Data Science for Insurance

Modeling dependence with copulas

Roberta Pappadà

a.a. 24-25

rpappada@units.it

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Background

Numerical summaries of (aspects of) dependence are known as *measures of association* and are mostly studied in the bivariate case

For a pair of rvs (X_1, X_2) we quantify the dependence by means of

- the usual linear Pearson's correlation coefficient

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- Copula-based measures $\begin{cases} \text{rank correlation coefficients} \\ \text{tail-dependence coefficients} \end{cases}$

Properties of Linear Correlation

Let (X_1, X_2) be a random vector whose components have finite variances. Then,

1. $\mathrm{Cor}(X_1, X_2) \in [-1, 1]$
2. $|\mathrm{Cor}(X_1, X_2)| = 1$ if and only if there exist $a, b \in \mathbb{R}$, $a \neq 0$, such that $X_2 = aX_1 + b$ almost surely (X_1 and X_2 are perfectly linearly dependent)
3. If X_1 and X_2 are independent, then $\mathrm{Cor}(X_1, X_2) = 0$
4. For any $a_1, a_2 > 0$, or any $a_1, a_2 < 0$, and for any $b_1, b_2 \in \mathbb{R}$,

$$\mathrm{Cor}(a_1X_1 + b_1, a_2X_2 + b_2) = \mathrm{Cor}(X_1, X_2)$$

   In particular, Pearson's correlation coefficient is invariant under strictly increasing linear transformations.

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Understanding the limitations of correlation

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Understanding the limitations of correlation

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Fallacy 2 (Invariance) $\text{Cor}(X_1, X_2)$ is invariant under strictly increasing transformations on $\text{ran}X_1$ or $\text{ran}X_2$.

Counterexample to Fallacies 1 and 2. If $X_1, X_2 \sim F(x) = 1 - x^{-3}, x \geq 1$ and X_1, X_2 ind., then $\text{Cor}(X_1, X_2) = 0$ but $\text{Cor}(X_1^3, X_2)$ does not exist since neither $E((X_1^3)^2)$ nor $E(X_1^3)$ are finite.

Understanding the limitations of correlation

Fallacy 3 (Uniqueness) The marginal distributions and the correlation coefficient uniquely determine the joint distribution

Fallacy 4 (Uncorrelatedness Implies Independence) $\mathrm{Cor}(X_1, X_2) = 0$ implies that X_1 and X_2 are independent.

Counterexamples to Fallacies 3 and 4. Fallacy 4 alone can be easily falsified by taking $X_1 \sim N(0,1)$ and $X_2 = X_1^2$; then $\mathrm{Cor}(X_1, X_2) = 0$ but X_1 and X_2 are dependent.

A counterexample to both Fallacies 3 and 4 can be constructed by a mixture of the two Fréchet-Hoeffding bounds W and M.

Example

Model 1: $\mathbf{X} = (X_1, X_2) \sim N_2(\mathbf{0}, I_2)$, i.e., (X_1, X_2) has $N(0,1)$ margins and zero correlation.
Model 2: $(Y_1, Y_2) = (X_1, VX_1)$, with X_1 as in Model 1 and V and independent discrete rv such that $P(V = 1) = P(V = -1) = 1/2$. Hence

$$\mathrm{Cor}(Y_1, Y_2) = \mathrm{Cov}(Y_1, Y_2) = \mathrm{E}(Y_1 Y_2) = \mathrm{E}(VX_1^2) = \mathrm{E}(V)\mathrm{E}(X_1^2) = 0$$

Conditional on $V = -1$ (respectively, $V = 1$), the copula C of (Y_1, Y_2) is the countermonotonicity W (respectively, the comonotonicity copula M):

$$C(u_1, u_2) = 0.5 \max(u_1 + u_2 - 1, 0) + (1 - 0,5) \min(u_1, u_2)$$

which is a *mixture* of the two-dimensional Fréchet-Hoeffding bounds.

