Data Science for Insurance

Introduction to Copulas

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Problem

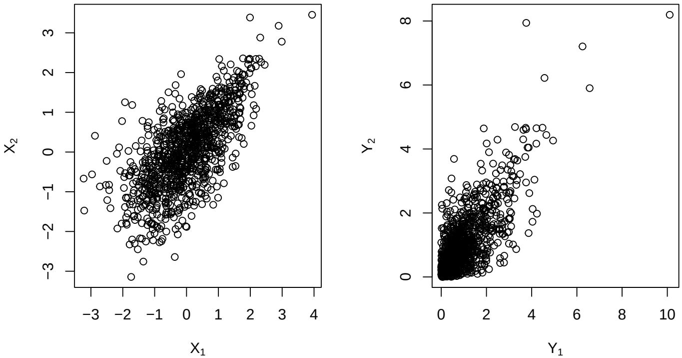
Consider two bivariate data sets

$$(x_{i1}, x_{i2}); \quad (y_{i1}, y_{i2}), \quad i \in \{1, \ldots, n\}$$

Each consists of $n = 1000$ independent observations (that is, a realization of independent copies) of a bivariate random vector (X_1, X_2) (respectively, (Y_1, Y_2)).

Comparing the two data sets in terms of dependence means comparing the way X_1 and X_2 are related with the way Y_1 and Y_2 are related.

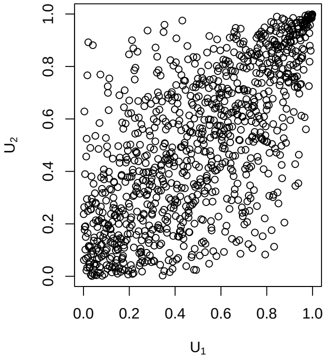
Scatter plots

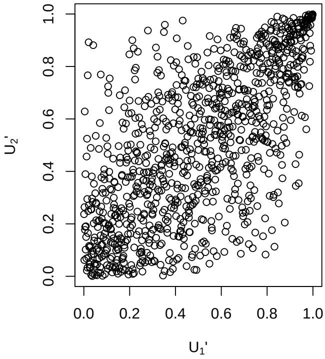


For which data is the dependence between the two variables *larger*?

Data transformation

$$(X_1, X_2) \quad \to \quad (U_1, U_2) \in [0,1]^2$$
$$(Y_1, Y_2) \quad \to \quad (U_1', U_2') \in [0,1]^2$$





The new observations give us insight in the actual dependence structure (copula) underlying our data sets **x** and **y**.

Rank transformation

- Consider the bivariate sample (x_{i1}, x_{i2}), $i = 1, \ldots, n$, from (X_1, X_2)
- Let $\hat{F}_{n,j}$ denote the (rescaled) empirical cumulative distribution function of the j-th margin ($j = 1, 2$)

$$\hat{F}_{n,j}(x) = \frac{1}{n+1} \sum_{i=1}^{n} \mathbf{1}_{\{X_{ij} \leq x\}}, x \in \mathbb{R}$$

A new sample (u_{i1}, u_{i2}), taking values in $[0, 1]^2$ is obtained from (x_{i1}, x_{i2}) as

$$u_{ij} = \hat{F}_{n,j}(x_{ij}) = \frac{R_{ij}}{n+1}, \; i = 1, \ldots, n$$

for $j = 1, 2$, where R_{ij} denotes the rank of x_{ij} among $x_{1j}, \ldots, x_{nj}$. Analogously, (u'_{i1}, u'_{i2}) is obtained from (y_{i1}, y_{i2}) (division by $n + 1$ keeps transformed points away from the boundary of the unit cube).

Pseudo-sample from the copula

Assume $\mathbf{X}_1, \ldots, \mathbf{X}_n$ form an iid data sample of a d-variate random vector of interest $\mathbf{X}$. Assuming $F_1, \ldots, F_d$ are all unknown,

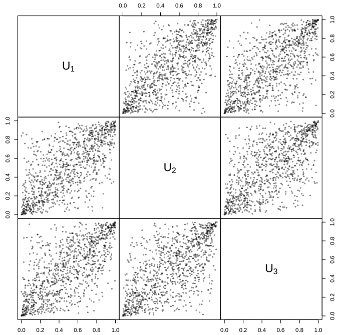
$$\mathbf{U}_i = (\hat{F}_{n,1}(X_{i1}), \ldots, \hat{F}_{n,d}(X_{id}))$$

where $i \in \{1, \ldots, n\}$, can be regarded as a consistently estimated version of the unobservable iid sample

$$(F_1(X_{i1}), \ldots, F_d(X_{id}))$$

$(\mathbf{U}_1, \ldots, \mathbf{U}_n)$ is frequently referred to as a sample of *pseudo-observations* from the copula of the data.

Note that the $\mathbf{U}_i$s are not independent, because $\hat{F}_{n,j}$ depends on the j-th component sample $X_{1j}, \ldots, X_{nj}$, $j \in \{1, \ldots, d\}$.



Informal notion of copula

The informal notion of dependence can be interpreted in terms of a **copula**, that is, a multivariate df with standard uniform univariate margins.

Going back to the example, the copula of (X_1, X_2) and the copula of (Y_1, Y_2) are simply the joint dfs of $(F_1(X_1), F_2(X_2))$ and $(G_1(Y_1), G_2(Y_2))$, respectively, where F_1, F_2, G_1, G_2 are the marginal dfs of X_1, X_2, Y_1, Y_2, respectively.

The statement that (X_1, X_2) *and* (Y_1, Y_2) *have the same dependence* can then be rephrased as (X_1, X_2) *and* (Y_1, Y_2) *have the same copula* C.

Where does the word ‘copula’ come from?

In 1959 the American mathematician Abe Sklar published a 3-page note, written in French [Sklar (1959)], showing that any multivariate distribution function can be expressed in terms of its univariate margins and a function C that he called ‘copula’ (usually linking subjects and predicates).

