$$\begin{cases} i\partial_t u_n = -P_n \Delta u_n + \lambda Q_n(|Q_n u_n|^{p-1} Q_n u_n) \\ u_n(0) = Q_n u_0 \end{cases}$$

Fisso

$$M > |u_0|_{H^1}$$

$$\vartheta_n = \sup\{\tau > 0 : |u_n(t)|_{H^1} < 2M \quad \text{per } |t| < \tau\}$$

$$u_n \in C^\infty(\mathbb{R}, H^1)$$

$$u_n \in C^{0,\frac{1}{2}}((-\vartheta_n, \vartheta_n), L^2)$$

con costante di Hölder $C(M)$

$t - s \geq 0$

$$|u_n(t) - u_n(s)|_{L^2} \leq |u_n(t) - u_n(s)|_{H^1}^{\frac{1}{2}} \, |u_n(t) - u_n(s)|_{H^{-1}}^{\frac{1}{2}}$$

$$0 = \frac{1}{2}\, 1 + \frac{1}{2}(-1)$$

$$s = (1-t)s_0 + t s_1$$

$$|f|_{H^s} \leq |f|_{H^{s_0}}^{1-t} \, |f|_{H^{s_1}}^{t}$$

$$\leq \left(|u_n(t)|_{H^1} + |u_n(s)|_{H^1}\right)^{\frac{1}{2}} (t-s)^{\frac{1}{2}} \sup_{\sigma \in [s,t]} |\partial_t u_n(\sigma)|_{H^{-1}}^{\frac{1}{2}}$$

$$\leq \sqrt{2} \sup_{|t'| < \vartheta_n} |u_n(t')|_{H^1}^{\frac{1}{2}} (t-s)^{\frac{1}{2}}$$

$$\cdot \quad \left\| -P_n \Delta u_n + \lambda Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right)\right\|^{\frac{1}{2}}_{L^\infty((-\vartheta_n,\vartheta_n), H^{-1})}$$

$$\leq \sqrt{2}\,\sqrt{2}\quad M \quad (t-s)^{\frac{1}{2}}$$

$$\left(\|\cancel{P_n}\Delta u_n\|_{L^\infty((-\vartheta_n,\vartheta_n), H^{-1})} + \left\| |Q_n u_n|^{p-1} Q_n u_n \right\|_{L^\infty((-\vartheta_n,\vartheta_n), H^{-1})} \right)^{\frac{1}{2}}$$

$$L^{\frac{p+1}{p}} \hookrightarrow H^{-1}, \quad H^1 \hookrightarrow L^{p+1}$$

$$\leq C_2 \quad M \quad (t-s)^{\frac{1}{2}} \Big(\|u_n\|_{L^\infty((-\vartheta_n,\vartheta_n), H^1)} +$$

$$+ \quad \left\| |Q_n u_n|^{p-1} Q_n u_n \right\|_{L^\infty((-\vartheta_n,\vartheta_n), L^{\frac{p+1}{p}})} \Big)^{\frac{1}{2}}$$

$$\leq C \quad M \,(t-s)^{\frac{1}{2}} \left(2M + \|\cancel{Q_n} u_n\|^p_{L^\infty((-\vartheta_n,\vartheta_n), L^{p+1})} \right)^{\frac{1}{2}}$$

$$H^1$$

$$\leq C \quad M\,(t-s)^{\frac{1}{2}} \Big(2M + \underbrace{\|u_n\|^p_{L^\infty((-\vartheta_n,\vartheta_n), H^1)}}_{\leq (2M)^p} \Big)^{\frac{1}{2}}$$

$$= C(M) \quad (t-s)^{\frac{1}{2}}$$

$\forall \quad -\vartheta_n < s < t < \vartheta_n \qquad C^{0,\frac{1}{2}}([-\vartheta_n, \vartheta_n], L^2)$

~~$t-s>0$~~

$$|u_n(t) - u_n(s)|_{L^2} \leq C(M)\,(t-s)^{\frac{1}{2}}$$

Vogliamo verificare che $\exists\ b>0$ t.c.

