_0 \approx 0.2\,\text{meV} \quad m_\nu \sim 50\,\text{meV} \right)$
- "dark radiation" $\rightsquigarrow$ we will see

Dark energy Can I have $w = -1$?

Yes: cosmological constant or scalar field potential

or zero-point energy

$$\Rightarrow \quad \underset{= M_p^2 \Lambda}{\rho = \text{const}} \qquad a \sim e^{Ht} \quad \leftarrow \text{exercise}$$

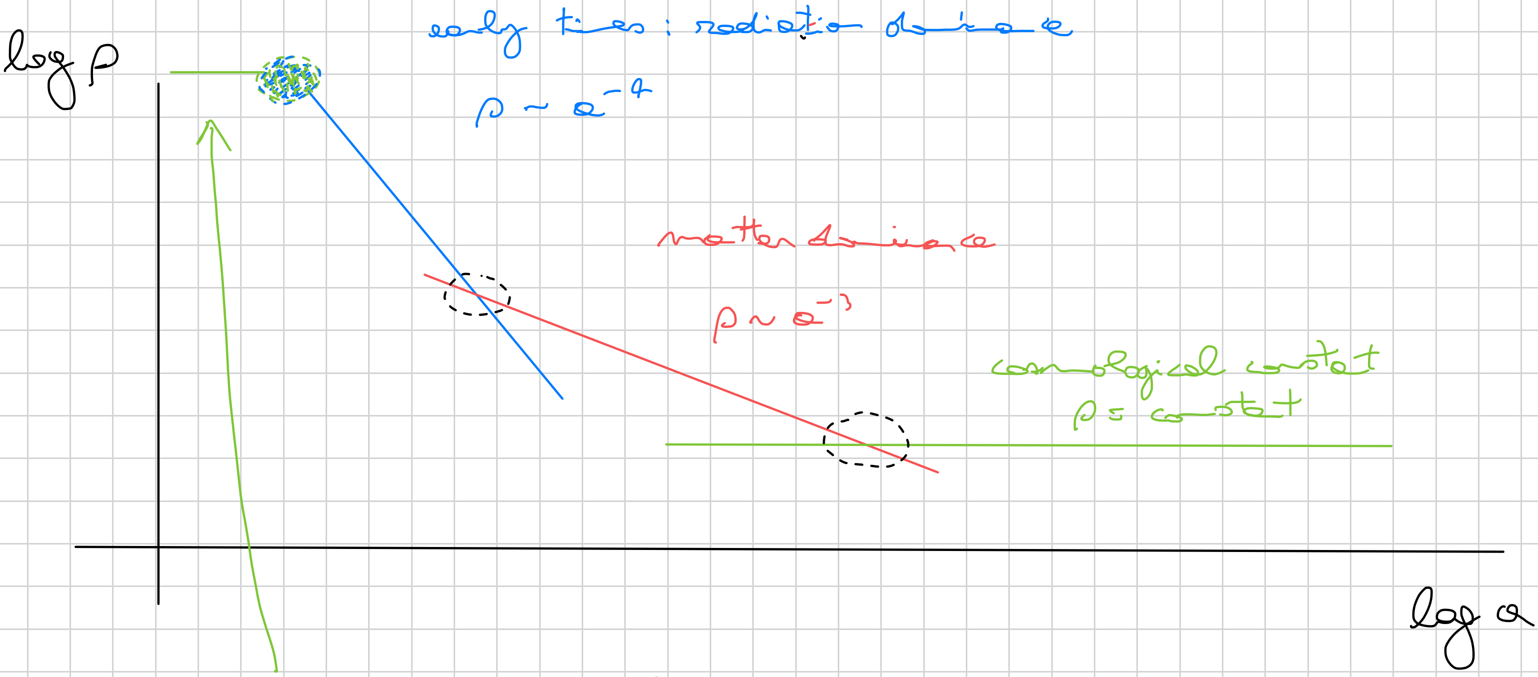
Kination Can I have $w = 1$?

Yes: if a classical field dominates the energy density

with $\frac{1}{2}\dot{\phi}^2 \gg V$

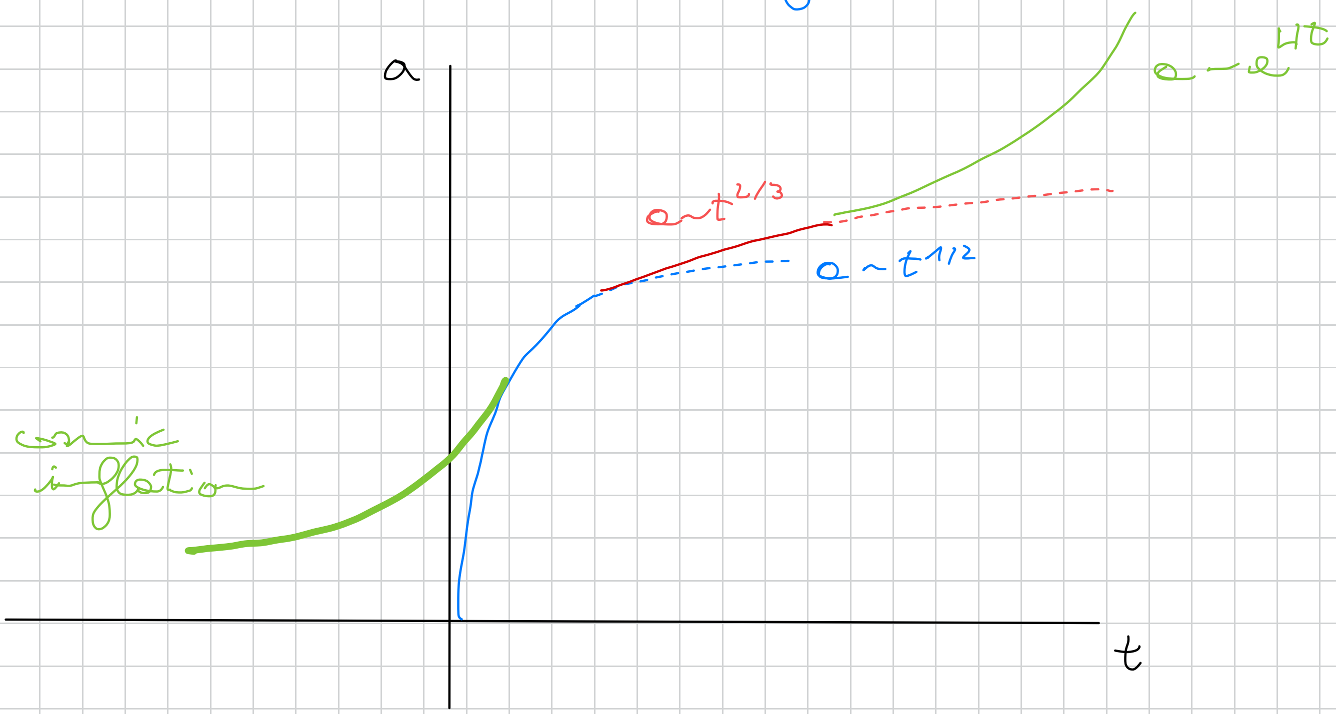
$\rightsquigarrow \rho \propto a^{-6} \qquad a \propto t^{1/3} \qquad H = \frac{1}{3t} \propto a^{-3}$

A first sketch of the cosmic history



early cc period: INFLATION

The energy that drives inflation must be converted into radiation : reheating



Here $a \to 0$, $\rho \to \infty$, $R \to \infty$: singularity problem

Historically, the first motivation for inflation

Big bang : the time at which the extrapolated $a(t) \to 0$

Critical density and overclosure

$$H^2 = \frac{\rho}{3M_p^2} - \frac{k}{a^2 R_0^2} \longrightarrow \frac{\rho_k}{3M_p^2} \quad \text{w/} \quad \rho_k = 3M_p^2 \frac{k}{a^2 R_0^2}$$

Define the critical density as the energy of a universe with zero spatial curvature

$$\rho_{crit,0} = \frac{3 H_0^2}{8\pi G} = 3 M_p^2 H_0^2 \simeq 2.8 \times 10^{11}\, h^2\, M_\odot / Mpc^3$$

$$\simeq 1.1 \times 10^{-5}\, h^2 \text{ protons} / cm^3$$

(w/ $H_0 = 100\, h$ km s^{-1} Mpc^{-1})

$$\simeq 1.1 \times 10^{-5}\, h^2\, GeV / cm^3$$

$$= 8.4\, h^2 \times 10^{47}\, GeV^4$$

compare w/ Avogadro's number:
$N_A = 6.023 \times 10^{23}\, mol^{-1}$
w/ 1 mol = 12g ^{12}C

Convenient to define ρ in terms of ρ_0

$$\Omega_{i,0} = \frac{\rho_{i,0}}{\rho_{crit,0}}$$

$$\Rightarrow \boxed{\frac{H^2}{H_0} = \Omega_{r,0}\, a^{-4} + \Omega_{m,0}\, a^{-3} + \Omega_{\Lambda,0} + \Omega_{k,0}\, a^{-2}}$$

w/ $\Omega_{k,0} = -\dfrac{k}{(R_0 H_0)^2}$

Today $H = H_0$, $a_0 = 1$

$$\Rightarrow \underbrace{\Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}}_{\Omega_0} + \Omega_{k,0} = 1$$