Two years later, he provides interesting historical background on the development of copula theory, explaining that he felt this word to be appropriate for a function linking marginal laws to a joint probability distribution (for a review of his work see Genest (2021))

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Interview Article Open Access
Special Issue in memory of Abe Sklar

Christian Genest*

A tribute to Abe Sklar

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This paper gives an account of the life and works of the American mathematician Abe Sklar. Born in Chicago on November 17, 1925, Sklar completed his PhD at the California Institute of Technology in 1956. He then joined the Illinois Institute of Technology, where he taught mathematics until his retirement in 1995. With his close friend and lifelong collaborator Berthold Schweizer (1929–2010), he was a pioneer of the theory of probabilistic metric spaces, which were introduced in 1942 by the Austro-American mathematician Karl Menger (1902–85). Together, Schweizer and Sklar made important contributions to the algebra of functions, the study of t-norms, and distributional chaos. Sklar is also credited for the notion of copula and for showing that any multivariate distribution function can be expressed in terms of its univariate margins and a copula. This result, known as Sklar’s representation theorem, is the bedrock of a widespread data analytical technique called copula modeling. Sklar passed away in Chicago on October 30, 2020.

Introduction

A motivating example

We are interested in how the dependence between the components of a random vector $\mathbf{X} \in \mathbb{R}^d$, $d \geq 2$, can be investigated and modelled.

A multivariate stochastic model is represented by means of the d-dimensional cumulative distribution function (cdf) describing the behavior of the random vector $\mathbf{X}$:

$$F_{\mathbf{X}}(x_1, \dots, x_d) := F(x_1, \dots, x_d) = P(X_1 \leq x_1, \dots, X_d \leq x_d)$$

Every joint d.f. for a random vector contains the description of

- the *marginal behavior* of the random variables (r.v.'s) X_is, i.e. the probabilistic knowledge of the single components of $\mathbf{X}$
- the dependence structure between the individual components (we will see that the copula approach provides a flexible way to describe complex dependence structures).

Multivariate risks: examples

Many real–world situations can be described by multivariate stochastic models.

- Portfolio Management: X_i's can represent (daily) returns on several assets
- Credit risk: X_i's can represent lifetimes (time-to-default) of financial institution exposed to some shock
- Insurance: X_i's represent potential losses in different lines of business for an insurance company
- Environmental Extremes: many phenomena are described in terms of two or more r.v.'s related to the same event (e.g., storm intensity-duration, flood peak-volume, etc.) or observed at different locations (rainfall maxima)

Dependent risks have been modeled with simplified assumptions (e.g., normality, independence) and/or numerical quantities (e.g., correlation coefficients) presenting well-known fallacies

The problem of the Gaussian assumption

The extensive use of the multidimensional Gaussian distribution and its generalizations is often not justified by the real situation that the model purported to describe.

In 1937 de Finetti wrote:

> *[...] the unjustified and harmful habit of considering the Gaussian distribution in too exclusive a way, as if it represented the rule in almost all the cases arising in probability and in statistics, and as if each non–Gaussian distribution constituted an exceptional or irregular case*

Two main features of the multivariate Gaussian distribution are often not supported in practice:

- the joint tails of the distribution do not assign enough weight to the occurrence of several extreme outcomes at the same time
- the distribution has a strong form of symmetry

Some references

Since their introduction (Sklar (1959)), the literature on copulas has considerably grew. Major references include

- Foundations
  - Genest et al. (1995)
  - Nelsen (2006)
  - Durante and Sempi (2016)
  - Joe (1997), and many others!
- Applications, Algorithms and simulation
  - Salvadori et al. (2007)
  - Patton (2013)
  - Hofert et al. (2018)
  - Aas et al. (2009)
  - Kojadinovic (2010), and many others!

Moreover, copula models have been largely implemented in various statistical software, see e.g. the `copula` R package provided by Hofert et al. (2014).

Background

To any df F is associated a *quantile function* $F^{\leftarrow} : \mathbb{I} = [0,1] \to \mathbb{R}$ defined by

$$F^{\leftarrow}(t) := \inf\{x \in \mathbb{R} : F(x) \geq t\}, t \in]0,1]$$

and $F^{\leftarrow}(0) := \inf\{x \in \mathbb{R} : F(x) > 0\}$. For continuous and strictly increasing dfs, $F^{\leftarrow}$ equals the ordinary inverse F^{-1}.

The following two classical results are fundamental:

- **Probability (integral) transform (PT).** Let X be a r.v. with df F. If F is continuous, then $F(X) \sim \mathrm{U}(0,1)$.
- **Quantile transform (QT).** If $U \sim \mathrm{U}(0,1)$, then $F^{\leftarrow}(U)$ has df equal to F, that is $P(F^{\leftarrow}(U) \leq x) = F(x)$.

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Definition: d-dimensional copula

Definition (Copula)

A d-dimensional copula is a distribution function on $\mathbb{I}^d = [0,1]^d$ with standard uniform marginal distributions.

Hence, the copula

$$C(\mathbf{u}) = C(u_1, \ldots, u_d)$$

is a mapping of the unit hypercube into the unit interval

$$C : [0,1]^d \to [0,1].$$

The set of d-copulas ($d \geq 2$) is denoted by $\mathcal{C}_d$.

C-volumes

In order to obtain a characterization of copulas we need the following additional definitions.