Example (cont)

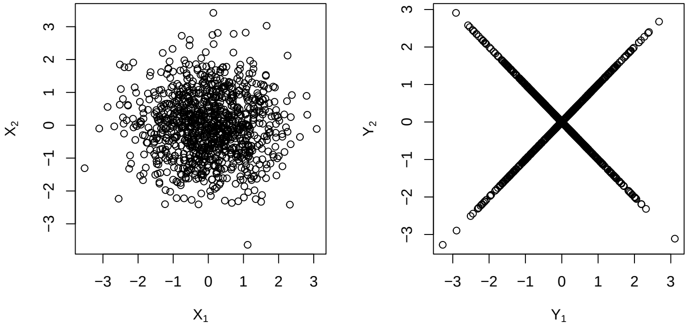

Figure: $n = 1000$ independent realizations from (X_1, X_2), whose copula is the independence copula (left) and (Y_1, Y_2), whose copula is a mixture between the Fréchet–Hoeffding bounds W and M (right); both have $N(0, 1)$ margins and zero correlation.

example (cont)

Assume X_1, X_2 from Model 1 and Y_1, Y_2 from Model 2 are losses, with $N(0,1)$ margins and zero correlation. it can be proved that for $\alpha > 0.75$, $VaR_\alpha(X_1 + X_2) = \sqrt{2}\Phi^{-1}(\alpha)$; $VaR_\alpha(Y_1 + Y_2) = 2\Phi^{-1}(2\alpha - 1)$

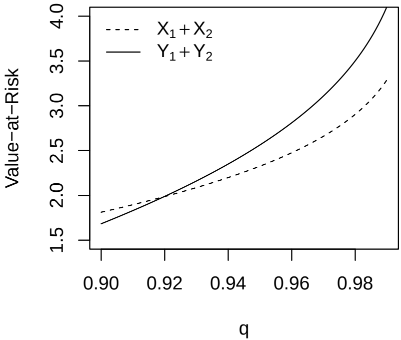

Figure: VaR for the risks $X_1 + X_2$ and $Y_1 + Y_2$. The VaR of a sum of risks is not determined by marginal distributions and pairwise correlations.

Attainable Correlations

Fallacy 5 (Attainable Correlations) Given margins F_1, F_2, all $\text{Cor}(X_1, X_2) \in [-1, 1]$ can be attained by choosing a suitable joint d.f for (X_1, X_2).

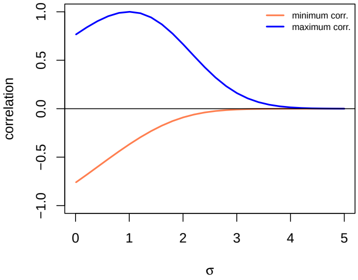

Figure: Correlation bounds for lognormal rvs X_1 and X_2 where $\ln X_1 \sim N(0,1)$ and $\ln X_2 \sim N(0,\sigma^2)$

Linear correlation: pitfalls

The main limitations of the linear correlation coefficient are:

1. $\mathrm{Cor}(X_1, X_2)$ does not exist for all random vectors (X_1, X_2) (only for those with finite second moments);
2. $\mathrm{Cor}(X_1, X_2)$ depends on the marginal dfs of (X_1, X_2) even when the latter are continuous
3. $\mathrm{Cor}(X_1, X_2)$ is invariant only under strictly increasing linear transformations (not under strictly increasing transformations in general)

By only depending on the underlying copula C in the case of continuous random vectors, *rank correlation coefficients* overcome several of the aforementioned issues.

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Rank correlation measures

Two important measures of concordance are Spearman's rho and Kendall's tau.

Definition (Spearman's Rho)

Let (X_1, X_2) be a random vector with continuous marginal dfs F_1 and F_2. The *(population version of) Spearman's rho* is defined as

$$\rho_S = 3[P((X_1 - V_1)(X_2 - W_2) > 0) - P((X_1 - V_1)(X_2 - W_2) < 0)]$$

for independent copies (X_1, X_2), (V_1, V_2), and (W_1, W_2) (note that V_1 and W_2 are independent).