$\Omega_k^0 = 1 - \Omega_0$ Measurements show $\Omega_0 \approx 1$

Energy budget of the Universe

Photons photons from the CMB, with a temperature of

$$T_\gamma \simeq 2.73\,K \Rightarrow \rho_\gamma \sim T_\gamma^4 \sim 10^{-52}\,GeV^4$$

$$\Omega_{\gamma,0} \simeq 9.4 \times 10^{-5}$$

Neutrinos we expect the existence of a cosmic ν background w/ $T_\nu \sim T_\gamma \Rightarrow \rho_\nu^0 \sim 10^{-15}\rho_c^0$

Baryons (ie normal particles)

quick estimate (last numbers are ~ random)

$$\rho_B = \frac{M_{galaxy}}{D^3_{galaxy}} \sim \frac{10^{12} M_\odot}{(10\,Mpc)^3} \sim 0.1 \frac{GeV}{m^3} \sim 10^{-2}\rho_c$$

(numbers are chosen in such a way to get the correct result)

more quantitatively

$$\Omega_{B,0} \simeq 0.05$$

(Why "baryons?" electrons in stars are non-relativistic. Most of the mass comes from p, n which are much heavier)

Dark matter

$$\Omega_{DM,0} \simeq 0.27$$

Dark energy $\Omega_\Lambda^0 \simeq 0.68$

Event	time t	redshift z	temperature T
Singularity	**0**	∞	∞
Quantum gravity	$\sim 10^{-43}$ s	–	$\sim 10^{18}$ GeV
Inflation	$\gtrsim 10^{-34}$ s	–	–
Baryogenesis	$\lesssim 20$ ps	$> 10^{15}$	> 100 GeV
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1100	0.26 eV
Reionization	100–400 Myr	10–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	**13.8 Gyr**	**0**	**0.24 meV**

Table 3.1: *Key events in the thermal history of the universe.*

2.73 K

first observable stars ~ 30 Myr ~ 65

first MW-sized galaxies ~ 400 Myr ~ 11

growth of structures happens in matter dominance

The Universe is (and was) flat

$$\Omega_R^0 + \Omega_m^0 + \Omega_\Lambda^0 + \Omega_k^0 = 1$$

observations: $|\Omega_k^0| \equiv \left| -\frac{k}{(R_0 H_0)^2} \right| \lesssim 2\times 10^{-3}$

At a time t: $|\Omega_k(t)| = \frac{|k|}{R_0^2} \frac{1}{a(t)^2 H(t)^2}$

* cosmological constant: $H \approx$ constant, $a \sim e^{Ht}$

$\Rightarrow$ $|\Omega_k(t)| \sim a^{-2}$ decreases in time

At matter $-\Lambda$ equality it was

$$\Omega_{k,\Lambda m} = \Omega_{k,0}\left(\frac{a_0}{a_{\Lambda m}}\right)^2 = \Omega_{k0}(1+z_{\Lambda m})^2 \approx \Omega_{k0}(1.4)^2 \approx 2\,\Omega_{k0}$$

* matter: $|\Omega_k(t)| \sim a$

at matter $-$ radiation equality it was

$$\Omega_{k,eq} = \Omega_{k,\Lambda m}\frac{a_{eq}}{a_{\Lambda m}} = \Omega_{k,\Lambda m}\frac{1+z_{\Lambda m}}{1+z_{eq}} = \frac{\Omega_{k,0}}{(1+z_{eq})(1+z_{\Lambda m})}$$

$$\approx 2\times 10^{-4}\,\Omega_{k0} \lesssim 4\times 10^{-7}$$

* radiation dominance: $|\Omega_k| \sim a^2$

$$\Omega_{k,r} = \Omega_{k,eq}\left(\frac{a}{a_{eq}}\right)^2 = \Omega_{k0}\frac{(1+z_{eq})}{(1+z_{\Lambda m})(1+z)^2}$$

$\Rightarrow$ as we go back in time Ω_k becomes smaller and smaller $\Rightarrow$ we can set it to zero

"Flatness problem": why was the curvature so small at the beginning? $\Rightarrow$ motivation for inflation.

if Ω_k was large at some point, an era of exponential expansion would have washed it away

fine-tuning

the problem is about $\rho = \rho_c$: why do the three components of energy (matter, radiation, Λ) sum up to one so precisely?

Age of the Universe

Naive estimate We have seen that galaxies recede at a average speed

$$v = H_0 d$$

Assuming constant v $\Rightarrow$ $d = v\, t_U = H_0\, d\, t_U$

$$\Rightarrow t_U = H_0^{-1} = \left(70 \text{ km s}^{-1} \text{Mpc}^{-1}\right)^{-1}$$

$$\approx \frac{10^6 \; 3 \; 10^{13} \text{ km}}{70 \text{ km s}^{-1}} \approx 4 \times 10^{17} \text{ s}$$

$$\approx 10^{10} \text{ yr} \qquad (10 \text{ billion years})$$

Integrate the Friedmann equation

- Pure matter Universe (Einstein - de Sitter)

$$a = \left(\frac{t}{t_0}\right)^{2/3} \qquad \sim \quad a_0 = a(t_0) = 1 \text{ today}$$

$$\Rightarrow H_0 = \frac{2}{3}\frac{1}{t_0} \quad \Rightarrow \quad t_0 = \frac{2}{3} H_0^{-1} \approx 9 \text{ billion years}$$

$\boxed{\text{age problem}}$ observations prove the existence of stars that were older than that

add reference here!!

- Our Universe (ΛCDM)

$$\frac{H(t)}{H_0} = \left(\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0} + \cancel{\Omega_{k,0} a^{-2}} \right)^{1/2}$$

can be neglected

$$\frac{da}{dt} = a H_0 \left(\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0} \right)^{1/2}$$

$$\Rightarrow t_U = \frac{1}{H_0} \int_0^1 \frac{da}{\left(\Omega_{m,0} a^{-1} + \Omega_{r,0} a^{-2} + \Omega_{\Lambda,0} a^2 \right)^{1/2}}$$

The integral is dominated by large a ⇒ can neglect Ω_r (which only affects the very early times)

$$\Rightarrow t_U = \frac{2}{3H_0} \frac{1}{\sqrt{1-\Omega_{m,0}}} \log \frac{1+\sqrt{1-\Omega_{m,0}}}{\sqrt{\Omega_{m,0}}}$$

$\Omega_{m,0} + \Omega_{\Lambda,0} = 1$

$$\approx 0.964\, H_0^{-1} \approx 14 \text{ Gyr}$$

Comments:

1) a cosmological constant solves the age problem
→ This was known already in the '90s, before the accelerated expansion was discovered

2) Radiation was relevant at early times.
Matter - radiation equality happens around 380 000 yrs → after that, radiation is negligible

COSMOLOGICAL CONSTANT

Recap so far :

we have seen that cosmological constant means

$\rho \sim M_p^2 \Lambda = \text{const}$ $\qquad a \sim e^{Ht}$ w/ H const

$ds^2 = -dt^2 + e^{2Ht} d\vec{x}^2$ $\qquad$ <u>de Sitter space</u>

Λ needed for accomodating stellar ages w/ age of the universe

<u>But</u> what is a cosmological constant?

Adding a cosmological constant

Einstein equation

$$G_{\mu\nu} = \frac{1}{M_p^2} T_{\mu\nu} \qquad \text{w/ } G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

implies $\nabla^\mu T_{\mu\nu} = 0$

I can add a constant piece $\Lambda g_{\mu\nu}$ without altering the conservation law

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{M_p^2} T_{\mu\nu} \qquad \left(\Rightarrow \nabla^\mu T_{\mu\nu} = 0\right)$$

exercise

(may also be added to the Einstein-Hilbert action)

Behaves like a fluid with

$$T^{\Lambda}_{\mu\nu} = -M_p^2 \Lambda g_{\mu\nu} \equiv -\rho_\Lambda g_{\mu\nu}$$

Compare with a fluid:

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}$$

$$P_\Lambda = -\rho_\Lambda \qquad w_\Lambda = -1$$

For $\Lambda > 0$ leads to accelerated expansion

$$a \propto e^{Ht} \qquad \text{w/} \qquad H^2 = \frac{\rho_\Lambda}{3 M_p^2} = \frac{\Lambda}{3}$$

observationally: $\rho_\Lambda \simeq 6 \times 10^{-10}\ \mathrm{J\,m^{-3}} \simeq (10^{-3}\ \mathrm{eV})^4$

The cosmological constant problem

"Astro" point of view: cc ♡ reconciles the age of the Universe with observation of very old stars

[For a summary: G. Efstathiou MNRAS 274, L73-76 (1995)]

"Theory" point of view: 

Field theory (classical): potential energy contributes to Λ

QFT: every particle arises as a fluctuation of a quantum field
Fields can be represented by a number of Fourier modes with frequency ω_k.
These modes satisfy the eom of a harmonic oscillator
$\to$ zero point fluctuations with energy $\frac{1}{2}\hbar\omega_k$.

Let's sum up these energies

$$\rho^{0}_{QFT} \sim \sum_k \frac{\frac{1}{2}\omega_k}{V} \sim k^3\omega_k \sim k^4 \to +\infty$$

Maybe put a cutoff at $k \sim M_*$ ie the highest energy at which we know QFT works

$$\rho_{QFT} \sim \sum_i m_i^4 \qquad m_i \sim \text{SM masses?}$$

With $\rho_{QFT} = (1\,\mathrm{TeV})^4 \sim 10^{60}\,\rho_\Lambda$

W/o gravity this doesn't affect the world (even though Casimir effect is a real thing)

But gravity couples to $T_{\mu\nu}$ $\Rightarrow$ ρ_{QFT} gravitates.