Definition (C-volume)

For any $\mathbf{a}, \mathbf{b} \in [0,1]^d$, $\mathbf{a} \leq \mathbf{b}$, let $(\mathbf{a}, \mathbf{b}]$ denote the *hyperrectangle* defined by $\mathbf{u} \in [0,1]^d : \mathbf{a} < \mathbf{u} \leq \mathbf{b}$. Then, for any hyperrectangle $(\mathbf{a}, \mathbf{b}]$, define its *C-volume* as

$$\Delta_{(\mathbf{a},\mathbf{b}]} C = \sum_{i \in \{0,1\}^d} (-1)^{\sum_{j=1}^{d} i_j} C(a_1^{i_1} b_1^{1-i_1}, \dots, a_d^{i_d} b_d^{1-i_d}) \tag{1}$$

where the summation is taken over all 2^d vectors $(i_1, \dots, i_d)$, $i_j \in 0, 1$. If

$$\Delta_{(\mathbf{a},\mathbf{b}]} C \geq 0 \ \text{ for all } \mathbf{a}, \mathbf{b} \in [0,1]^d, \mathbf{a} \leq \mathbf{b}$$

then C is called *d-increasing*. When $d = 2$, (1) becomes

$$\Delta_{(\mathbf{a},\mathbf{b}]} C = C(b_1, b_2) - C(b_1, a_2) - C(a_1, b_2) + C(a_1, a_2)$$

Characterization

The function $C : [0,1]^d \to [0,1]$ is a copula if and only if

1. C is grounded, that is,

$$C(u_1, \ldots, u_d) = 0 \text{ if } u_j = 0 \text{ for at least one } j \in \{1, \ldots, d\}$$

2. C has standard uniform univariate margins, that is,

$$C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j \text{ for all } u_j \in [0,1] \text{ and } j \in \{1, \ldots, d\}$$

3. C is d-increasing, that is, any C-volume $\Delta_{(\mathbf{a},\mathbf{b}]} C$ is nonnegative, for all $\mathbf{a} = (a_1, \ldots, a_d)$, $\mathbf{b} = (b_1, \ldots, b_d) \in [0,1]^d$, $a_i \leq b_i$

Note that, for $2 \leq k < d$, the k-dimensional margins of a d-dimensional copula are themselves copulas.

Copula density

A copula C is called *absolutely continuous* if it admits a density, that is, if

$$c(\mathbf{u}) = c(u_1, \dots, u_d) = \frac{\partial^d}{\partial u_d \dots \partial u_1} C(u_1, \dots, u_d), \quad \mathbf{u} \in (0,1)^d$$

exists and is integrable.

Remark: If the density c is nonnegative for all $\mathbf{u} \in (0,1)^d$ then C is d-increasing.
Example: the independence copula Π_d is absolutely continuous with constant density $c(\mathbf{u}) = 1, \mathbf{u} \in (0,1)^d$.

Example: Independence Copula

One of the simplest copulas is the *independence copula*

$$\Pi_d(\mathbf{u}) = \prod_{j=1}^{d} u_j, \quad \mathbf{u} \in [0,1]^d$$

Π_d is the df which is the df of a random vector $\mathbf{U} = (U_1, \ldots, U_d)$ with independent components $U_1, \ldots, U_d \sim \mathrm{U}(0,1)$:

For any $\mathbf{u} \in [0,1]^d$,

$$P(\mathbf{U} \leq \mathbf{u}) = P(U_1 \leq u_1, \ldots, U_d \leq u_d) = \prod_{j=1}^{d} P(U_j \leq u_j) = \prod_{j=1}^{d} u_j = \Pi_d(\mathbf{u})$$

Example: Independence Copula/ 2

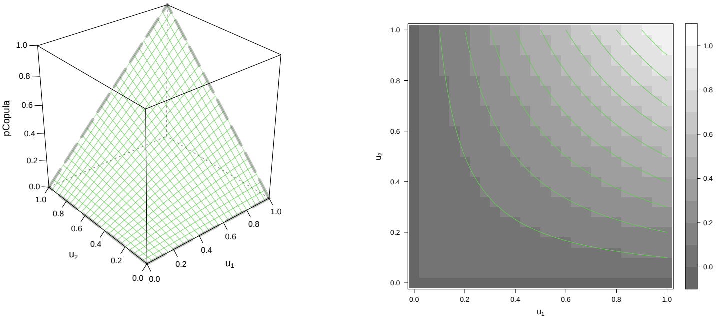


Figure: (Left) Surface (or perspective) plot and (right) contour plot of the independence copula for $d = 2$.

Remark: Π_2 is zero on all edges of the unit square which start at $(0, 0)$, $\Pi_2(u_1, 1) = u_1$ and $\Pi_2(1, u_2) = u_2 \, \forall u_1, u_2 \in [0, 1]$, i.e. the copula is *grounded* ($C(\mathbf{u}) = 0$ if $u_j = 0$ for at least one j) and has *standard uniform univariate margins* ($C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$, $\forall u_j$)

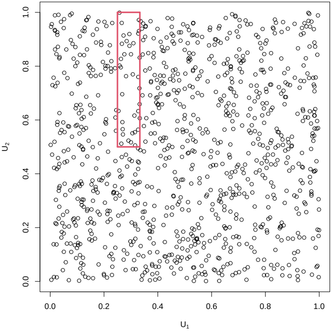
Example: Independence Copula/ 3

Let $C = \Pi_2 = u_1 u_2$. It can be shown that $\Delta_{(\mathbf{a},\mathbf{b}]} C = P(\mathbf{U} \in (\mathbf{a}, \mathbf{b}])$. Using (1),

$$\begin{aligned}
&\Delta_{(a_1,a_2),(b_1,b_2)]} C \\
&= b_1 b_2 - b_1 a_2 - a_1 b_2 + a_1 a_2 \\
&\quad = (b_1 - a_1)(b_2 - a_2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&P(\mathbf{U} \in (\mathbf{a}, \mathbf{b}]) \\
&= P(a_1 < U_1 \leq b_1) P(a_2 < U_2 \leq b_2) \\
&= (b_1 - a_1)(b_2 - a_2)
\end{aligned}$$



Approximation of the Π_2-volume of the hyperrectangle with lower end point $\mathbf{a} = (1/4, 1/2)$ and upper end point $\mathbf{b} = (1/3, 1)$ based on 1000 independent observations of $\mathbf{U} \sim \Pi_2$.