Rank correlation measures (cont.)

The Spearman's rho for the vector $(X_1, X_2) \sim C$ is given by

$$\begin{aligned} \rho_S = \rho_S(X_1, X_2) =& 12 \int_0^1 \int_0^1 u_1 u_2 \mathrm{d}C(u_1, u_2) - 3 \\ =& 12 \int_{[0,1]^2} C(\mathbf{u}) \mathrm{d}\mathbf{u} - 3 \end{aligned} \tag{2}$$

If $X_1 \sim F_1$, $X_2 \sim F_2$, and we let $U_1 = F_1(X_1), U_2 = F_2(X_2)$, then

$$\begin{aligned} \rho_S = \rho_S(X_1, X_2) &= 12 \int_{[0,1]^2} u_1 u_2 \mathrm{d}C(u_1, u_2) - 3 = 12E(U_1 U_2) - 3 \\ &= \frac{E(U_1 U_2) - 1/4}{1/12} = \frac{Cov(U_1, U_2)}{\sqrt{V(U_1)}\sqrt{V(U_2)}} \\ &= \mathrm{Cor}(F_1(X_1), F_2(X_2)) \end{aligned}$$

Rank correlation measures (cont.)

Definition (Kendall's Tau)

Let (X_1, X_2) be a random vector with continuous marginal dfs F_1 and F_2. The *(population version of) Kendall's tau* is defined as the probability of concordance minus the probability of discordance

$$\tau = P((X_1 - X_1')(X_2 - X_2') > 0) - P((X_1 - X_1')(X_2 - X_2') < 0)$$

where (X_1', X_2') is an independent copy (same distribution) of (X_1, X_2).

Rank correlation measures (cont.)

In terms of the copula C of (X_1, X_2), Kendall's tau can be written as

$$\tau = \tau(X_1, X_2) = 4 \int_{[0,1]^2} C(u_1, u_2) \mathrm{d} C(u_1, u_2) - 1 \tag{3}$$

The integral above is the expected value of the random variable $C(U_1, U_2)$, where $U_1, U_2 \sim U(0, 1)$ with joint df C, that is

$$\tau(X_1, X_2) = 4E(C(U_1, U_2)) - 1$$

Rank correlation VS linear correlation

Spearman's rho and Kendall's tau only depend on the underlying copula; they can be viewed as moments of the copula. Moreover,

- they always exist, and are not limited to continuous random vectors with finite second moments
- are invariant under strictly increasing transformations
- if κ denotes Kendall's tau or Spearman's rho for two continuous r.v.s X and Y, then
  - $\kappa(X, Y) = 1 \iff C = M$;
  - $\kappa(X, Y) = -1 \iff C = W$;

Method-of-moments estimators

Consider a one-parameter family of copulas $\{C_\theta : \theta \in \Theta\}$, where $\Theta \subseteq \mathbb{R}$. For many such copula families, the functions

$$g_{\rho_S}(\theta) = \rho_S(C_\theta) \quad \text{and} \quad g_\tau(\theta) = \tau(C_\theta), \quad \theta \in \Theta$$

are one-to-one. For example:

- Clayton family: $g_\tau(\theta) = \theta/(\theta+2), \quad \theta \in (0, \infty)$;
- Gumbel-Hougaard family: $g_\tau(\theta) = 1 - 1/\theta, \quad \theta \in [1, \infty)$;
- Normal family: for $\theta \in [-1, 1]$
  $g_{\rho_S}(\theta) = (6/\pi) \arcsin(\theta/2); \quad g_\tau(\theta) = (2/\pi) \arcsin \theta$

From $g_{\rho_S}^{-1}$ and g_τ^{-1} one can obtain the unique value of θ corresponding to an admissible value of ρ_S or τ.

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From $g_{\rho_S}^{-1}$ and g_τ^{-1} one can obtain the unique value of θ corresponding to an admissible value of ρ_S or τ.

