Prop $\forall\, 1 < p < d^*$ e $\forall\, u_0 \in H^1$ $\exists\, T = T(\|u_0\|_{H^1}) > 0$

ed una unica soluzione

$$u \in \underbrace{C^0([0,T], H^1(\mathbb{R}^d)) \cap L^q([0,T], W^{1,p+1}(\mathbb{R}^d))}_{\|\cdot\|^{(1)}_T}$$ dove

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$$

Ho scritto che $\exists$ V_{u_0} un intorno di u_0 in $H^1(\mathbb{R}^d)$ t.c.

$$v_0 \longrightarrow v$$

$$V_{u_0} \longrightarrow C^0([0,T], H^1(\mathbb{R}^d)) \cap L^q([0,T], W^{1,p+1}(\mathbb{R}^d))$$

~~Lipchitz~~ Continuo

Lineare - Ponce

Dim di quest'ultimo punto.

$V_{u_0} \in H^1$

$\forall \quad v_0 \in V_{u_0} \quad \rightsquigarrow \quad v \in \underbrace{C^0([0,T], H^1) \cap L^q([0,T], W^{1,p+1})}_{|\cdot|^{(1)}_T}$

$$|v-u|^{(1)}_T = |v-u|_T + |\nabla(v-u)|_T$$

1) Per $T = T(|u_0|_{H^1})$ opportuno $\exists \quad C_{d,p}$

$$|v-u|_T \leq C_{d\,p} \, |v_0 - u_0|_{H^1}$$

$$|\nabla u - \nabla v|_T \le c_0 |u_0 - v_0|_{H^1} + c_0 \left| |u|^{p-1}\nabla u - |v|^{p-1}\nabla v \right|_{L^{q'}([0,T],\, L^{\frac{p+1}{p}})}$$

$$\le c_0 |u_0 - v_0|_{H^1} + \underbrace{c_0 \left| \left(|u|^{p-1} - |v|^{p-1}\right)\nabla u \right|_{L^{q'}([0,T],\, L^{\frac{p+1}{p}})}}_{I}$$

$$+ \underbrace{c_0 \left| |v|^{p-1}(\nabla u - \nabla v) \right|_{L^{q'}([0,T],\, L^{\frac{p+1}{p}})}}_{II}$$

$$\frac{p-1}{\beta} + \frac{1}{q} = \frac{1}{q'}$$

$$II \le |v|^{p-1}_{L^{\beta}([0,T],\, L^{p+1})} \underbrace{|\nabla u - \nabla v|_{L^{q}([0,T],\, L^{p+1})}}_{\le |u-v|^{(1)}_T}$$

$$\le T^{\frac{p-1}{\beta}} |v|^{p-1}_{L^\infty([0,T],\, H^1)} |u-v|^{(1)}_T$$

$$\le C_p T^{\frac{p-1}{\beta}} \Big(\underbrace{|u|^{p-1}_{L^\infty([0,T],\, H^1)}}_{C(|u_0|_{H^1})} + |u-v|^{p-1}_{L^\infty([0,T],\, H^1)} \Big) |u-v|^{(1)}_T$$

$$II \le C(|u_0|_{H^1})\, T^{\frac{p-1}{\beta}} |u-v|^{(1)}_T + C_p T^{\frac{p-1}{\beta}} \left(|u-v|^{(1)}_T\right)^{p}$$

$$I = \left\| (|u|^{p-1} - |v|^{p-1}) \nabla u \right\|_{L^{q'}([0,T], L^{\frac{p+1}{p}})}$$

$$\frac{p-1}{\beta} + \frac{1}{q} = \frac{1}{q'} \qquad \frac{p}{p+1} = \frac{p-1}{p+1} + \frac{1}{p+1}$$

$$\leq \left\| |u|^{p-1} - |v|^{p-1} \right\|_{L^{\frac{\beta}{p-1}}([0,T], L^{\frac{p+1}{p-1}})} \underbrace{\|\nabla u\|_{L^{q}([0,T], L^{p+1})}}_{\leq C(\|u_0\|_{H^1})}$$

$p \geq 2$, $\boxed{1 < p < 2}$

Se $1 < p < 2 \iff 0 < p-1 < 1$

$v \to |v|^{p-1}$ è concava

$$\left| |u|^{p-1} - |v|^{p-1} \right| \leq \left| |u| - |v| \right|^{p-1} \leq |u-v|^{p-1}$$

(Se $f : [0,+\infty) \to [0,+\infty)$ è concava con $f(0) = 0$,
allora $f(x+y) \leq f(x) + f(y) \quad \forall\, x, y \geq 0$)