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Theorem: Fréchet-Hoeffding Bounds

Any d-dimensional copula C is pointwise bounded from below by the lower Fréchet-Hoeffding bound W and from above by the upper Fréchet-Hoeffding bound M

$$W(\mathbf{u}) \leq C(\mathbf{u}) \leq M(\mathbf{u}), \quad \mathbf{u} \in [0,1]^d$$

where

$$W(\mathbf{u}) = \max\left\{\sum_{j=1}^{d} u_j - d + 1, 0\right\} \quad \text{and} \quad M(\mathbf{u}) = \min_{1 \leq j \leq d}(u_j)$$

For $d = 2$,

$$W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}, \quad M(u_1, u_2) = \min\{u_1, u_2\}$$

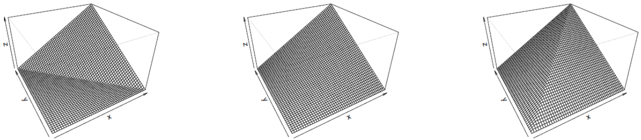
Note that W is a copula only if $d = 2$ whereas M is a copula for all $d \geq 2$.

Graphical visualization of copulas

The investigation of a given copula may require the preliminary assessment of its behavior via a suitable graphical representation, at least in the two–dimensional case.

Definition (Graph of a Copula)

The graph of a copula $C \in \mathcal{C}_d$ is the set of all points $\mathbf{x} \in \mathbb{I}^{d+1}$ that can be expressed as $\mathbf{x} = (\mathbf{u}, C(\mathbf{u}))$ for $\mathbf{u} \in \mathbb{I}^d$.



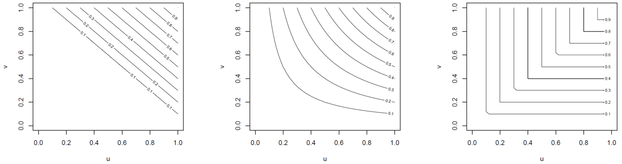
3-d graphs of the basic copulas W_2 (left), Π_2 (center) and M_2 (right).

Graphical visualization of copulas (cont)

Let C belong to $\mathcal{C}_d$ and let t be in $\mathbb{I}$. The t-level set

$$\mathcal{L}_C^t = \{\mathbf{u} \in \mathbf{I}^d : C(\mathbf{u}) = t\}$$

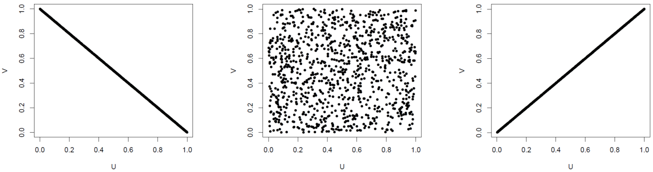
is the set of all points $\mathbf{u} \in \mathbf{I}^d$ such that the copula has value t. Notice that, for every $t \in \mathbb{I}$, all the points of type $(t, 1, \ldots, 1)$, $(1, t, 1, \ldots, 1)$, ..., $(1, 1, \ldots, 1, t)$ belong to $\mathcal{L}_C^t$.



Levels plots of the basic copulas W_2 (left), Π_2 (center) and M_2 (right).

Graphical visualization of copulas (cont)

Since a copula is the d.f. of a random vector $\mathbf{U}$, with uniform margins, we may also visualize its behavior by random sampling points that are identically distributed as $\mathbf{U}$:



Scatter-plots of 1000 random points simulated from W_2 (left), Π_2 (center) and M_2 (right).

Graphical visualization of copulas (cont)

The scatter plot from a copula C can help the visual identification of the following features:

- Symmetry (Exchangeability): the sample cloud is symmetric with respect to the line joining $(0, 0)$ with $(1, 1)$;
- Radial symmetry: The sample cloud is symmetric with respect to the line joining $(1, 0)$ with $(0, 1)$;
- Concordance: small (respectively large) values of one variable are associated with small (respectively large) values of the other variable;
- Tail dependence: The points tend to cluster near some of the corners of the copula domain

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Sklar's Theorem: preliminaries

Sklar's Theorem Sklar (1959) is the main result of copula theory: it explains how copulas determine the dependence between the components of a random vector.

Some notation:

- given a univariate df F, ran$F = \{F(x) : x \in \mathbb{R}\}$ denotes the range of F
- $F^{\leftarrow}$ denotes the quantile function associated with F (this is the ordinary inverse F^{-1} if F is continuous and strictly increasing.

Sklar's Theorem

Theorem (Sklar)

1. *For any d-dimensional df H with univariate margins $F_1, \dots, F_d$, there exists a d-dimensional copula C such that*

$$H(\mathbf{x}) = C(F_1(x_1), \dots, F_d(x_d)), \quad \mathbf{x} \in \mathbb{R}^d. \tag{2}$$

   The copula C is uniquely defined on $ranF_1 \times \cdots \times ranF_d = \prod_j ranF_j$:

$$C(\mathbf{u}) = H(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d)), \quad \mathbf{u} \in \prod_{j=1}^{d} ranF_j \tag{3}$$

2. *Conversely, given a d-dimensional copula C and univariate dfs $F_1, \dots, F_d, H$ defined by (2) is a d-dimensional df with margins $F_1, \dots, F_d$.*

Sklar's Theorem: remarks

Part [1] of Sklar's Theorem states the decomposition of any d-dimensional df H into its univariate margins $F_1, \ldots, F_d$ and a copula C.

Let $\mathbf{X} = (X_1, \ldots, X_d) \sim H$ and continuous margins $F_1, \ldots, F_d$. Hence, $U_i = F_i(X_i) \sim \mathrm{U}(0,1)$ (**PT**).
Let C denote the df of $(U_1, \ldots, U_d)$, then

$$\begin{aligned} H(x_1, \ldots, x_d) &= P(X_1 \leq x_1, \ldots, X_d \leq x_d) \\ &= P(F_1^{\leftarrow}(U_1) \leq x_1, \ldots, F_d^{\leftarrow}(U_d) \leq x_d) \\ &= P(U_1 \leq F_1(x_1), \ldots, U_d \leq F_d(x_d)) \\ &= C(F_1(x_1), \ldots, F_d(x_d)) \end{aligned}$$

If the margins are continuous, then C is unique; otherwise C is uniquely determined on $\text{ran}F_1 \times \cdots \times \text{ran}F_d$.