Scalar field potential energy

Consider a scalar field with lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$$

Classical fields: a field displaced from the minimum of the potential behaves classically

$$[\phi, \dot\phi] \approx 0$$

Energy momentum tensor:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\,\frac{\delta(\sqrt{-g}\,\mathcal{L})}{\delta g_{\mu\nu}}$$

$$T_{00} = \rho_\phi = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi) + \dot\phi^2$$

$$T_{ij} = P_\phi\, g_{ij} = g_{ij}\left(-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right) + \partial_i\phi\,\partial_j\phi$$

Neglecting spatial derivatives (uniform field)

$$\rho_\phi = \frac{1}{2}\dot\phi^2 + V(\phi)$$

$$P_\phi = \frac{1}{2}\dot\phi^2 - V(\phi)$$

$$\Rightarrow w = \frac{P}{\rho} = \frac{\frac{1}{2}\dot\phi^2 - V(\phi)}{\frac{1}{2}\dot\phi^2 + V(\phi)}$$

Now assume $\dot\phi \approx 0$ (for example if the field sits in a minimum)

$$\Rightarrow w = \frac{-V}{V} = -1$$

$\Rightarrow$ scalar potential behaves as a CC.

Fine tuning and phase transitions

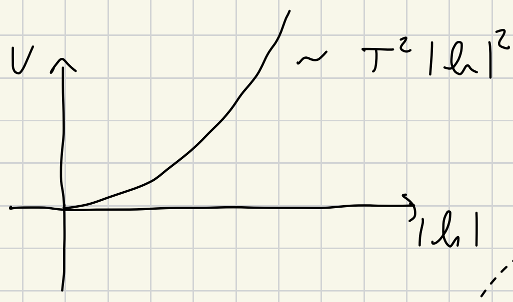
One can try to solve the cc problem by adding a non-zero cc to vacuum energy

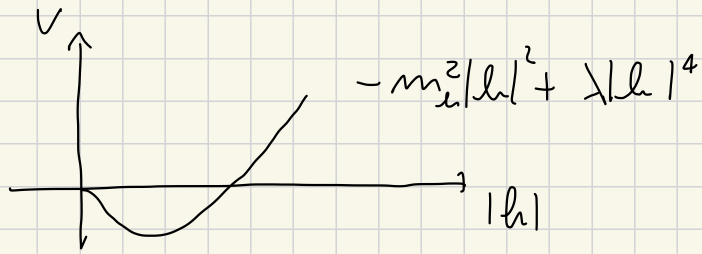
$$\rho_\Lambda = \rho_{QFT} + \rho_0$$

First problem is fine-tuning: ρ_Λ and ρ_0 have 60 digits in common but differ at the 61st digit

Second problem is phase transitions (more at the end of the course)

- Higgs boson has a T-dependent potential:





$\Delta V \sim (200\ \text{GeV})^4$

- QCD confinement (around few $\times$ 100 MeV)
  $\Delta V \sim (300\ \text{MeV})^4$

$\Rightarrow$ fine tune Λ today, not at the initial conditions

- bounds from NS (QCD is deconfined at the core)

Vacuum energy in QFT & Casimir effect

Are vacuum fluctuations a real thing?

Scalar field $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

conjugate momentum $\pi = \frac{\delta\mathcal{L}}{\delta\dot\phi}$

Hamiltonian $\mathcal{H} = \pi\dot\phi - \mathcal{L}$ (density)

$$H = \int d^3x\,\mathcal{H} = \int d^3x\ \frac{1}{2}\pi^2 + \frac{1}{2}(\vec\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

eom $\left(\dot\phi = \frac{\partial H}{\partial\pi} \qquad \dot\pi = -\frac{\partial H}{\partial\phi}\right)$

Quantization:
$$\phi = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left[a_{\vec k}\,e^{i\vec k\cdot\vec x} + a^\dagger_{\vec k}\,e^{-i\vec k\cdot\vec x}\right]$$

$$\pi = \int\frac{d^3k}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec k}}{2}}\left[a_{\vec k}\,e^{i\vec k\cdot\vec x} - a^\dagger_{\vec k}\,e^{-i\vec k\cdot\vec x}\right]$$

$$[\phi,\pi] = i\,\delta^3(\vec x-\vec y) \qquad [a_{\vec p}, a^\dagger_{\vec q}] = (2\pi)^3\,\delta^{(3)}(\vec p-\vec q)$$

Compute the Hamiltonian:

$$H = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,\omega_{\vec p}\left[a^\dagger_{\vec p}a_{\vec p} + a_{\vec p}a^\dagger_{\vec p}\right]$$

$$= \int\frac{d^3p}{(2\pi)^3}\,\omega_{\vec p}\left[\underbrace{a^\dagger_{\vec p}\,a_{\vec p}} + \frac{1}{2}(2\pi)^3\,\delta^{(3)}(0)\right]$$

normal ordered $:H: = \int\frac{d^3p}{(2\pi)^3}\,\omega_{\vec p}\,a^\dagger_{\vec p}\,a_{\vec p}$

Vacuum energy:

$$\langle 0|H|0\rangle = 0 + \int d^3p\ \frac{1}{2}\omega_{\vec p}\ \delta^{(3)}(0)$$ (doubly divergent)

$\delta^{(3)}(0)$: IR divergence → it merely represents a volume

$$(2\pi)^3 \delta^{(3)}(0) = \left(\lim_{L\to+\infty} \int_{-L/2}^{L/2} d^3x \, e^{i\vec{p}\cdot\vec{x}} \right)_{\vec{p}=0} = \lim_{L\to\infty} \int d^3x = V$$

⇒ Energy density: $\mathcal{E}_0 = \frac{E_0}{V} = \int d^3p \, \frac{1}{2}\omega_p$ (UV divergent)

With no gravity: just subtract the constant (divergent) piece

$$:H: = \int \frac{d^3p}{(2\pi)^3} \, \omega_p \, a^\dagger_{\vec{p}} a_{\vec{p}}$$

$$E_0 = \langle 0| :H: |0\rangle = 0$$

Is the other term physically relevant?

If the 2nd term can not give any effect without gravity, I could simply say it doesn't exist. But there are situations in which it does make a difference:

Compute H in detail [skip]

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right)$$

$$= \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \Big[\frac{1}{2}(-i)^2 \sqrt{\frac{\omega_k}{2}\frac{\omega_p}{2}} \left(a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} - a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right)$$

$$\left(a_{\vec{p}} e^{i\vec{p}\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\vec{x}} \right) +$$

$$\frac{1}{2}\sqrt{\frac{1}{4\omega_{\vec{k}}\omega_{\vec{p}}}} \left(i\vec{k} a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} - i\vec{k} a^\dagger_{\vec{k}} e^{-i\vec{k}\vec{x}} \right)\left(i\vec{p} a_{\vec{p}} e^{i\vec{p}\vec{x}} - i\vec{p} a^\dagger_{\vec{p}} e^{-i\vec{p}\vec{x}} \right)$$

$$+ \frac{1}{2}\sqrt{\frac{1}{4\omega_{\vec{k}}\omega_{\vec{p}}}}\, m^2 \left(a_{\vec{k}} e^{i\vec{k}\vec{x}} + a^\dagger_{\vec{k}} e^{-i\vec{k}\vec{x}} \right)\left(a_{\vec{p}} e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\vec{x}} \right)$$

$$= \int d^3x \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3}$$

$$- \frac{1}{4}\sqrt{\omega_k\omega_p} \left(a_{\vec{k}} a_{\vec{p}} e^{i(\vec{k}+\vec{p})\vec{x}} + a^\dagger_{\vec{k}} a^\dagger_{\vec{p}} e^{-i(\vec{k}+\vec{p})\vec{x}} - a_{\vec{k}} a^\dagger_{\vec{p}} e^{i(\vec{k}-\vec{p})\vec{x}} - a^\dagger_{\vec{k}} a_{\vec{p}} e^{-i(\vec{k}-\vec{p})\vec{x}} \right)$$

$$+ \frac{1}{4}\frac{\vec{k}\cdot\vec{p}}{\sqrt{\omega_k\omega_p}} \left(- a_{\vec{k}} a_{\vec{p}} e^{i(\vec{k}+\vec{p})\vec{x}} - a^\dagger_{\vec{k}} a^\dagger_{\vec{p}} e^{-i(\vec{k}+\vec{p})\vec{x}} + a_{\vec{k}} a^\dagger_{\vec{p}} e^{i(\vec{k}-\vec{p})\vec{x}} + a^\dagger_{\vec{k}} a_{\vec{p}} e^{-i(\vec{k}-\vec{p})\vec{x}} \right)$$

$$+ \frac{1}{4}\frac{m^2}{\sqrt{\omega_k\omega_p}} \left(a_{\vec{k}} a_{\vec{p}} e^{i(\vec{k}+\vec{p})\vec{x}} + a^\dagger_{\vec{k}} a^\dagger_{\vec{p}} e^{-i(\vec{k}+\vec{p})\vec{x}} + a_{\vec{k}} a^\dagger_{\vec{p}} e^{i(\vec{k}-\vec{p})\vec{x}} + a^\dagger_{\vec{k}} a_{\vec{p}} e^{-i(\vec{k}-\vec{p})\vec{x}} \right)$$