Se $|u| \geq |v|$

$$|u| = (|u| - |v|) + |v|$$

$$|u|^{p-1} \leq \left| |u| - |v| \right|^{p-1} + |v|^{p-1}$$

$$0 \leq |u|^{p-1} - |v|^{p-1} \leq \left| |u| - |v| \right|^{p-1} \leq |u-v|^{p-1}$$

$$I \le |u-v|^{p-1}_{L^\beta([0,T],L^{p+1})} \; C(|u_0|_{H^1})$$

$$\le T^{\frac{p-1}{\beta}} \, |u-v|^{p-1}_{L^\infty([0,T],L^{p+1})} \; C(|u_0|_{H^1})$$

$L^{p+1}(\mathbb{R}^d)$ $\qquad \frac{1}{p+1} = \frac{1}{2} - \frac{s}{d}$ $\qquad 0<s<1$

$s = \frac{d}{2} - \frac{d}{p+1} = \frac{2}{q}$ $\qquad$ Con Gagliardo Nirenberg

$$|u-v|_{L^{p+1}} \le |u-v|^{1-s}_{L^2} \; |\nabla(u-v)|^{s}_{L^2}$$

$$I \le T^{\frac{p-1}{\beta}} \, |u-v|^{(1-s)(p-1)}_{L^\infty([0,T],L^2)} \; |\nabla(u-v)|^{s(p-1)}_{L^\infty([0,T],L^2)} \; C(|u_0|_{H^1})$$

$$\le T^{\frac{p-1}{\beta}} \, |u-v|^{(1-s)(p-1)}_{T} \; |\nabla(u-v)|^{s(p-1)}_{T} \; C(|u_0|_{H^1})$$

$$\le C(|u_0|_{H^1}) \, T^{\frac{p-1}{\beta}} \, |u_0-v_0|^{(1-s)(p-1)}_{H^1} \left(|u-v|^{(1)}_{T} \right)^{s(p-1)}$$

$\gamma = \frac{1}{s(p-1)}$ $\qquad \gamma' = \frac{\gamma}{\gamma-1}$

$$\le C(|u_0|_{H^1}) \, T^{\frac{p-1}{\beta}} \left(\frac{|u-v|^{(1)}_{T}}{\gamma} + \frac{|u_0-v_0|^{(1-s)(p-1)\gamma'}_{H^1}}{\gamma'} \right)$$

$$|u-v|_T^{(1)} \le C_0 |u_0 - v_0|_{H^1} + C(|u_0|_{H^1}) |u_0 - v_0|_{H^1}^{(1-s)(p-1)\gamma'}$$

$$+ C(|u_0|_{H^1}) T^{\frac{p-1}{\beta}} |u-v|_T^{(1)} + C(|u_0|_{H^1}) T^{\frac{p-1}{\beta}} \left(|u-v|_T^{(1)}\right)^{p-1}$$

$|u+v|_T^{(1)}$

$$A = \min\{1, (1-s)(p-1)\gamma'\}$$

$$\gamma' = \frac{1}{s(p-1)}$$

$$s = \frac{2}{9}$$

Suppongo che

$$|u-v|_T^{(1)} \le M |u_0 - v_0|_{H^1}^A$$

$$M |u_0 - v_0|_{H^1}^A \ge M |u_0 - v_0|_{H^1}$$

$$|u-v|^{(1)}_T \le M\, |u_0-v_0|^{A}_{H^1}$$

$$|u-v|^{(1)}_T \le C_0 |u_0-v_0|_{H^1} + C(|u_0|_{H^1})\, |u_0-v_0|^{(1-s)(p-1)\gamma'}_{H^1}$$

$$+\underbrace{C(|u_0|_{H^1})\, T^{\frac{p-1}{3}} \left(1 + M^{p-1} |u_0-v_0|^{A(p-1)}_{H^1}\right)}_{<\frac{1}{2}} |u-v|^{(1)}_T$$

$$|u-v|^{(1)}_T \le \underbrace{2C_0}_{<\frac{M}{4}} |u_0-v_0|_{H^1} + \underbrace{2C(|u_0|_{H^1})}_{<\frac{M}{4}} |u_0-v_0|^{(1-s)(p-1)\gamma'}_{H^1}$$

$$|u-v|^{(1)}_T \le \left(\frac{M}{4} + \frac{M}{4}\right) |u_0-v_0|^{A}_{H^1}$$

$$\le \frac{M}{2} |u_0-v_0|^{A}_{H^1}$$

$$|u-v|^{(1)}_T \le M\, |u_0-v_0|^{A}_{H^1} \Longrightarrow |u-v|^{(1)}_T \le \frac{M}{2} |u_0-v_0|^{A}_{H^1}$$

$$\underbrace{(1-s)\overbrace{(p-1)}^{\to 0}\gamma'}_{\downarrow\, p\to 1^+ \atop 0}$$