Sklar's Theorem: remarks

The explicit representation of the copula of $\mathbf{X}$ can be obtained by evaluating (2) at the arguments $x_i = F_i^{\leftarrow}(u_i)$, $0 \leq u_i \leq 1, i = 1, \ldots, d$

$$\begin{aligned} C(u_1, \ldots, u_d) &= C(F_1(F_1^{\leftarrow}(u_1)), \ldots, F_d(F_d^{\leftarrow}(u_d))) \\ &= H(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d)) \end{aligned}$$

For a given continuous multivariate df, part [1] of Sklar's Theorem implies that the underlying unknown copula is unique, which justifies its estimation from available data.

If $\mathbf{X} \sim H$ with margins F_j and the decomposition (2) holds, we say that $\mathbf{X}$ (or H) has copula C. Moreover, the copula expresses the dependence on a quantile scale

$$C(u_1, \ldots, u_d) = P(X_1 \leq F_1^{\leftarrow}(u_1), \ldots, X_d \leq F_d^{\leftarrow}(u_d))$$

Sklar's Theorem: remarks

From [1], it also follows that H is absolutely continuous if and only if C and the F_is are absolutely continuous. In that case, the density of H satisfies

$$h(\mathbf{x}) = c(F_1(x_1), \ldots, F_d(x_d)) \prod_{j=1}^{d} f_j(x_j), \quad \mathbf{x} \in \prod_{j=1}^{d} \mathrm{ran} X_j$$

where, for any $j \in \{1, \ldots, d\}$, $\mathrm{ran} X_j$ is the range of the rv X_j, f_j denotes the density of F_j and c denotes the density of C. Hence, c can be recovered from h via

$$c(\mathbf{u}) = h(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d)) \left(\prod_{j=1}^{d} f_j(F_j^{\leftarrow}(u_j)) \right)^{-1}, \quad \mathbf{u} \in (0,1)^d$$

and used in likelihood-based copula estimation methods.

Sklar's Theorem: remarks

Part [2] of Sklar's Theorem:

Given any copula C and univariate dfs $F_1, \dots, F_d$, a multivariate df H can be composed via (2) which then has univariate margins $F_1, \dots, F_d$ (continuous if H is continuous) and 'dependence structure' C

Let $\mathbf{U} \sim C$ and set $\mathbf{X} := (F_1^{\leftarrow}(U_1), \dots, F_d^{\leftarrow}(U_d))$. Then

$$\begin{aligned} P(\mathbf{X} \le \mathbf{x}) &= P(F_1^{\leftarrow}(U_1) \le x_1, \dots, F_d^{\leftarrow}(U_d) \le x_d) \\ &= P(U_1 \le F_1(x_1), \dots, U_d \le F_d(x_d)) \quad (QT) \\ &= C(F_1(x_1), \dots, F_d(x_d)) = H(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \end{aligned}$$

- New multivariate dfs can be constructed with given univariate margins
- Copulas can be used to formulate dependence scenarios and to evaluate risk measures of interest by means of simulation.

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Invariance property

Let $\mathbf{X} \sim H$ with continuous margins F_j ($j \in \{1, \ldots, d\}$) and (unique) copula C. If $T_1, \ldots, T_d$ are strictly increasing functions, then

$$(T_1(X_1), \ldots, T_d(X_d)) \sim C$$

that is, copulas are invariant under strictly increasing transformations (on the ranges) of the underlying random variables.

Invariance property

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$$(T_1(X_1), \ldots, T_d(X_d)) \sim C$$

that is, copulas are invariant under strictly increasing transformations (on the ranges) of the underlying random variables.

$$\begin{aligned} C(u_1, \ldots, u_d) &= P(X_1 \leq F_1^{\leftarrow}(u_1), \ldots, X_d \leq F_d^{\leftarrow}(u_d)) \\ &= P(T_1(X_1) \leq T_1(F_1^{\leftarrow}(u_1)), \ldots, T_d(X_d) \leq T_d(F_d^{\leftarrow}(u_d))) \\ &= P\left(T_1(X_1) \leq F_{T_1(X_1)}^{\leftarrow}(u_1), \ldots, T_d(X_d) \leq F_{T_d(X_d)}^{\leftarrow}(u_d)\right) \end{aligned}$$

Invariance property/ 2

The invariance property allows us to transform $\mathbf{X} = (X_1, \ldots, X_d)$ to $\mathbf{U} = (F_1(X_1), \ldots, F_d(X_d))$ without changing the underlying copula

$$\mathbf{X} \text{ has copula } C \iff (F_1(X_1), \ldots, F_d(X_d)) \sim C.$$

that is, $\mathbf{X}$ and $\mathbf{U}$ have the same copula!
Hence, regardless of the marginals, we can study the dependence between $X_1, \ldots, X_d$ by studying the dependence between the components of $\mathbf{U}$

Assume $d = 2$, and $(X_1, X_2) \sim H$ with continuous margins F_1, F_2. Then

$$(U, V) = (F_1(X_1), F_2(X_2))$$

gives the corresponding copula defined on $[0, 1]^2$.

Invariance property: Examples

From bivariate normal to normal copula

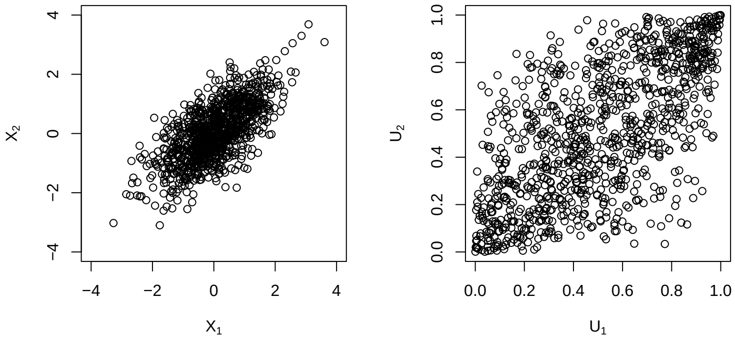

Figure: (Left) Scatter plot of $n = 1000$ independent observations from (X_1, X_2) having a joint bivariate Gaussian distribution $\mathcal{N}_2(\mathbf{0}, P)$, $P = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$.
(Right) The corresponding (probability transformed) sample from the Gaussian copula is obtained by applying the df Φ (the F_j's here) to each pair of points.