The $a_{\vec{k}} a_{\vec{p}}$ and $a^\dagger_{\vec{k}} a^\dagger_{\vec{p}}$ pieces are 0 on $|0\rangle$. In general, the $L \to \infty$ limit is innocuous and gives $\delta^3(\vec{k}+\vec{p})$

$$= \int \frac{d^3p}{(2\pi)^3} \left(-\frac{1}{4}\omega_p + \frac{1}{4}\frac{p^2}{\omega_p} + \frac{m^2}{4\omega_p} \right)\left(a_{\vec{p}} a_{-\vec{p}} + a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}} \right)$$

$$+ \text{ terms w/ } a_{\vec{k}} a^\dagger_{\vec{p}} \text{ and } a^\dagger_{\vec{k}} a_{\vec{p}}$$

but $-\omega_p + \dfrac{p^2+m^2}{\omega_p} = -\omega_p + \omega_p = 0$ thus the first term is null

I'm left with

$$H = \int d^3x \, \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{4} \left(\sqrt{\omega_k \omega_p} + \frac{\vec{k}\cdot\vec{p} + m^2}{\sqrt{\omega_k \omega_p}} \right) \times$$

$$\times \left(a_{\vec{k}} a^{+}_{\vec{p}} e^{i(\vec{k}-\vec{p})\vec{x}} + a^{+}_{\vec{k}} a_{\vec{p}} e^{-i(\vec{k}-\vec{p})\vec{x}} \right)$$

$$= \int d^3x \, \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{4} \left(\sqrt{\omega_k \omega_p} + \frac{\vec{k}\cdot\vec{p} + m^2}{\sqrt{\omega_k \omega_p}} \right) \times$$

$$\left[a^{+}_{\vec{k}} a_{\vec{p}} \underbrace{\left(e^{-i(\vec{k}-\vec{p})\vec{x}} + e^{i(\vec{k}-\vec{p})\vec{x}} \right)}_{2\cos[(\vec{k}-\vec{p})\cdot\vec{x}]} + (2\pi)^3 \delta^3(\vec{k}-\vec{p}) e^{i(\vec{k}-\vec{p})\cdot\vec{x}} \right]$$

The first term is the nice one. I can integrate in d^3x on infinite space and get a $\delta^3(\vec{k}-\vec{p})$

$$H = \int \frac{d^3p \, d^3k}{(2\pi)^6} \frac{1}{4} \left(\sqrt{\omega_k \omega_p} + \frac{\vec{k}\cdot\vec{p} + m^2}{\sqrt{\omega_k \omega_p}} \right) 2 (2\pi)^3 \delta^3(\vec{k}-\vec{p}) \, a^{+}_{\vec{k}} a_{\vec{p}}$$

$$+ \int d^3x \, \frac{d^3p \, d^3k}{(2\pi)^6} \frac{1}{4} \left(\sqrt{\omega_k \omega_p} + \frac{\vec{k}\cdot\vec{p} + m^2}{\sqrt{\omega_k \omega_p}} \right) (2\pi)^3 \delta^3(\vec{k}-\vec{p}) \, e^{i(\vec{k}-\vec{p})\cdot\vec{x}}$$

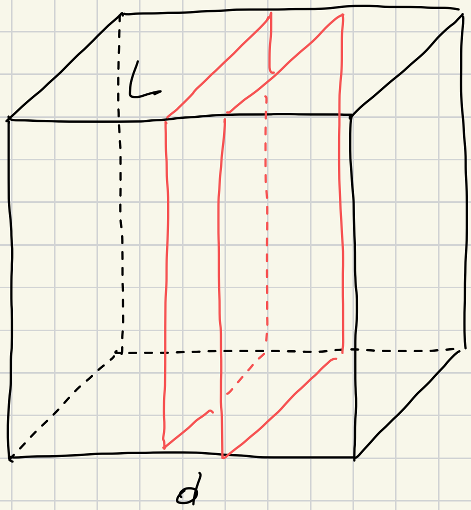
$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \, a^{+}_p a_p + \underbrace{\int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \omega_p}_{= \mathrm{Vol} \times \left(\int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2} \right)}$$

QFT in finite volume

Box of finite size $\Delta x = L$ (infinite for simplicity in y, z)
Periodic bc

$$\phi(\vec{x}) = \phi(\vec{x} + L\hat{x})$$

Insert two parallel, reflecting plates separate by a distance $d \ll L$



NB: L is not important here, it is just there to avoid the IR diverging $\delta(0)$. It naturally disappears from the calculation and can be sent to $+\infty$

$\phi = 0$ at the plates (eg the plates are mirrors and ϕ is the electric field)

$$\vec{p} = \left(n\frac{\pi}{d}, p_y, p_z\right) \qquad n \in \mathbb{Z}^+ \qquad (\mathbb{Z} \setminus \{0\})$$

Energy per unit surface (between the plates)

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y\, dp_z}{(2\pi)^2} \frac{1}{2} \sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2}$$

Total energy:

$$\frac{E}{A} \simeq \frac{E(d)}{A} + \frac{E(L-d)}{A}$$

not exact, but only small deviation for the boundary conditions

$E(d)/A$ is infinite (UV) due to arbitrarily high momenta

Not physical: a mirror cannot be perfect above some frequency!

This is a first example of renormalization (p << plasma frequency)

Mathematically: we want to cut-off the integral at some high frequency, neglecting modes $p \gg a^{-1}$, w/ $a \ll d$.

The procedure is somewhat arbitrary, thus the results must not depend on a

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y\, dp_z}{(2\pi)^2} \frac{1}{2} \sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2}\; e^{-a\sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2}}$$

(gives back the previous result for $a \to 0$)

Define $\quad p_y = p\cos\varphi \qquad p_z = p\sin\varphi$

$$\frac{E(d)}{A} = \sum \int \frac{p\, dp\, d\varphi}{(2\pi)^2} \frac{1}{2} \sqrt{\left(\frac{n\pi}{d}\right)^2 + p^2}\; e^{-a\sqrt{}}$$

$$p^2 + \left(\frac{n\pi}{d}\right)^2 = u^2 \qquad\qquad p = \sqrt{u^2 - \left(\frac{n\pi}{d}\right)^2}$$

$$p\,dp = \frac{1}{2}\, dp^2 = \frac{1}{2}\, du^2 = u\, du$$

$$\frac{E(d)}{A} = \frac{2\pi}{2(2\pi)^2} \sum_{n=1}^{\infty} \int du\; u^2 e^{-au} = \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{d^2}{da^2} \int du\; e^{-au}$$

$$= \frac{1}{4\pi} \sum \left[\frac{d^2}{da^2}\left(-\frac{e^{-au}}{a}\right)\right]_{\frac{n\pi}{d}}^{\infty} = \frac{1}{4\pi} \sum_n \frac{d^2}{da^2} \frac{e^{-n\frac{a\pi}{d}}}{a}$$

$$= \frac{1}{4\pi} \frac{d^2}{da^2} \left(\frac{1}{a} \frac{1}{1 - e^{-\frac{a\pi}{d}}}\right) \overset{a \ll d}{\approx} \frac{3}{2\pi^2} \frac{d}{a^4} + \frac{1}{4\pi a^3} - \frac{\pi^3}{1440\, d^3}$$

The total energy is

$$\frac{E}{A} = \frac{E(d) + E(L-d)}{A} = \frac{3}{2\pi^2}\frac{L}{a^4} + \frac{1}{2\pi a^3} + \frac{\pi^2}{1440}\left(\frac{1}{d^3} - \frac{1}{(L-d)^3}\right)$$

$E/A \to \infty$ for $a \to 0$, as it should. But the force is finite

$$\frac{F}{A} = -\frac{\partial E/A}{\partial d} = +\frac{\pi^2}{480 d^4} - \frac{\pi^2}{480(L-d)^4} \simeq \frac{\pi^2}{480\, d^4}$$

Putting back $\hbar$, c, and multiplying by 2 for a photon,

$$\frac{F}{A} = \frac{\pi^2}{240}\frac{\hbar c}{d^4} \simeq \frac{0.016 \times 10^{-5}}{d^4}\, N \frac{\mu m^4}{cm^2}$$

Comments:

- predicted in 1948, first measurement attempted in 1958, measured in 1996 (Lamoreaux, PRL 78 (1997) 5-8, PRL 81 (1998) 5475-5476 erratum)
- the sign (not only the magnitude) depends on the geometry
- somehow related to Van der Waals force, but I have no idea how it works.
- only photon field matters:
  - cut-off frequency $a^{-1} \sim \mathcal{O}(\omega_p) \sim 10\,eV \ll m_e$
  - neutrinos are not reflected
  - other particles are simply too heavy

Conclusions: 1) Vacuum energy exists.

2) W/o gravity, only energy differences are important

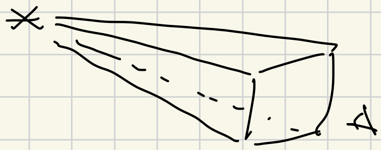
3) CC is a real question!

Measuring the accelerated expansion

How do I measure the distance of a far-away object?

I can measure it's apparent luminosity

$$\text{flux} = \frac{L}{4\pi d_L^2}$$



Need a "standard candle": it's a star I know the intrinsic luminosity of.