In the `R` package **copula** the functions ρ_S, $g_{\rho_S}^{-1}$, τ, g_τ^{-1} are `rho()`, `iRho()`, `tau()` and `iTau()`

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Tail dependence

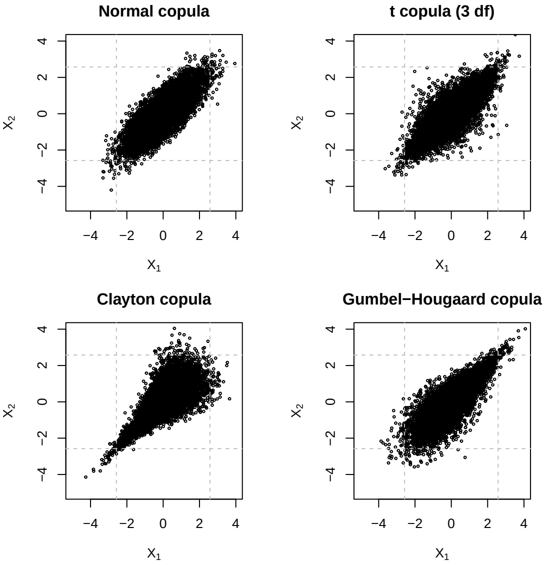
Coefficients of tail dependence aim at summarizing the *extremal dependence*, i.e., the dependence in the (joint) tails of bivariate distributions.

Formally, the coefficients of tail dependence (TD) are limits of conditional probabilities of quantile exceedances (see Nelsen, 2006).

Scatter plots from bivariate distributions with $N(0,1)$ margins and the same Kendall's tau but different copulas can exhibit very different *tail behaviors*.

Tail dependence: example

The plots show generated random samples of size $n = 10000$ from four bivariate distributions in the Fréchet class, with standard normal margins, to investigate how the copula affects the dependence in the tails (Kendall's tau is 0.6 for all of them)



Tail dependence coefficients

Definition (Upper and Lower TDC)

Let (X_1, X_2) be a random vector with marginal dfs F_1 and F_2. Provided that the limits exist, the coefficient of *lower* and *upper tail dependence* of X_1 and X_2 are defined by

1. $\lambda_l = \lambda_l(X_1, X_2) = \lim_{q \to 0^+} P(X_2 \leq F_2^{\leftarrow}(q) | X_1 \leq F_1^{\leftarrow}(q))$
2. $\lambda_u = \lambda_u(X_1, X_2) = \lim_{q \to 1^-} P(X_2 > F_2^{\leftarrow}(q) | X_1 > F_1^{\leftarrow}(q))$

respectively.
If $\lambda_l \in (0, 1]$ (respectively, $\lambda_u \in (0, 1]$), then X_1 and X_2 are said to be lower (respectively, upper) tail dependent;

If $\lambda_l = 0$ (respectively, $\lambda_u = 0$) they are *asymptotically independent* in the lower (respectively, upper) tail.

Representation of TDC

If F_1 and F_2 are continuous dfs, then we get simple expressions for λ_l and λ_u in terms of the unique copula C of (X_1, X_2):

$$\lambda_l = \lambda_l(C) = \lim_{q \to 0^+} \frac{P(X_2 \leq F_2^{\leftarrow}(q), X_1 \leq F_1^{\leftarrow}(q))}{P(X_1 \leq F_1^{\leftarrow}(q))} = \lim_{q \to 0^+} \frac{C(q,q)}{q}$$

$$\lambda_u = \lambda_u(C) = \lim_{q \to 1^-} \frac{\hat{C}(1-q, 1-q)}{1-q} = \lim_{q \to 1^-} \frac{1 - 2q + C(q,q)}{1-q}$$

where $\hat{C}$ is the survival copula of C. Hence, for radially symmetric copulas we must have $\lambda_u = \lambda_l$, since $C = \hat{C}$ and $\lambda_u = \lim_{q \to 0^+} \dfrac{\hat{C}(q,q)}{q}$.

