$u(t,x)$ $\overline{u(-t,x)}$

Prop L^2

se $1 < p < 1 + \frac{4}{d}$ $\forall\, u_0 \in L^2(\mathbb{R}^d)$

$\exists\, T$ ed una soluzione unica

$u \in C^0([0,T), L^2(\mathbb{R}^d)) \cap L^q([0,T], L^{p+1}(\mathbb{R}^d))$

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$$

$T \geq T(\|u_0\|_{L^2}) > 0$

Inoltre $\exists\, V \ni u_0$ intorno in $L^2(\mathbb{R}^d)$ t.c.

$V \ni v_0 \longrightarrow v(t,x)$

$V \longrightarrow C^0([0,T), L^2(\mathbb{R}^d)) \cap L^q([0,T], L^{p+1}(\mathbb{R}^d$

è Lipsch.

Prop $1 < p < d^* = \begin{cases} +\infty & \text{se } d \leq 2 \\ \frac{d+2}{d-2} & d \geq 3 \end{cases}$

$\forall\, u_0 \in H^1(\mathbb{R}^d)$ $\exists!$ u soluzione di

$$u = \underbrace{e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds}_{\phi(u)},$$

con

$$u \in C^0([0,T], H^1(\mathbb{R}^d)) \cap L^q([0,T], W^{1,p+1}(\mathbb{R}^d))$$

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$$

$$T \geq T(\|u_0\|_{H^1}) > 0$$

ed $\exists$ $V \ni u_0$ in $H^1(\mathbb{R}^d)$ t.c.

$$V \ni v_0 \longrightarrow v(t)$$

$$V \to C^0([0,T], H^1(\mathbb{R}^d)) \cap L^q([0,T], W^{1,p+1}(\mathbb{R}^d))$$

Dim Sketch

$$E^1(T,a) = \{ v \in C^0([0,T], H^1(\mathbb{R}^d)) \cap L^q([0,T], W^{1,p+1}(\mathbb{R}^d)) :$$

$$|v|_T^{(1)} := |v|_{L^\infty([0,T), H^1)} + |v|_{L^q([0,T), W^{1,p+1})} \leq a \}$$

si fa in modo che si abbia

$$\phi : E^1(T, a) \circlearrowleft$$

e che la mappa sia una contrazione

$$|v|_T^{(1)} = |v|_T + |\nabla v|_T$$

$$|\nabla \phi(u)|_T = \left| e^{it\Delta} \nabla u_0 - i\lambda \int^t e^{i(t-s)\Delta} \nabla(|u|^{p-1} u)\, ds \right|_T$$

$$\leq c_0 |u_0|_{H^1} + c_0 \left| |u|^{p-1} \nabla u \right|_{L^{q'}([0,T], L^{\frac{p+1}{p}})}$$

$$\leq c_0 |u_0|_{H^1} + c_0 \left| \| u |^{p-1}_{L^{p+1}_x} |\nabla u|_{L^{p+1}_x} \right|_{L^{q'}((0,T))}$$

$$\frac{p}{p+1} = \frac{p-1}{p+1} + \frac{1}{p+1}$$

$$\leq c_0 |u_0|_{H^1} + c_0 \left| |u|_{L^{p+1}_x} \right|^{p-1}_{L^{\beta}(0,T)} |\nabla u|_{L^q((0,T), L^{p+1})}$$

$$\frac{1}{q'} = \frac{p-1}{\beta} + \frac{1}{q}$$

$$= c_0 |u_0|_{H^1} + c_0 |u|^{p-1}_{L^{\beta}((0,T), L^{p+1}_x)} |\nabla u|_{L^q((0,T), L^{p+1})}$$

$$|\nabla \phi(u)|_T \leq$$

$$\leq c_0 |u_0|_{H^1} + c_0 |u|^{p-1}_{L^{\beta}((0,T), L^{p+1}_x)} |\nabla u|_{L^q((0,T), L^{p+1})}$$

$$p < 1 + \frac{4}{d} \implies \beta < q$$

$$u \in C^0((0,T), H^1) \qquad 1 < p < d^*$$

$$\Rightarrow \qquad H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$$

$$u \in C^0([0,T], H^1) \longrightarrow L^\beta((0,T), L^{p+1}_x)$$

$$|u|_{L^\beta((0,T), L^{p+1}_x)} = \Big| \, |u|_{L^{p+1}_x} \Big|_{L^\beta(0,T)}$$

$$\le c_{sob} \Big| \, |u|_{H^1_x} \Big|_{L^\beta(0,T)} \le$$

$$\le C_{sob} \, |u|_{L^\infty((0,T], H^1_x)} \, T^{\frac{1}{\beta}}$$

$$|\nabla \phi(u)|_T \le$$

$$\le C_0 |u_0|_{H^1} + C_0 \, |u|^{p-1}_{L^\beta((0,T), L^{p+1}_x)} \, |\nabla u|_{L^q((0,T), L^{p+1})}$$