Type Ia Supernovae are "standardizable" candles

- matter accreting onto a WD, when it crosses the Chandrasekar limit it explodes
- well defined mass $M_c \simeq 1.4 \odot$ $\Rightarrow$ well defined intrinsic luminosity
- adjustments needed (and possible) due to WD atmosphere
- need calibration through a procedure called cosmic ladder

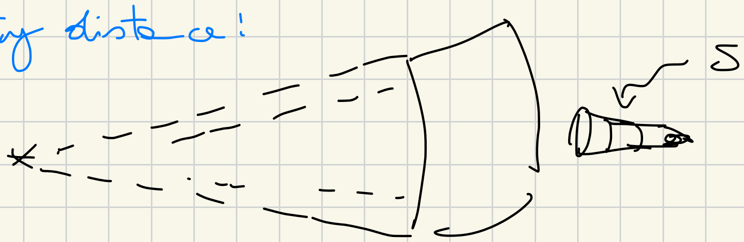
Chandrasekhar limit:

a WD is sustained by electron degeneracy pressure, ie Pauli exclusion principle: no two electrons can occupy the same state, thus they cannot have all zero kinetic energy $\Rightarrow$ creates enough pressure to prevent collapse if $M_{WD} < 1.4\, M_\odot$

d_L luminosity distance: attenuation of light coming from a distant source

Depends on $a(t)$ along the path

Luminosity distance:



- area covered by my detector: $\dfrac{S}{4\pi a_0^2 r^2}$

- from emission until observation individual photons have a decrease in energy of $(1+z)^{-1}$

- photons emitted once every δt are detected once every $\delta t(1+z)$

$\Rightarrow$ detected power is redshifted as $(1+z)^{-2}$

$$\Rightarrow \quad d_L^2 = a_0^2 r^2 (1+z)^2$$

NB: r can be computed as function of the redshift from

$$r = \int_{t_*}^{t_0} \frac{dt'}{a(t')} \qquad \text{and} \qquad \frac{a(t)}{a_0} = \frac{1}{1+z}$$

Acceleration parameter, redshift, luminosity distance

For a photon $ds^2 = 0 \iff dt^2 = a^2(t)\, dr^2$ $(k=0)$

time of flight of a photon:

$$\int_t^{t_0} \frac{dt}{a(t)} = r$$

For $a \sim a_0$ (not too far back in the past)

$$a(t) = a(t_0)\left[1 + \frac{\dot a(t_0)}{a(t_0)}(t-t_0) + \frac{1}{2} \frac{\ddot a(t_0)}{a(t_0)}(t-t_0)^2 + \dots\right]$$

$$= a(t_0)\left[1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \dots\right]$$

w/ $q(t) = -\dfrac{a(t)\,\ddot a(t)}{\dot a^2(t)} = -\dfrac{\ddot a}{\dot a}\dfrac{1}{H}$

Relate time of flight and redshift

$$\frac{1}{1+z} = \frac{a(t)}{a_0} \implies z = (t_0-t)H_0 + \left(1+\frac{1}{2} q_0\right)H_0^2(t_0-t)^2 + \dots$$

$$\implies t_0 - t = \frac{1}{H_0}\left[z - \left(1+\frac{1}{2} q_0\right)z^2 + \dots\right]$$

Compute comoving position of the emitting star

$$\implies r = \int_t^{t_0}\frac{dt'}{a(t')} = \frac{1}{a_0}\int_t^{t_0}\frac{dt'}{1+(t'-t_0)H_0+\dots} \approx \frac{1}{a_0}\int_t^{t_0} dt'\left[1+(t_0-t')H_0+\dots\right]$$

$$= \frac{1}{a_0}\left[(t_0-t) + t_0 H_0(t_0-t) - \frac{1}{2}(t_0^2-t^2)H_0 + \dots\right]$$

$$= \frac{1}{a_0}\left[(t_0-t)+\frac{1}{2}(t_0-t)^2H_0+\dots\right]$$

$$\implies r = \frac{1}{a_0H_0}\left[z - \frac{1}{2}(1+q_0)z^2+\dots\right]$$

Substituting r:

$$d_L = \frac{1}{H_0}\Big[\underbrace{z}_{\text{Hubble's law}} + \underbrace{\tfrac{1}{2}(1-q_0)z^2}_{\text{acceleration}} + \dots\Big]$$

Measure d_L at increasing $z \Rightarrow$ I can measure q_0

$\Rightarrow$ $q_0 < 0$: accelerated expansion

Evaluate q_0

- For a CC dominated universe

$$a(t) = e^{H(t-t_0)} \qquad \dot{a} = Ha \qquad \ddot{a} = H^2 a$$

$$\Rightarrow q_0 = -1$$

- Our universe (matter + CC)

$$\frac{H(t)}{H_0} = \left(\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0} + \Omega_{k,0}a^{-2}\right)^{1/2}$$

$$\dot{a} = H_0\left(\Omega_{r,0}a^{-2} + \Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2 + \Omega_{k,0}\right)^{1/2}$$

$$\ddot{a} = \frac{1}{2}H_0(\quad)^{-1/2}\left(-2\Omega_{r,0}a^{-3} - \Omega_{m,0}a^{-2} + 2\Omega_{\Lambda,0}a\right)\dot{a}$$

$$\Rightarrow \left(\frac{\ddot{a}}{\dot{a}}\right)_0 = \frac{1}{2}H_0\underbrace{(\quad)^{-1/2}}_{=1}\left(-2\Omega_{r,0} - \Omega_{m,0} + 2\Omega_{\Lambda,0}\right)$$

$$\boxed{q_0 = -\frac{a_0\ddot{a}_0}{\dot{a}_0^2} = -\frac{1}{H_0}\frac{\ddot{a}_0}{a_0} = -\frac{1}{2}\left(2\Omega_{\Lambda,0} - \Omega_{m,0} - \cancel{\Omega_{r,0}}\right)}$$

Anthropic principle

[Carroll, astro-ph/0004075]

"We live where we can live"

Tautology: intelligent observers can only exist in a Universe which allow the existence of observers
(for example, stars must have formed)

Theory side: requires the existence of alternative conditions
(separate in space, time, or branches of the wavefunction)

$\Rightarrow$ Our "local" conditions arise as

$$\circledast \quad \left(\frac{\text{Volume of that portion of space}}{\text{Total volume}} \right) \times P\left(\text{hosting life with those conditions} \right)$$

Structure formation theory: overdense regions do not collapse if Λ dominates the universe

$\Rightarrow$ Λ cannot dominate before $z \sim 4$

$$\Rightarrow \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} < a_{\text{galaxies}}^{-3} = (1+z_{gal})^3 \sim O(10^2)$$

More stringent requirement: a universe w/ $\frac{\Omega_{\Lambda 0}}{\Omega_{m,0}} \sim 1$ is more likely to host life than one with ~ 100

$\rightsquigarrow$ requires a knowledge of the prior in $\circledast$

THE UNIVERSE
IN THERMAL EQUILIBRIUM

Thermal equilibrium

As the Universe was very small and dense, particles were interacting at a very large rate

$\Rightarrow$ we expect particles to be in thermal equilibrium

Example: weak interactions (at $p < m_W$)

$\sigma \simeq G_F^2 p^2$ at temperature $T \longrightarrow \sigma = G_F^2 T^2$

interaction rate $\Gamma = n \sigma |v|$

we can guess: $n = N/a^3$

$p \sim a^{-1}$ but $p \sim T$

$\Rightarrow n \sim T^3$

$\Rightarrow \Gamma \sim G_F^2 T^5$

condition for equilibrium:

interaction rate $\times$ age of the universe $\gg 1$

$$\boxed{\Gamma / H \gg 1}$$

Some statistical mechanics

Distribution function: prob for a particle to be found at position $\vec{x}$, w/ momentum $\vec{p}$, at the time t

$$f(t, \vec{x}, \vec{p}) \xrightarrow[\substack{\text{homogeneity: } \cancel{\vec{x}} \\ \text{isotropy: } \vec{p} \to p \\ \text{thermal eq: } \cancel{t},\ T}]{} f(p, T)$$

A gas of particles at thermal eq follows the Bose-Einstein or Fermi-Dirac distributions

$$f(p,T) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1} \qquad \begin{array}{ll} + & FD \\ - & BE \end{array}$$

μ: chemical potential (response of the system to a change in N)

Density of states: for particles in a box of size L, in QM

$$\vec{p} = \frac{h}{L}\left(n_1 \hat{x} + n_2 \hat{y} + n_3 \hat{z}\right)$$

the density in $\{\vec{p}\}$ space is: $\frac{L^3}{h^3} = \frac{V}{h^3}$

in $\{\vec{x}, \vec{p}\}$: $\frac{1}{h^3}$

w/ g internal dof (eg spin states): $\frac{g}{h^3} = \frac{g}{(2\pi)^3} \leftarrow \hbar = \frac{h}{2\pi} = 1$

Thermodynamics:

$$\boxed{\begin{aligned} n(T) &= \frac{g}{(2\pi)^3}\int d^3p \; f(pT) \\ \rho(T) &= \frac{g}{(2\pi)^3}\int d^3p \; f(pT)\; E(p) \\ P(T) &= \frac{g}{(2\pi)^3}\int d^3p \; f(pT)\; \frac{p^2}{3E(p)} \end{aligned}}$$

Chemical potential can be set to zero (see Baumann and discussion below)

For most of the evolution $\mu_i \ll T$, for photons $\mu_\gamma = 0$ by definition

Calculation:

$$n(T) = \frac{g}{(2\pi)^3} \int d^3p \; f(p\,T) = \frac{g}{2\pi^2} \int dp \, \frac{p^2}{\exp \sqrt{p^2+m^3}/T \pm 1}$$

define $x = \frac{m}{T}$ $\quad z = \frac{p}{T}$

$$\Rightarrow n = \frac{g}{2\pi^2} I_\pm(x)\, T^3 \quad \text{and} \quad I_\pm(x) = \int_0^\infty dz \, \frac{z^2}{\exp(\sqrt{z^2+x^2}) \pm 1}$$

$$\rho = \frac{g}{2\pi^2} J_\pm(x)\, T^4 \qquad J_\pm(x) = \int_0^\infty dz \, \frac{z^2 \sqrt{z^2+x^2}}{\exp(\sqrt{z^2+x}) \pm 1}$$

$$P = \frac{g}{2\pi^2} K_\pm(x)\, T^4 \qquad K_\pm(x) = \frac{1}{3}\int_0^\infty dz \, \frac{z^4}{\left[\exp(\sqrt{z^2+x^2}) \pm 1\right] \sqrt{z^2+x^2}}$$