$$|u_n(t)|^2_{H^1} \leq |u_0|^2_{H^1} + C(M)|t|^b$$

$$\forall \quad t \in (-\vartheta_n, \vartheta_n)$$

$$E_n(u_n(t)) = \frac{1}{2}|\cancel{P_n} \nabla u_n(t)|^2_{L^2} + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n u_n(t)|^{p+1}\, dx$$

$$= E_n(Q_n u_0) =$$

$$= \frac{1}{2}|\nabla Q_n u_0|^2_{L^2} + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n^2 u_0|^{p+1}\, dx$$

$$Q(u_n(t)) = \frac{1}{2}|u_n(t)|^2_{L^2} = \frac{1}{2}|Q_n u_0|^2_{L^2}$$

$$E_n(u_n(t)) + Q(u_n(t)) = E_n(Q_n u_0) + Q(Q_n u_0)$$

$$|u|_{H^1} = \sqrt{|\nabla u|^2_{L^2} + |u|^2_{L^2}}$$

$$\frac{1}{2}|u_n(t)|^2_{H^1} + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n u_n|^{p+1}\, dx =$$

$$= \frac{1}{2} \| Q_n u_0 \|_{H^1}^2 + \frac{\lambda}{p+1} \int_{\mathbb{R}^d} |Q_n^2 u_0|^{p+1} dx$$

$$\|u_n(t)\|_{H^1}^2 \leq \|Q_n u_0\|_{H^1}^2 + \frac{2}{p+1} \int_{\mathbb{R}^d} \left| |Q_n^2 u_0|^{p+1} - |Q_n u_n(t)|^{p+1} \right| dx$$

$$\leq \|u_0\|_{H^1}^2 + C \int_{\mathbb{R}^d} \left(|Q_n u_n| + |Q_n^2 u_0| \right)^p |Q_n u_n(t) - Q_n^2 u_0|$$

$$1 = \frac{1}{p+1} + \frac{p}{p+1}$$

$$\leq \|u_0\|_{H^1}^2 + C \left\| |Q_n u_n| + |Q_n^2 u_0| \right\|_{L^{p+1}}^p \|\cancel{Q}_n u_n(t) - Q_n^{\cancel{2}} u_0\|_{L^{p+1}}$$

$$\leq \|u_0\|_{H^1}^2 + C \underbrace{\left(\|\cancel{Q}_n u_n(t)\|_{H^1} + \|\cancel{Q}_n^2 u_0\|_{H^1} \right)^p \left(\|u_n(t)\|_{H^1} + \|Q_n u_0\|_{H^1} \right)^\alpha}_{C'(M)} \|u_n(t) - Q_n u_0\|_{L^2}^{1-\alpha}$$

$$\|u_n(t) - u_n(0)\|_{L^2} \leq C(M) |t|^{\frac{1}{2}}$$

$|t| < \vartheta_n$

$$\|u_n(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + C(M) |t|^{\frac{1}{2}(1-\alpha)} \qquad \frac{1}{2}(1-\alpha) = b > 0$$

$$\|u_n(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + C(M) |t|^b$$

$$|u_n(t)|^2_{H^1} \le M^2 + C(M)\,|t|^b$$

$$C(M)\,T^b(M) = 2M^2$$

Verifichiamo che $0 < \boxed{T(M) < \vartheta_n}$ $\forall n$

Altrimenti $T(M) \ge \vartheta_n$ ma allora

per $|t| < \vartheta_n$ avremmo

$$|u_n(t)|^2_{H^1} \le M^2 + C(M)\,T^b(M) = 3M^2$$

$$\Rightarrow \quad |t| < \vartheta_n \Rightarrow \quad |u_n(t)|_{H^1} \le \sqrt{3}\,M < 2M$$

$+\infty > T(M) \ge \vartheta_n \Rightarrow$

$$\vartheta_n = \max\{\tau > 0 : |u_n(t)|_{H^1} < 2M \text{ per } |t| < \tau\}$$

così se $\tilde{\vartheta}_n > \vartheta_n \Rightarrow \sup_{|t| < \tilde{\vartheta}_n} |u_n(t)|_{H^1} \ge 2M$

Ma da $\sup_{|t| < \vartheta_n} |u_n(t)|_{H^1} \le \sqrt{3}\,M$ e da

$u_n \in C^0(\mathbb{R}, H^1) \Rightarrow \exists \; \tilde{\vartheta}_n > \vartheta_n$

t.c. $\sup_{|t| < \tilde{\vartheta}_n} |u_n(t)|_{H^1} < 2M$ e si

ottiene un assurdo.