$$\le C_0 |u_0|_{H^1} + C_0 \, C_{sob}^{p-1} \, |u|^{p-1}_{L^\infty((0,T), H^1_x)} \, T^{\frac{p-1}{\beta}} \, |\nabla u|_{L^q((0,T), L^{p+1})}$$

$$|\nabla \phi(u)|_T \le C_0 |u_0|_{H^1} + C_0 C_{sob}^{p-1} T^\alpha \left(|u|^{(1)}_{T} \right)^p$$

$$|\phi(u)|^{(1)}_T \le C_0 |u_0|_{H^1} + C_d \, T^\alpha \left(|u|^{(1)}_T \right)^p$$

$$|\phi(u)|_T^{(1)} \leq c_0 |u_0|_{H^1} + C_d T^\alpha a^p$$

$$C_d T^\alpha a^{p-1} < \frac{1}{2}$$

$$\leq c_0 |u_0|_{H^1} + \frac{a}{2} < \frac{a}{2} + \frac{a}{2} = a$$

$$a > 2 c_0 |u_0|_{H^1}$$

$$\phi : E^{(1)}(T, a) \circlearrowleft$$

scegliendo eventualmente T un po' più piccolo si ottiene che ϕ è una contrazione

$$\phi(u) - \phi(v) = -i\lambda \int_0^t e^{(t-s)\Delta} \left(|u|^{p-1}u - |v|^{p-1}v \right) ds$$

$$|\phi(u) - \phi(v)|_T^{(1)} = |\phi(u) - \phi(v)|_T + |\nabla(\phi(u) - \phi(v))|_T$$

$$|\nabla(\phi(u) - \phi(v))|_T = \left| \int_0^t e^{(t-s)\Delta} \left(\nabla(|u|^{p-1}u) - \nabla(|v|^{p-1}v) \right) ds \right|_T$$

$$\leq c_0 \left| \nabla(|u|^{p-1}u) - \nabla(|v|^{p-1}v) \right|_{L^{q'}([0,T], L^{\frac{p+1}{p}})}$$

$$= c_0 \left| |u|^{p-1} \nabla u - |v|^{p-1} \nabla v \right|_{L^{q'}((0,T), L^{\frac{p+1}{p}})}$$

$$= C_0 \left\| \left(|u|^{p-1}\nabla u - |u|^{p-1}\nabla v\right) + \left(|u|^{p-1}\nabla v - |v|^{p-1}\nabla v\right)\right\|_{L^{q'}((0,T), L^{\frac{p+1}{p}})}$$

$$\leq C_0 \left\| |u|^{p-1}(\nabla u - \nabla v)\right\|_{L^{q'}((0,T), L^{\frac{p+1}{p}})} + \left\| \left(|u|^{p-1} - |v|^{p-1}\right)\nabla v\right\|_{L^{q'}((0,T), L^{\frac{p+1}{p}})}$$

$$\left\| |u|^{p-1}(\nabla u - \nabla v)\right\|_{L^{q'}((0,T), L^{\frac{p+1}{p}})} \leq \left\| \left\| |u|^{p-1}(\nabla u - \nabla v)\right\|_{L_x^{\frac{p+1}{p}}}\right\|_{L^{q'}(0,T)}$$

$$\leq \left\| \; \|u\|_{L^{p+1}}^{p-1} \|\nabla u - \nabla v\|_{L^{p+1}}\right\|_{L^{q'}(0,T)}$$

$$\leq \|u\|_{L^{\beta}((0,T), L^{p+1})}^{p-1} \qquad \|\nabla u - \nabla v\|_{L^{q}((0,T), L^{p+1})}$$

$$\leq C_{sob}^{p-1} \|u\|_{L^{\infty}((0,T), H^1)}^{p-1} \; T^{\frac{p-1}{\beta}} \; \|\nabla u - \nabla v\|_{L^{q}((0,T), L^{p+1})}$$

<u>Prop</u> Sia $u(t)$ la soluzione di quest'ultimo proposizione. Allora

$$E(u(t)) = E(u_0)$$

$$Q(u(t)) = Q(u_0) = \frac{1}{2}\int |u_0|^2 \, dx$$

$$P_j(u(t)) = P_j(u_0)$$

$$P_j(u) = \pm\frac{1}{2}\langle i\,\partial_j u, u\rangle$$

$\underline{\text{Dim}}$ $u \in C^0((-S^*, T^*), H^1(\mathbb{R}^d))$ massimale

dimostreremo che $\exists$ $T > 0$ t.c. le

$*$ sono vere in $[-T, T]$. Questo implicherà

che $E(u(t))$, $Q(u(t))$, $P_j(u(t))$ sono

loc costanti ~~in~~ t e ucciso

$\cdot$ $E, Q, P_j \in C^0(H^1, \mathbb{R})$

e $\underset{\mathbb{R}}{t} \to u(t) \in H^1$ è continua

questo sono continue e quindi costanti in $(-S^*, T^*)$

$\frac{d}{dt} Q(u(t))$

Tronchiamo la NLS

$\varphi \in C_c^\infty(\mathbb{R}, [0,1])$ $\qquad \varphi = 1$ vicino a 0

$\varphi|_{[-1,1]} = 1$ $\qquad$ supp $\varphi \subseteq [-2, 2]$

$$Q_n = \varphi\left(\frac{\sqrt{-\Delta}}{n}\right) : L^p(\mathbb{R}^d) \circlearrowleft \quad \chi_{[0,1]}\left(\frac{\sqrt{-\Delta}}{n}\right)$$