Relativistic limit   light particles : $m \ll T$ $x = \frac{m}{T} \to 0$, $p = E$

$$I_\pm(0) = \int_0^\infty dz \, \frac{z^2}{e^z \pm 1} \, , \qquad J_\pm(0) = \int dz \, \frac{z^3}{e^z \pm 1}$$

$$\begin{cases} n = \frac{\zeta(3)}{\pi^2} \, g T^3 \begin{cases} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{cases} \\ \rho = \frac{\pi^2}{30} \, g T^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases} \\ P = \frac{\rho}{3} \end{cases}$$

eg CMB $T = 2.73\,K$

$n_{\gamma 0} = 410$ photons cm^{-3}

$\rho_{\gamma 0} = 4.6 \, 10^{-34} \, g \, cm^{-3}$

$\Omega_\gamma h^2 = 2.5 \times 10^{-5}$

Non-relativistic limit   heavy particles / low T $x = \frac{m}{T} \gg 1$

$$I_\pm(x) = \int_0^\infty dz \, \frac{z^2}{\exp(\sqrt{z^2 + x^2} \pm 1)} \approx \int_0^\infty dz \, \frac{z^2}{\exp\sqrt{z^2 + x^2}}$$

fermions = bosons

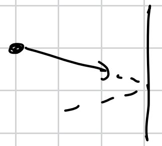
$$J_\pm(x) = \int_0^\infty dz \, \frac{z^2 \sqrt{z^2 + x^2}}{\exp(\sqrt{z^2 + x^2}) \pm 1} \approx \int dz \, \frac{z^2 \sqrt{z^2 + x^2}}{\exp\sqrt{z^2 + x^2}} \approx \int dz \, \frac{z^2 x \left(1 + \frac{z^2}{2x^2}\right)}{e^x \, e^{z^2/2x}}$$

$K_\pm(x) = \ldots$

$$\begin{cases} n = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \\ \rho = m n + \frac{3}{2} n T \\ P = n T \ll \rho \end{cases}$$

perfect gas! $\begin{cases} U = \frac{3}{2} N k_B T \\ PV = N k_B T \end{cases}$

Derivation of the pressure equation

Force exerted:  a particle bouncing off the wall

exchanges a momentum $= 2p_x$

In a time dt and for velocity v_x, particles in a volume $v_x\, dt\, dA$

hit the wall in an area dA, and those are in number

$$dN = dn\, dV = \frac{1}{2} \frac{g}{(2\pi)^3} f(p)\, A\, v_x\, dt$$

only those w/ $v_x > 0$

$$\Rightarrow dP = \frac{1}{A}\frac{dp}{dt} = \frac{1}{2} \times 2 \frac{g}{(2\pi)^3} f(p,T) \frac{p_x^2}{E}$$

$v_x = \dfrac{p_x}{E}$

The total pressure is

$$P = \frac{g}{(2\pi)^3} \int d^3p \; f(p,T) \frac{p_x^2}{E}$$

This is the only term that depends on the direction. I can replace it w/ its average

$$\frac{g}{(2\pi)^3} \int d^3p \; f(p,T) \frac{p^2}{3E}$$

Computation of the thermodynamic quantities

Relativistic limit $\qquad x = \frac{m}{T} \to 0$

$$I_\pm(0) = \int_0^\infty dz \, \frac{z^2}{e^z \pm 1}$$

$$\frac{1}{e^z \pm 1} = \frac{e^{-z}}{1 \pm e^{-z}} = e^{-z} \sum_{j=0}^{+\infty} (\mp e^{-z})^j = \sum_{j=1}^{\infty} (\mp 1)^{j-1} e^{-jz}$$

$$\Rightarrow I_\pm(0) = \sum_{j=1}^{\infty} (\mp 1)^{j-1} \int_0^\infty dz \, z^2 e^{-jz} = \sum_{j=1}^{\infty} (\mp 1)^{j-1} \frac{2}{j^3}$$

$$\int_0^\infty z^2 e^{-jz} dz = \frac{1}{j^3} \int_0^\infty z^2 e^{-z} dz = - \frac{e^{-z}}{j^3} z^2 \Big|_0^\infty + \frac{2}{j^3} \int e^{-z} z = \frac{2}{j^3} \int e^{-z} z$$

$$= - \frac{2 e^{-z} z^2}{j^3} \Big|_0^\infty + \frac{2}{j^3} \int e^{-z} = \frac{2}{j^3} \int e^{-z} = \frac{-2}{j^3} e^{-z} \Big|_0^\infty = \frac{2}{j^3}$$

Bosons: $I_-(0) = 2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} \dots\right) = 2\zeta(3) = 2 \times 1.202\dots$

Fermions $I_+(0) = 2\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} \dots\right) =$

$$= 2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} \dots\right) - 4\left(\frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{6^3} \dots\right)$$

$$= 2\zeta(3) - \frac{4}{2^3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots\right)$$

$$= \left(2 - \frac{1}{2}\right)\zeta(3) = \frac{3}{2}\zeta(3) = \frac{3}{4} I_-(0)$$

$$\left(\text{or} \quad \frac{1}{e^z+1} = \frac{1}{e^z-1} - \frac{2}{e^{2z}-1} \Rightarrow I_+(0) = I_-(0) - 2\left(\frac{1}{2}\right)^3 I_-(0) = \frac{3}{4} I_-(0) \right)$$

$$\Rightarrow n = \frac{\zeta(3)}{\pi^2} g T^3 \begin{cases} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{cases}$$

$$J_\pm(x) = \int_0^\infty dz \, \frac{z^2\sqrt{z^2+x^2}}{\exp\left(\sqrt{z^2+x^2}\right)\pm 1} \simeq \int_0^\infty dz \, \frac{z^3}{e^z \pm 1}$$

$$= \sum_{j=1}^\infty (\mp 1)^{j-1} \int_0^\infty z^3 e^{-jz} dz$$

$$\int_0^\infty z^3 e^{-jz} dz = \frac{1}{j^4}\int dz\, z^3 e^{-z} = -\frac{e^{-z} z^3}{j^4}\Big|_0^\infty + \frac{3}{j^4}\int_0^\infty e^{-z} z^2 dz =$$

$$= 0 + \frac{3}{j}\left(\frac{1}{j^3}\int e^{-z} z^2 dz\right) = \frac{6}{j^4}$$

$$\Rightarrow J_\pm(0) = 6\sum_{j=1}^\infty (\mp 1)^{j-1}\frac{1}{j^4}$$

Bosons: $J_-(0) = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right) = 6\,\zeta(4)$

Fermions: $J_+(0) = 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4}\dots\right) = 6\,\zeta(4) - 12\left(\frac{1}{2^4} + \frac{1}{4^4} + \dots\right)$

$$= 6\,\zeta(4) - \frac{12}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right) = \left(6 - \frac{12}{16}\right)\zeta(4) = \frac{21}{4}\zeta(4)$$

recall: $\zeta(4) = \frac{\pi^4}{90}$

Thus

$$\rho = \frac{g}{2\pi^2}\frac{\pi^4}{90} T^4 \begin{cases} 6 \\ \frac{21}{4} \end{cases} = \frac{\pi^2}{30} g T^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases}$$

$$P = \frac{g}{(2\pi)^3}\int d^3p\, f(p,T)\frac{p^2}{3E} \underset{p \gg m}{\approx} \frac{g^2}{(2\pi)^3}\int d^3p\, f(p,T)\frac{E}{3} = \frac{1}{3}\rho$$

Non-relativistic limit

$x \gg 1$: $\quad I_\pm(x) = \int_0^\infty dz \, \frac{z^2}{\exp(\sqrt{z^2+x^2} \pm 1)} \simeq \int_0^\infty dz \, \frac{z^2}{\exp\sqrt{z^2+x^2}}$

Main contribution comes from $z \ll x \Rightarrow \sqrt{z^2+x^2} \approx x\left(1+\frac{z^2}{2x^2}\right)$

$$I_\pm(x) \approx \int_0^{+\infty} dz \, \frac{z^2}{e^x \, e^{z^2/2x}} = e^{-x} \int dz \, z^2 \, e^{-z^2/2x}$$

$$= (2x)^{3/2} e^{-x} \int dz \, z^2 \, e^{-z^2} = \sqrt{\frac{\pi}{2}} \, x^{3/2} \, e^{-x}$$

$$\Rightarrow \quad n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

$$J_\pm(x) = \int_0^\infty dz \, \frac{z^2\sqrt{z^2+x^2}}{\exp(\sqrt{z^2+x^2}) \pm 1} \approx \int dz \, \frac{z^2\sqrt{z^2+x^2}}{\exp\sqrt{z^2+x^2}} \approx \int dz \, \frac{z^2 x\left(1+\frac{z^2}{2x^2}\right)}{e^x \, e^{z^2/2x}}$$

$$= \frac{x}{e^x} \int dz \left(z^2 + \frac{z^4}{2x^2}\right) e^{-z^2/2x} = x \, I_\pm(x) + \frac{e^{-x}}{2x} \int dz \, z^4 \, e^{-z^2/2x}$$