TDCs: Examples

For some parametric copulas with a simple closed form, calculation of the coefficients of tail dependence λ_l and λ_u is simple:

Family	λ_l	λ_u
Clayton	$2^{-1/\theta}$	0
Frank	0	0
Gumbel	0	$2 - 2^{1/\theta}$
Normal	0	0
t	$2t_{\nu+1}(\gamma)$	$2t_{\nu+1}(\gamma)$,
		$\gamma = -\sqrt{(\nu+1)(1-\rho)/(1+\rho)}$

Table: Tail dependence coefficients for some parametric copulas

In the R package **copula** the function for computing the coefficients of tail dependence is `lambda()`

Example: Student-t Copulas

For the t copula $C^t_{\rho,\nu}$ with correlation ρ and degrees of freedom ν, the plots display the graphs of the TDC $\lambda = \lambda_u = \lambda_l$ as a function of ρ and ν

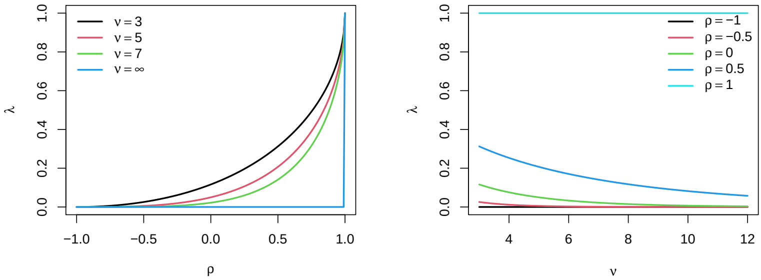


Figure: For fixed, finite ν, tail dependence increases as ρ increases (left). For fixed $|\rho| < 1$, tail dependence increases as ν decreases (right).

Example: Student-t Copulas (cont)

Coefficients of tail dependence for the t-copula are tabulated below various values of ρ and ν.

ν/ρ	-0.5	0	0.5	0.9	1
2	0.06	0.18	0.39	0.72	1
4	0.01	0.08	0.25	0.63	1
10	0.0	0.01	0.08	0.46	1
∞	0	0	0	0	1

Joint quantile exceedances

Looking at joint exceedances of *finite high quantiles* can help to understand the practical consequences of the differences between the extremal behaviours of different models.

Example: Daily returns. Suppose $\mathbf{X} = (X_1, \dots, X_5)$ represent a vector of five daily negative log-returns with fixed continuous marginal dfs and fixed common pairwise Kendall's tau equal to $1/3$. In addition, suppose that we are unsure whether a normal or a t copula should be used as underlying dependence structure C.
Under the normal copula (with parameter ρ) the probability that, on any day, all five negative log-returns lie above their $u = 0.99$ quantiles is

$$\begin{aligned} P(X_1 > F_1^{\leftarrow}(u), \dots, X_5 > F_5^{\leftarrow}(u)) &= P(F_1(X_1) > u, \dots, F_5(X_5) > u) \\ &= C_\rho^n(1-u, \dots, 1-u), \end{aligned}$$

where the last equality follows by radial symmetry.

Joint quantile exceedances/ 2

Assuming 260 trading days in a year, his calculation can be carried out using the following code

R code:

```
>set.seed(231)
>d<-5
>rho<-iTau(normalCopula(), tau=1/3) #0.5
>u<-0.99
>prob<-pCopula(rep(1-u, d),copula=normalCopula(rho, dim=d))
>1/(260*prob) # 51.42 years
```

Hence, the event of joint exceedances above the 99% quantile for the five daily negative log-returns happens about once every 51.42 years.

Joint quantile exceedances/ 3

If the copula of $\mathbf{X}$ is assume to be a 5-dimensional t copula $C^t_{0.5,3}$, such an event will happen approximately 9.31 times more often (roughly once every 5.63 years)

R code:

```
> prob.t<-pCopula(rep(1-u, d), copula=tCopula(rho, dim=d,
df=3))
> 1/(260*prob.t)
[1] 5.625567
```