$\forall p \in \forall d$

$$\widehat{Q_n u} = \varphi\left(\frac{|\xi|}{n}\right) \hat{u}$$

$\chi_{[0,1]}(\sqrt{-\Delta}) : \cancel{L^p(\mathbb{R}^d) \circlearrowleft}$ se $p \neq 2$ $\quad d > 1$

$$\Phi_n u(x) = \int_{\mathbb{R}^d} e^{ix\xi}\, \varphi\left(\frac{|\xi|}{n}\right) \hat{u}(\xi)\, d\xi =$$

$$= \check{\Phi}\left(\frac{\cdot}{n}\right) * u \quad (x)$$

$$= n^d\, \check{\Phi}(n\,\cdot) * u$$

$$= n^d \int_{\mathbb{R}^d} \check{\Phi}(n(x-y))\, u(y)\, dy$$

$$\Phi(\xi) = \varphi(|\xi|)$$

$$\Phi\left(\frac{\xi}{n}\right) = \varphi\left(\frac{|\xi|}{n}\right)$$

$$\left[\Phi\left(\frac{\cdot}{n}\right)\right]^{\vee}(x) = n^d\, \check{\Phi}(n x)$$

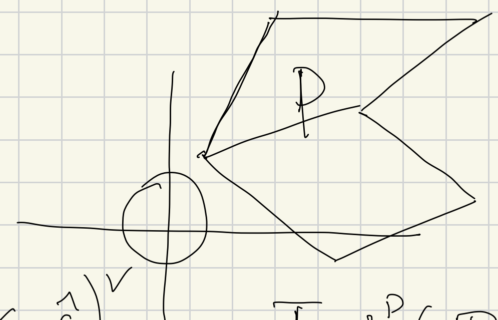
$$|\Phi_n u|_{L^p(\mathbb{R}^d)} \le n\, |\check{\Phi}(n\,\cdot)|_{L^1(\mathbb{R}^d)}\, \|u\|_{L^p(\mathbb{R})}$$

$$= |\check{\Phi}|_{L^1(\mathbb{R}^d)}\, \|u\|_{L^p(\mathbb{R})}$$

$1 < p < +\infty$

$p \neq 2$

$f \in L^p(\mathbb{R}^2)$



$$Tf = \left(\chi_D \hat{f}\right)^{\vee} \qquad T : L^p(\mathbb{R}^2)$$

$$\begin{cases} i\,\partial_t u = -\Delta u + \lambda\, |u|^{p-1} u \\ u|_{t=0} = u_0 \end{cases}$$

$$P_{2n}\, u = \left(\chi_{D(0,2n)}\, \hat{u}\right)^{\vee}$$

$$\begin{cases} i\,\partial_t u_n = \underbrace{-P_{2n}\Delta u_n + \lambda\, Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right)}_{F_n(u_n)} \\ u_n|_{t=0} = Q_n u_0 \end{cases}$$

Queste sono delle ODE in ambito infinito dimensionale

$F_n : H^1 \longrightarrow H^1$ Loc Lipschitz

$u \to P_{2n}\Delta u$ è un op lineare limitato in H^1

$$\|P_{2n}\Delta u\|_{H^1} = \|\langle\xi\rangle\, \chi_{D(0,2n)}(\xi)\, |\xi|^2\, \hat{u}\|_{L^2}$$

$$\le n^2\, \|\langle\xi\rangle\, \hat{u}\|_{L^2} = n^2\, \|u\|_{H^1}$$

$$\begin{tikzcd} u \arrow[d] & H^1 \arrow[d] & \\ Q_n u & u \in H^1 \arrow[r] \arrow[dr] & |u|^{p-1}u \in H^{-1} \arrow[d, "Q_n(|u|^{p-1}u)"'] \\ & & H^1 \end{tikzcd}$$

è ~~loc~~ lipsch

$u \longrightarrow Q_n\left(|Q_n u|^{p-1} Q_n u\right)$ è loc Lipsch

$\exists$ delle soluzioni massimale

$$u_n \in \underbrace{C^1\left(\left(-T_1(n), T_2(n)\right), H^1\right)}$$

$$[-S, T] \subseteq \left(-T_1(n), T_2(n)\right)$$

$$u \longleftarrow u_n$$

Se $\qquad T_2(n) < +\infty$

$$\Longrightarrow \quad \lim_{t \to T_2(n)^-} \|u_n(t)\|_{H^1} = +\infty$$