$$= x \, I_\pm(x) + \frac{3\sqrt{\pi}}{8} (2x)^{3/2} e^{-x} = x \, I_\pm(x) + \frac{3}{2}\sqrt{\frac{\pi}{2}} \, x^{3/2} e^{-x} =$$

$$= \left(x + \frac{3}{2}\right) I_\pm(x)$$

$$\int x^{2n} e^{-bx^2} = \frac{(2n-1)!!}{b^n \, 2^{n+1}} \sqrt{\frac{\pi}{b}}$$

$$\int z^4 e^{-z^2/(2x)} = \frac{3!!}{2^3\left(\frac{1}{2x}\right)^2} \sqrt{\frac{\pi}{1/(2x)}} = \frac{3\sqrt{\pi}}{8} (2x)^{5/2}$$

$$\Rightarrow \quad \rho = m n + \frac{3}{2} n T$$

$$K_\pm(x) = \frac{1}{3}\int_0^\infty dz \frac{z^4}{\left[\exp\left(\sqrt{z^2+x^2}\right) \pm 1\right]\sqrt{z^2+x^2}} \approx \frac{1}{3}\int_0^\infty dz \frac{z^4}{\left(\exp\sqrt{z^2+x^2}\right)\sqrt{z^2+x^2}}$$

$$\approx \frac{1}{3}\int_0^\infty dz \frac{z^4}{\left[\exp x\left(1+\frac{z^2}{2x^2}\right)\right] x \left(1+\frac{z^2}{2x^2}\right)} \approx \frac{e^{-x}}{3x}\int dz\, z^4 e^{-\frac{z^2}{2x}}\left(1-\frac{z^2}{2x^2}\right)$$

$$= \frac{e^{-x}}{3x}\int_0^\infty dz\, e^{-\frac{z^2}{2x}}\left(z^4 - \frac{z^6}{2x^2}\right)$$

$$\int_0^\infty dz\, e^{-\frac{z^2}{2x}} z^4 = \frac{3\sqrt{\pi}}{8}(2x)^{5/2}$$

$$\int z^6 e^{-z^2/(2x)} = \frac{5!!}{2^4\left(\frac{1}{2x}\right)^3}\sqrt{\frac{\pi}{1/(2x)}} = \frac{15\sqrt{\pi}}{16}(2x)^{7/2}$$

$$\Rightarrow K_\pm(x) \approx \frac{e^{-x}}{3x}\left(\frac{3\sqrt{\pi}}{8}(2x)^{5/2} - \frac{1}{2x^2}\frac{15\sqrt{\pi}}{16}(2x)^{7/2}\right)$$

$$= e^{-x}\sqrt{\pi}\left(\frac{x^{3/2}}{\sqrt{2}} - \frac{5}{2\sqrt{2}}x^{1/2}\right)$$

$$= I(x)\left(1 - \frac{5}{2}x^{-1}\right)$$

$$\Rightarrow \quad P = nT - \frac{5}{2}n\frac{T^2}{m}$$

Chemical potential

(M. Schwartz lecture notes)

μ : introduced for thermodynamical systems that can exchange particles (ie N not fixed) described by the grand-canonical ensemble.

$$\mu \equiv -T\left(\frac{\partial S}{\partial N}\right)_{E,V}$$

More particle species $\Rightarrow$ more chemical potentials

$$\frac{\partial S(E,V,N_1,N_2,\dots)}{\partial N_1} = -\frac{\mu_1}{T} \qquad \frac{\partial S(E,V,N_1,N_2,\dots)}{\partial N_2} = -\frac{\mu_2}{T} \quad \dots$$

(but one can define it equally well in the microcanonical!)

$$dS = \left(\frac{\partial S}{\partial E}\right)dE + \left(\frac{\partial S}{\partial V}\right)dV + \left(\frac{\partial S}{\partial N}\right)dN = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$$

$$\Rightarrow dE = TdS - PdV + \mu dN$$

$$\Rightarrow \mu = \left(\frac{\partial E}{\partial N}\right)_{S,V}$$

useful, but the condition of constant S makes it not intuitive

In the grand canonical

$$-k_B T \log Z = \langle E \rangle - TS - \mu\langle N \rangle$$

Ideal gas: $\mu = k_B T \log(n\lambda^3)$ $\qquad \lambda = \dfrac{h}{\sqrt{2\pi m k_B T}}$ thermal de Broglie wavelength

$$p = \frac{h}{\lambda} = \sqrt{\frac{2\pi}{3}}\,p_{rms}$$

air: $\lambda \sim 1.87 \times 10^{-11}\,m$, $n \sim (3.35 \times 10^{-9}\,m)^{-3}$ $\Big\}$ $n\lambda^3 \ll 1$

$\mu < 0$ when $n < \lambda^{-3}$ (ie when classical statistical mechanics applies)

$$\Rightarrow \quad n = \frac{1}{\lambda^3} \exp\left(\frac{\mu}{k_B T}\right) \qquad \text{for } n \to 2n, \ \mu \to \mu + \log 2$$

denser sistem: μ raises to 0 (for $n \approx \lambda^{-3}$)

dilute gas: $\mu < 0$ and interparticle interactions can be ignored

Ground state energy

Include eg binding energy (important eg if I have different molecules)

or rest energy mc^2 (important in the early universe)

If ε is the ground energy of a molecule. Say S_0, E_0 are computed w/ $\varepsilon = 0$.

$$\Rightarrow \quad S(E) = S_0(E_0) = S_0(E - N\varepsilon)$$

The chemical potential shifts as ε: $\mu = -T\left(\frac{\partial S}{\partial N}\right)_{E,V} = \mu_0 + \varepsilon$

and for an ideal gas $\quad n = \frac{1}{\lambda^3} \exp\left(\frac{\mu - \varepsilon}{k_B T}\right)$

μ is a potential: relative to the ground state

Chemical reactions

Include $1\ \mu_i$ for each specie

$$dE = T\,dS - P\,dV + \sum \mu_j\, dN_j$$

Example: $\quad 3H_2 + N_2 \longleftrightarrow 2NH_3$

- if I create 1 N_2 particle, I create 3 H_2 and 2 NH_3 disappear

$$dN_{H_2} = 3\, dN_{N_2} \qquad dN_{NH_3} = -2\, dN_{N_2}$$

entropy change: $\quad dS = \frac{\partial S}{\partial N_{H_2}} dN_{H_2} + \frac{\partial S}{\partial N_{NH_3}} dN_{NH_3} + \frac{\partial S}{\partial N_{N_2}} dN_{N_2}$

at equilibrium:

$$0 = dS = \frac{\partial S}{\partial N_{H_2}} dN_{H_2} + \frac{\partial S}{\partial N_{NH_3}} dN_{NH_3} + \frac{\partial S}{\partial N_{N_2}} dN_{N_2}$$

$$= \left(3\mu_{H_2} - 2\mu_{NH_3} + \mu_{N_2}\right) dN_{N_2}$$

$$\Rightarrow \boxed{3\mu_{H_2} + \mu_{N_2} = 2\mu_{NH_3}}$$

For eg monoatomic ideal gas $\quad N_x = \frac{1}{\lambda^3} \exp\left(-\frac{\varepsilon_x - \mu_x}{k_B T}\right)$

$$\frac{N_{H_2}^3 N_{N_2}}{N_{NH_3}^2} \approx \frac{\lambda_{NH_3}^6}{\lambda_{H_2}^9 \; \lambda_{N_2}^3} \underbrace{\exp\left(-\frac{3\varepsilon_{H_2} + \varepsilon_{N_2} - 2\varepsilon_{NH_3}}{k_B T}\right)}_{\exp - \frac{\Delta\varepsilon}{k_B T}} \underbrace{\exp\left(-\frac{3\mu_{H_2} + \mu_{N_2} - 2\mu_{NH_3}}{k_B T}\right)}_{1}$$

Proton content of the Universe (Matter - antimatter asymmetry)

$$p^+ + p^- \longleftrightarrow \gamma\gamma$$

$\Delta\varepsilon = 2 m_p c^2 = 2\,\text{GeV} \qquad (k_B T \gg \varepsilon \text{ for } T \gg 2\times 10^{13}\,K)$

Photon number not conserved: eg $\quad \gamma e^- \leftrightarrow e^- \gamma\gamma$

$$\Rightarrow \mu_\gamma + \mu_e = 2\mu_\gamma + \mu_e \quad \Rightarrow \mu_\gamma = 0$$

$\Rightarrow$ { for particles not associated w/ any conservation number, 3 can set $\mu = 0$

Suppose now $p^+ p^-$ are only produced from $\gamma\gamma \to p^+ p^-$.

Then $\mu_{p^+} + \mu_{p^-} = 0$ and at equilibrium (FD distribution) $m_{p^+} = m_{p^-}$

$$\Rightarrow \mu_{p^+} = \mu_{p^-} \quad \Rightarrow \mu_{p^+} = \mu_{p^-} = 0$$

Thus we should expect

$$n_{p^+} = n_{p^-} = \frac{1}{\lambda^3} e^{-\frac{\Delta E}{2 k_B T}} = \left(\frac{2\pi m_p k_B T}{h}\right)^{3/2} e^{-\frac{2 m_p c^2}{k_B T}} =$$

$$\approx 10^{-(10^{13})} \approx 0$$

But actually the reaction freezes-out at some point

$$\Gamma_{ann} = n \sigma v \qquad w/ \quad \sigma \sim m_p^{-2}$$

$$\stackrel{!}{=} (2\pi m_p k_B T)^{3/2} e^{-\frac{m_p c^2}{k_B T}} \frac{1}{m_p^2} \sqrt{\frac{3 k_B T}{m_p}}$$

$$H \approx \frac{k_B^2 T^2}{M_p}$$

$$\Rightarrow \quad \Gamma_{ann} < H \quad \text{for } T < T_f = 2.4 \times 10^{11} K$$

At T_f: $\quad n_{p^+} = n_{p^-} = 10^{23} m^{-3}$

$\Rightarrow$ Today $\quad n^0_{p^+} = n^0_{p^-} = 10^{23} m^{-3} \left(\frac{T_0}{T_f}\right)^3 = 10^{-10} m^{-3}$

Observation: $\quad n^0_{p^+} \approx 0.26 \, m^{-3}$

That means, at freeze out, ...