Invariance property: Examples

From normal copula to meta-Gaussian sample with exponential margins

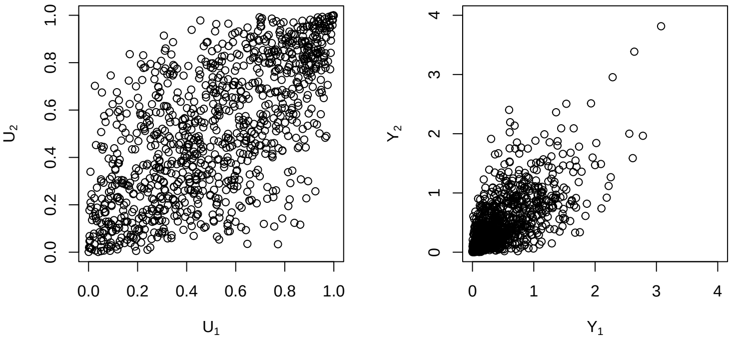

Figure: (Left) Same Gaussian copula scatter plot as before. (Right) The corresponding (quantile transformed) sample having a Gaussian copula and exponentially distributed marginals $F_j \sim \exp(2)$ (apply $F_j^{-1}(u) = -log(1-u)/2$ to each pair of points on the left plot.)

Simulation of Copula and Meta-C Model

Algorithm 1: Sample from C (C is defined by (3) in Sklar's Th)

1. Sample $\mathbf{X} \sim H$, where H is a d-dimensional df with continuous margins $F_1, \ldots, F_d$
2. Return $\mathbf{U} = (F_1(X_1), \ldots, F_d(X_d))$

Algorithm 2: sample from a Meta-C model

1. Sample $\mathbf{U} \sim C$
2. Return $\mathbf{X} = (F_1^{\leftarrow}(U_1), \ldots, F_d^{\leftarrow}(U_d))$

Survival copula

Suppose the marginal dfs F_i are continuous and strictly increasing. Then

$$\begin{aligned}\bar{H}(x_1,\ldots,x_d) &= P(X_1 > x_1,\ldots,X_d > x_d)\\ &= P(1-F_1(X_1)\le \bar{F}_1(x_1),\ldots,1-F_d(X_d)\le \bar{F}_d(x_d))\\ &= \hat{C}(\bar{F}_1(x_1),\ldots,\bar{F}_d(x_d))\end{aligned}$$

where $\hat{C}$ is the *survival copula* of $X_1,\ldots,X_d$, that is, the df of $\mathbf{1}-\mathbf{U}=(1-F_1(X_1),\ldots,1-F_d(X_d))$.
A representation of $\hat{C}$ is

$$\hat{C}(u_1,\ldots,u_d)=\bar{H}(\bar{F}_1^{-1}(u_1),\ldots,\bar{F}_d^{-1}(u_d))$$

Survival copula (cont.)

Let $d = 2$. We want to compute the survival function of (X_1, X_2) and the survival copula $\hat{C}$ of C:

$$\begin{aligned}
\bar{H}(x_1, x_2) &= P(X_1 > x_1, X_2 > x_2) \\
&= 1 - (P(X_1 \leq x_1) + P(X_2 \leq x_2) - P(X_1 \leq x_1, X_2 \leq x_2)) \\
&= 1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2) \\
&= 1 - (1 - \bar{F}_1(x_1)) - (1 - \bar{F}_2(x_2)) + C(1 - \bar{F}_1(x_1), 1 - \bar{F}_2(x_2)) \\
&= \bar{F}_1(x_1) + \bar{F}_2(x_2) - 1 + C(1 - \bar{F}_1(x_1), 1 - \bar{F}_2(x_2))
\end{aligned}$$

The survival copula is $\hat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2)$.
Notice in particular that this function is not equal to the survival function $\bar{C}$ corresponding to the copula C:

$$\begin{aligned}
\bar{C}(u_1, u_2) &= P(U_1 > u_1, U_2 > u_2) \\
&= P(1 - U_1 \leq 1 - u_1, 1 - U_2 \leq 1 - u_2) \\
&= \hat{C}(1 - u_1, 1 - u_2) = C(u_1, u_2) - u_1 - u_2 + 1
\end{aligned}$$

Radial Symmetry and Exchangeability

1. A random vector $\mathbf{X}$ is called radially symmetric about $\mathbf{a} \in \mathbb{R}^d$ if $\mathbf{X} - \mathbf{a} \stackrel{d}{=} \mathbf{a} - \mathbf{X}$, that is, if $\mathbf{X} - \mathbf{a}$ and $\mathbf{a} - \mathbf{X}$ are equal in distribution

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   If $C(u_{j1}, \dots, u_{jd}) = C(u_1, \dots, u_d)$ for all $u_1, \dots, u_d \in [0, 1]$ and all permutations $(j_1, \dots, j_d)$ of $\{1, \dots, , d\}$, we call C *exchangeable*

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- $W_{d=2}$, Π, and M are both radially symmetric and exchangeable

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- **Extreme value copulas** emerges as the class of natural limiting dependence structures for multivariate maxima: the Gumbel copula provides an example of a parametric EV copula family (see McNeil et al. (2015))

Elliptical Copulas

A copula is elliptical if it is the copula of an elliptical distribution

- $\mathbf{Z} \sim \mathcal{N}_d(\mathbf{0}, \Sigma)$
- $\mathbf{T} \sim \mathit{Student}_d(\mathbf{0}, \Sigma, \nu)$

Without loss of generality, we can assume that

- $\mu = 0$
- Σ is a correlation matrix, we denote it by P

Note that when $\nu \to \infty$, then the Student-t tends to the Gaussian distribution

Gaussian copulas

The Gaussian copula is the copula of $\mathbf{Z} \sim \mathcal{N}_d(\mathbf{0}, P)$

$$\begin{aligned} C_P^{Ga}(\mathbf{u}) &= P(\Phi(Z_1) \leq u_1, \ldots, \Phi(Z_d) \leq u_d) \\ &= P(Z_1 \leq \Phi^{-1}(u_1), \ldots, Z_d \leq \Phi^{-1}(u_d)) \\ &= {}_P(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)) \end{aligned}$$

where ${}_P$ is the joint df of $\mathbf{Z}$, and Φ is the cdf of $\mathcal{N}(0, 1)$.