$p \to 1^+$

$s \to 0$

$|u|^{p-1}u$

$$\frac{1}{p+1} = \frac{1}{2} - \frac{s}{d}$$

$$s = \frac{2}{q} \qquad q \xrightarrow{p\to 1^+} +\infty$$

$$\gamma = \frac{1}{s(p-1)} \xrightarrow{p\to 1^+} +\infty \Rightarrow \gamma' \longrightarrow 1$$

$\varphi \in C_0^\infty(\mathbb{R}, [0,1])$ $\quad \varphi \equiv 1$ vicino a 0, poi

$\operatorname{supp} \varphi \subseteq [-1,1]$

$Q_n = \varphi\left(\frac{\sqrt{-\Delta}}{n}\right)$ $\quad n \in \mathbb{N}$ $\quad P_n = \chi_{[0,1]}\left(\frac{\sqrt{-\Delta}}{n}\right)$

$$\begin{cases} i\partial_t u_n = -P_n \Delta u_n + \lambda Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right) \\ u_n(0) = Q_n u_0 \end{cases}$$

$e^{i\vartheta_0}$

$(1-P_n)$

$u_n = P_n u_n$

$u_n \in C^1((-T_1(n), T_2(n)), H^1)$

se $T_2(n) < +\infty \implies \lim_{t \to T_2(n)^-} |u_n(t)|_{H^1} = +\infty$

$$E_n(v) = \frac{1}{2}|P_n \nabla v|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n v|^{p+1}\,dx$$

$Q(v)$ $\qquad Q(u_n(t))$

$P_j(v)$ $\qquad P_j(u_n(t))$ $\qquad$ sono costanti del moto

$$|u_n(t)|_{H^1} = |P_n u_n(t)|_{H^1}$$

$u_n(t) = P_n u_n(t)$

$(1-P_n)u_n(t)$

$$\begin{cases} -i\partial_t (1-P_n)u_n = \underbrace{(1-P_n)(-1)P_n \Delta u_n}_{0} + \\ \qquad + (1-P_n)Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right) \\ (1-P_n)u_n(0) = (1-P_n)Q_n u_0 \end{cases}$$

$(1-P_n)P_n = 0$ $\qquad \widehat{(1-P_n)P_n f} = $

$$= \left(1 - \chi_{[0,1]}\left(\frac{|\xi|}{n}\right)\right)\underbrace{\chi_{[0,1]}\left(\frac{|\xi|}{n}\right)\hat{f}}_{0}$$

$$\widehat{(1-P_n)Q_n f} = \overbrace{\left(1-\chi_{[0,1]}\left(\frac{|\xi|}{n}\right)\right)}^{\chi_{\mathbb{R}^d \setminus D(0,n)}} \underbrace{\varphi\left(\frac{|\xi|}{n}\right)}_{\text{ha supporto in } |\xi| \le n} \hat{f} = 0$$

$$\begin{cases} \partial_t (1-P_n)u_n = 0 \\ (1-P_n)u_n(0) = 0 \end{cases}$$

$u_n = P_n u_n$

$$|u_n(t)|_{H^1} = |P_n u_n(t)|_{H^1} = \left|\langle \xi \rangle \chi_{D(0,n)} \hat{u}_n(t)\right|_{L^2}$$

$$\le \langle n \rangle |u_n(t)|_{L^2} = \langle n \rangle |Q_n u_0|_{L^2}$$

$$= \langle n \rangle \left|\varphi\left(\frac{\xi}{n}\right) u_0\right|_{L^2} \le \langle n \rangle |u_0|_{L^2}$$

Questo esclude il blow up: le u_n sono globalmente definite

Fissiamo $M > |u_0|_{H^1}$ e poniamo

$$\vartheta_n = \sup\left\{\tau > 0 : |u_n(t)|_{H^1} \le 2M \ \ \forall t \in [0,\tau]\right\}$$

Dimostreremo che $\exists$ $T(M) > 0$ t.c.

$\vartheta_n \ge T(M)$ $\forall n$.

Dimostriamo che $u_n \to u$ in $C^0([0,T(M)], H^1)$

$$Q(u(t)) = \lim_{n \to +\infty} Q(u_n(t))$$