Relativistic species

At early enough times, all particles were relativistic $(SM: T \gtrsim 200\,GeV)$

$$\rho = \sum_i \frac{g_i}{2\pi^2} T_i^4 J_\pm(x_i)$$

define g_*:

$$\rho = \sum_i \frac{\pi^2}{30} g_i T_i^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases}$$

$$= \frac{\pi^2}{30} T^4 \underbrace{\left[\sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=f} g_i \left(\frac{T_i}{T}\right)^4 \right]}_{g_*}$$

Thermal eq $T_i = T \ \forall i$

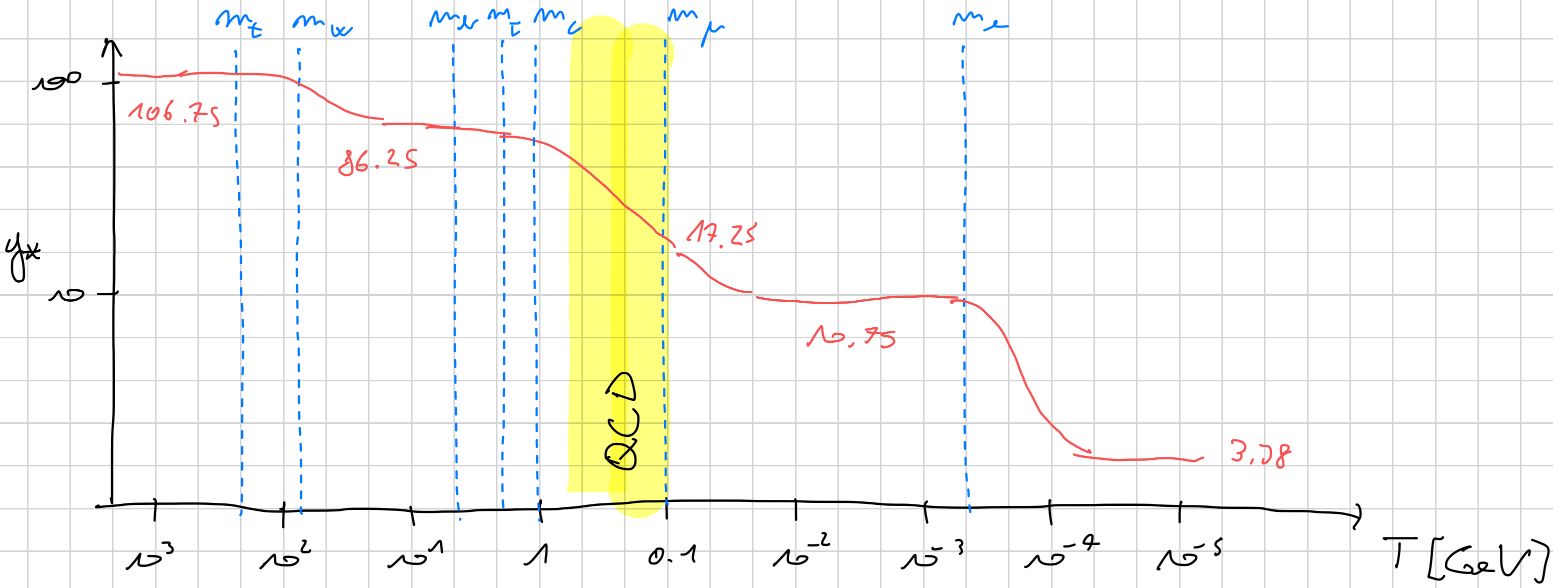
In the SM $(T \gtrsim \text{few} \times 10^2\,GeV)$

	G_μ^a	W_μ^a	B_μ	Q	u^c	d^c	L	e^c	H
family	1	1	1	3	3	3	3	3	1
spin / helicity	2	2	2	1	1	1	1	1	1
particle / antiparticle	1	1	1	2	2	2	2	2	2
weak isospin	1	3	1	2	1	1	2	1	2
color	8	1	1	3	3	3	1	1	1
statistics	1	1	1	7/8	7/8	7/8	7/8	7/8	1
g_*	16	6	2	63/2	63/4	63/4	21/2	21/4	4

$\boxed{106.75}$

$\Rightarrow$ at $T \gtrsim 200$ GeV $\qquad g_b = 28 \qquad g_f = 90$

$$g_* = g_b + \frac{7}{8} g_f = 106.75$$



Around mass thresholds and phase transitions, the distinction between relativistic and non-relativistic becomes less clear. There, one needs to use

$$g_* = \frac{\rho}{\frac{\pi^2}{30} T^4}$$

Neutrinos $\qquad$ SM: massless, LH neutrinos

$\Rightarrow$ ν has 1 dof $\qquad \nu, \bar{\nu} \to 2$

But neutrinos have mass

- Majorana mass term: only LH exist, ok
- Dirac mass term: RH neutrinos must decouple early enough (before BBN)

Entropy

Entropy is more useful than energy because it is normally conserved

Thermodynamics: $T dS = dU + P dV - \mu_i dN_i$

$$T\, d(sV) = d(\rho V) + P\, dV - \mu_i\, d(n_i V)$$

$$(Ts - \rho - P + \mu_i n_i)\, dV + V\left(T \frac{ds}{dT} - \frac{d\rho}{dT} + \mu_i \frac{dn_i}{dT}\right) dT = 0$$

$$\Rightarrow \quad s = \frac{\rho + P - \mu_i n_i}{T} \qquad \frac{ds}{dT} = \frac{1}{T}\left(\frac{d\rho}{dT} - \mu_i \frac{dn_i}{dT}\right)$$

Now compute

$$\frac{d(s a^3)}{dt} = a^3 \frac{ds}{dt} + 3 a^3 s H = 3a^3 s H + \frac{dT}{dt}\frac{ds}{dT} =$$

$$= 3a^3 s H + a^3 \frac{dT}{dt} \frac{1}{T}\left(\frac{d\rho}{dT} - \mu_i \frac{dn_i}{dT}\right) = 3a^3 s H + a^3 \frac{1}{T}\frac{d\rho}{dt} - a^3 \frac{\mu_i}{T}\frac{dn_i}{dt}$$

use continuity eq: $\frac{d\rho}{dt} = -3H(\rho + P) = -3H(Ts + \mu_i n_i)$

$$= 3a^3 s H - 3H a^3 s - 3H a^3 \frac{\mu_i}{T} n_i - a^3 \frac{\mu_i}{T}\frac{dn_i}{dt}$$

$$\Rightarrow \quad \frac{d(s a^3)}{dt} = -\frac{\mu_i}{T}\frac{d(n_i a^3)}{dt}$$

If $\mu = 0$ $\Rightarrow$ $\frac{d(s a^3)}{dt} = 0$ entropy is conserved

In most cases, $\mu = 0$. There are other cases of entropy non-conservation, eg the decay of a heavy particle which was out of equilibrium

Entropy & relativistic species

Collection of particle species: $s = \sum_i \frac{\rho_i + P_i}{T_i}$

For a single, relativistic specie: $P = \rho/3 \Rightarrow s = \frac{2\pi^2}{45} g\, T^3$

Define $s \equiv \frac{2\pi^2}{45} g_{*S} T^3$

$$g_{*S} = \sum_{b} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{f} g_i \left(\frac{T_i}{T}\right)^3$$

If all relativistic particles are in thermal eq $\Rightarrow$ $g_{*S} = g_*$

(for the SM, valid for $t \lesssim 1s$ ie $T \gtrsim 1\,MeV$ (neutrino decoupling))

Useful: $s \simeq 1.8\, g_*\, n_\gamma$ (before e^+e^- annihilation)

Yield: For a particle ψ out of thermal equilibrium

$$Y_\psi \equiv \frac{n_\psi}{s}$$

If there are no number-changing interactions of ψ

$\Rightarrow n_\psi \propto a^{-3} \Rightarrow n_\psi a^3 = \text{const}$

$$\Rightarrow Y_\psi = \frac{n_\psi}{s} = \text{const}$$

useful eg for DM, or baryogenesis $\frac{n_B}{s} = \frac{n_b - n_{\bar{b}}}{s}$

or more often $\eta = \frac{n_B}{n_\gamma} = 1.8\, g_{*S} \frac{n_B}{s}$

Temperature behaviour

$S = \text{const} \Rightarrow g_{*s} T^3 a^3 = \text{const} \Rightarrow T \propto g_{*s}^{-1/3} a^{-1}$

Away from mass thresholds $g_* \approx$ constant $\Rightarrow T \propto a^{-1}$

Expansion history

From the formulas we saw above

$$H^2 = \frac{\rho}{3 M_p^2} = \frac{\pi^2}{90} g_* \frac{T^4}{M_p^2}$$

Away from mass thresholds $a \propto t^{1/2}$ (in rad domination)

Temperature scales as $\frac{T}{1\,\text{MeV}} \approx 1.5\, g_*^{-1/4} \left(\frac{1\,s}{t}\right)^{1/2}$