- if $d = 2$, then $C_P^{Ga} \equiv C_\rho^{Ga}$, where $\rho = \mathrm{corr}(Z_1, Z_2)$
- $P = I_d$ gives independence
- If $P = J_d$, a $d \times d$ matrix of ones, then C is the comonotonicity copula (M)
- For $d = 2$ and $\rho = -1$, C_ρ^{Ga} is the countermonotonicity copula (W)

Gaussian copulas

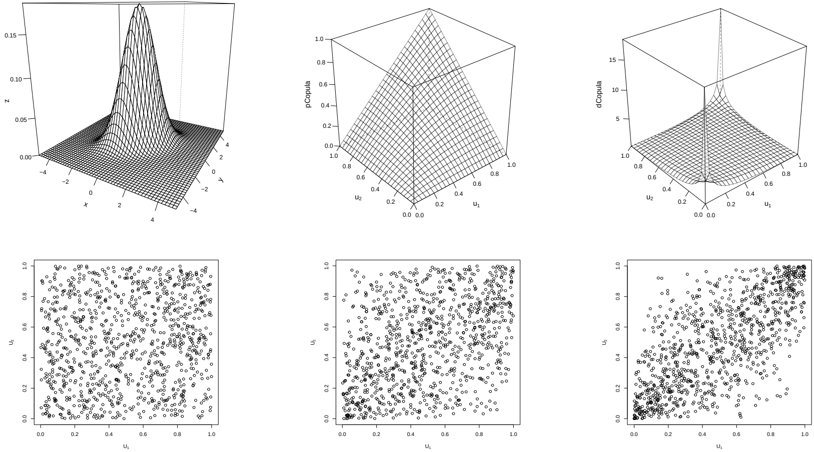


Figure: (Top) Density of the bivariate normal df with $\rho = 0.5$ (left), perspective plot of C_ρ^{Ga} (middle), and corresponding copula density c_ρ^{Ga} (right). (Bottom) Sample of size 1000 from C_ρ^{Ga} with $\rho = 0.1, 0.5, 0.7$ (from left to right).

t copulas

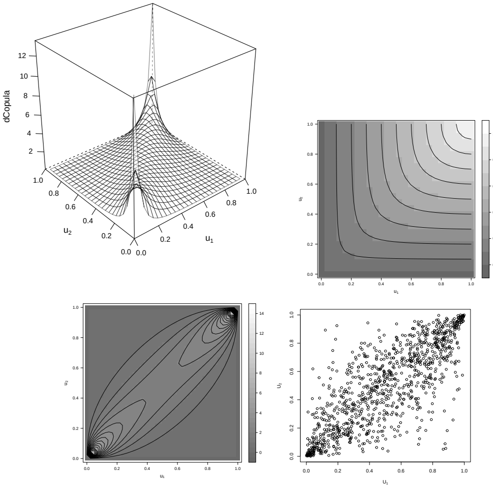
The t copula is the copula of $\mathbf{T} \sim Student_d(\mathbf{0}, \Sigma, \nu)$ with location vector $\mathbf{0}$, scale matrix P, and $\nu > 0$ degrees of freedom:

$$
\begin{aligned}
C^t_{P,\nu}(\mathbf{u}) &= P(t_\nu(T_1) \leq u_1, \dots, t_\nu(T_d) \leq u_d) \\
&= t_{P,\nu}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d))
\end{aligned}
$$

where t_ν is the univariate Student-t distribution with ν degrees of freedom and $t_{P,\nu}$ is the d-variate t distribution.

- For $d = 2$, $C^t_{-1,\nu}$ is the lower Fréchet-Hoeffding bound W,
- For $d \geq 2$, if P only consists of entries equal to 1, $C^t_{P,\nu}$ is the upper Fréchet–Hoeffding bound M
- $P = I_d$ does not lead to the independence copula

t copulas /2



Left: Density plot of $c^t_{\rho,\nu}$ for $\rho \approx 0.81$ (Kendall'tau $\tau = 0.6$) and $\nu = 4$ degrees of freedom, contour plot of $c^t_{\rho,\nu}$
Right: Scatter plot of a sample of size $n = 1000$ from $C^t_{\rho,\nu}$ and contour plot of $C^t_{\rho,\nu}$

Bivariate t-copulas are both radially symmetric and exchangeable

Explicit copulas

A number of copula families have simple closed forms.

Some examples:

Gumbel-Hougaard Copula

(d=2) $C_\theta^{Gu}(u_1, u_2) = \exp(-((-log(u_1))^\theta + (-log(u_2))^\theta)^{1/\theta})$

$\theta \geq 1$: $\theta = 1$ gives independence; $\theta \to \infty$ gives comonotonicity

Clayton copula (d=2) $C_\theta^C(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \theta > 0$

$\theta \to 0$ gives independence; $\theta \to \infty$ gives comonotonicity

Frank copula $C_\theta^F(u_1, u_2) = -\frac{1}{\theta} \log\left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right)$

$\theta \to 0$ gives independence; $\theta \to \infty$ gives comonotonicity

Comparison of some copulas

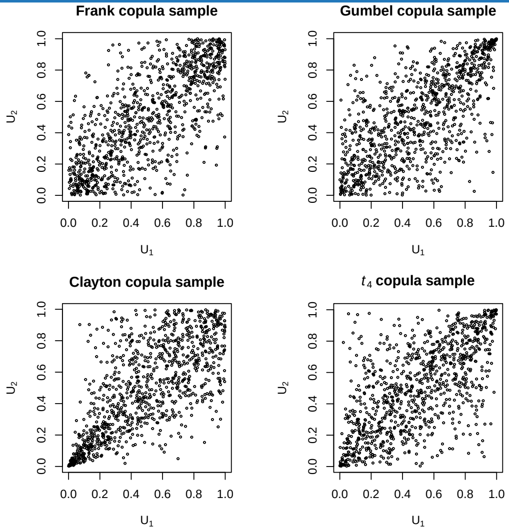
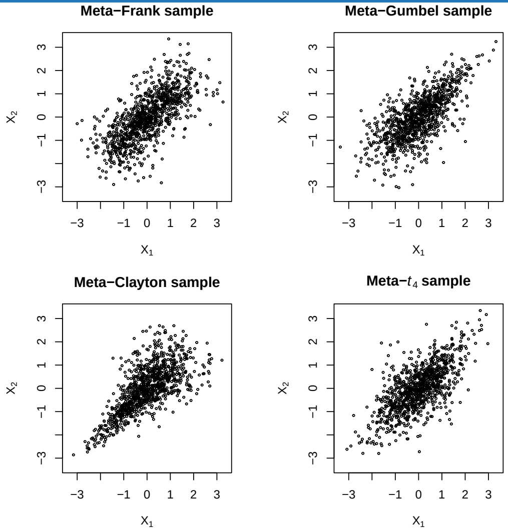


Figure: Copula parameters are chosen such that linear correlation between the (quantile transformed) $N(0, 1)$ margins is roughly 0.7

Comparison of some copulas/ 2



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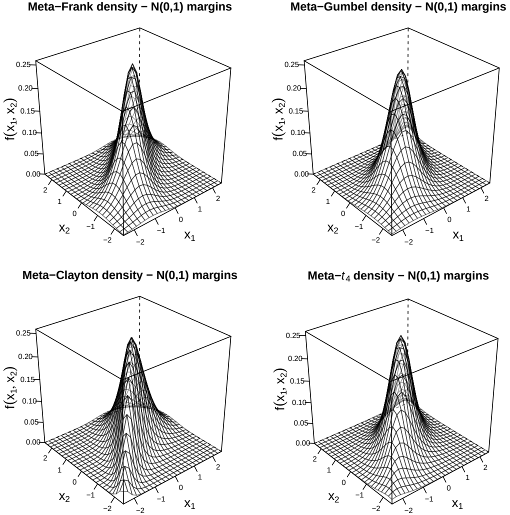
Comparison of some copulas/ 3



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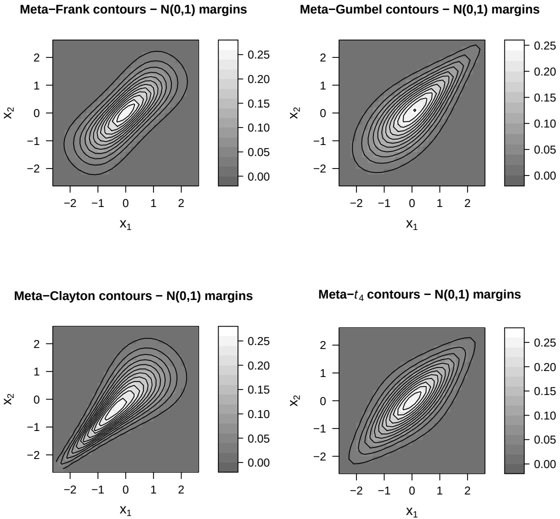
Comparison of some copulas/ 4



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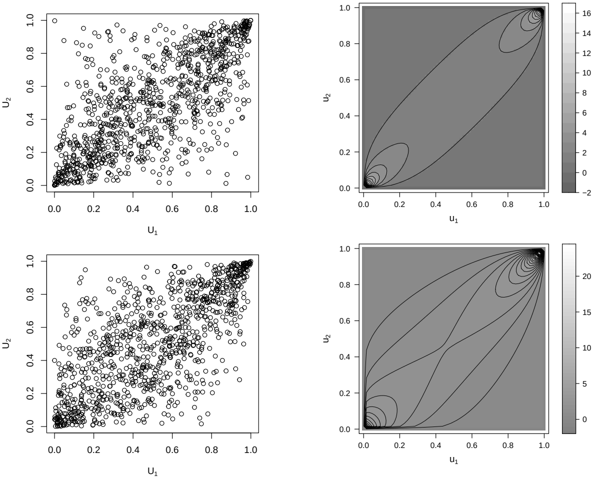
Assessing Symmetry/Exchangeability



Bivariate t copulas $(\rho = 0.7, \nu = 3.5)$ are both radially symmetric (symmetry wrt the point $(1/2, 1/2)$) and exchangeable;

The copulas in the Gumbel-Hougaard family (here $\theta = 2$) are exchangeable (symmetry of the density with respect to the main diagonal) but not radially symmetric

Conditional distributions of copulas

Suppose $(U_1, U_2) \sim C$. Recall that a copula is an increasing continuous function in each argument. Hence

$$\begin{aligned} C_{U_2|U_1}(u_2|u_1) &= P(U_2 \leq u_2 | U_1 = u_1) \\ &= \lim_{\delta \to 0} \frac{C(u_1 + \delta, u_2) - C(u_1, u_2)}{\delta} = \frac{\partial}{\partial u_1} C(u_1, u_2) \end{aligned}$$

(see Nelsen (2006)). The conditional distribution $C_{U_2|U_1}(u_2|u_1)$ is a df on $[0, 1]$ which is uniform only in the case $C = \Pi$.

Interpretation in Risk management. (X_1, X_2) is a pair of two continuous risks having (unique) copula C. Then

$$\begin{aligned} 1 - C_{U_2|U_1}(q|p) &= 1 - P(U_2 \leq q | U_1 = p) \\ &= P(U_2 > q | U_1 = p) \\ &= P(X_2 > F_2^{-1}(q) | X_1 = F_1^{-1}(p)) \end{aligned}$$

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