Abbiamo dimostrato che $\exists \; T(M) > 0$ t.c.

$$\| u_n(t) \|_{L^\infty([-T(M), T(M)], H^1)} \quad 2M$$

$$\begin{cases} i \partial_t u_n = -P_n \Delta u_n + \lambda Q_n(|Q_n u_n|^{p-1} Q_n u_n) \\ u_n(0) = Q_n u_0 \end{cases}$$

$$E_n(u_n) = \frac{1}{2} \|\nabla u_n\|_{L^2}^2 + \frac{\lambda}{p+1} \int_{\mathbb{R}^d} |Q_n u_n|^{p+1} dx$$

$$H^1(\mathbb{R}^d, \mathbb{C}) \qquad \langle f, g \rangle = \mathrm{Re} \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, dx$$

$$\Omega(f,g) = \langle if, g \rangle = \langle g, if \rangle = -\langle ig, f \rangle$$

$$\| \quad -\Omega(g,f)$$

$$F : C^1(H^1(\mathbb{R}^d), \mathbb{R})$$

$$dF(u)X = \frac{d}{dt} F(u+tX) \Big|_{t=0} = \langle \nabla F(u), X \rangle$$

$$dF(u) \in \mathcal{L}(H^1(\mathbb{R}^d), \mathbb{R})$$

$$\nabla F(u) \in H^{-1}(\mathbb{R}^d)$$

$$\nabla E_n(u) = -P_n \Delta u + \lambda Q_n(|Q_n u|^{p-1} Q_n u)$$

$$\frac{d}{dt} E_n(u+tX)\Big|_{t=0} =$$

$$\frac{d}{dt}\left[\frac{1}{2}\|\nabla(u+tX)\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n(u+tX)|^{p+1}\,dx\right]\Big|_{t=0}$$

$$\frac{d}{dt}\left[\frac{1}{2}\|\nabla u\|_{L^2}^2 + t\langle \nabla u, \nabla X\rangle + \frac{t^2}{2}\|\nabla X\|_{L^2}\right]\Big|_{t=0}$$

$$+\frac{\lambda}{p+1}\int_{\mathbb{R}^d} \frac{d}{dt}|Q_n(u+tX)|^{p+1}\Big|_{t=0}\,dx$$

$$\langle a,b\rangle_{\mathbb{C}} = \operatorname{Re} a\bar{b}$$

$$= \langle -\Delta u, X\rangle$$

$$+\frac{\lambda}{p+1}\int_{\mathbb{R}^d} \frac{d}{dt}\Big|_{t=0}\left(\langle Q_n u + tQ_nX, Q_n u + tQ_nX\rangle_{\mathbb{C}}\right)^{\frac{p+1}{2}} dx$$

$$= \langle -\Delta u, X\rangle + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} dx\, \frac{d}{dt}\Big|_{t=0}\left(|Q_n u|^2 + 2t\langle Q_n u, Q_n X\rangle_{\mathbb{C}} + t^2|Q_nX|^2\right)^{\frac{p+1}{2}}$$

$$= \langle -\Delta u, X\rangle + \frac{\lambda}{\cancel{p+1}}\,\cancel{2}\,\frac{\cancel{p+1}}{\cancel{2}}\int_{\mathbb{R}^d} dx\; |Q_n u|^{p-1}\langle Q_n u, Q_n X\rangle_{\mathbb{C}}\, dx$$

$$= \langle -\Delta u, X\rangle + \lambda \operatorname{Re}\int_{\mathbb{R}^d} dx\; |Q_n u|^{p-1} Q_n u\, \overline{Q_n X}\, dx$$

$$= \langle -\Delta u, X\rangle + \lambda\langle Q_n(|Q_n u|^{p-1}Q_n u), X\rangle$$

$$= \langle -\Delta u + \lambda Q_n(|Q_n u|^{p-1}Q_n u), X\rangle$$

$$= \langle \nabla E_n(u), X \rangle$$

$H^1(\mathbb{R}^d, \mathbb{C})$ $\qquad$ $\Omega = \langle i\,\cdot, \cdot \rangle$

$$F \in C^1(H^1, \mathbb{R})$$

X_F $\qquad$ $\forall X$

$$dF(u)X = \Omega(X_F(u), X)$$
$$= \langle i X_F(u), X \rangle$$
$$= \langle \nabla F(u), X \rangle$$

$$\Rightarrow \quad X_F(u) = -i \nabla F(u)$$

$$i \partial_t u_n = \nabla E_n(u_n)$$

$$\partial_t u_n = -i \nabla E_n(u_n) = X_{E_n}(u_n)$$

$$\frac{d}{dt} E_n(u_n(t)) = dE_n(u_n(t))\, \dot{u}_n(t)$$
$$= \langle \nabla E_n(u_n(t)), \dot{u}_n(t) \rangle =$$
$$= \langle \nabla E_n(u_n(t)), -i \nabla E_n(u_n(t)) \rangle = 0$$

$$\Omega$$

$$1 = -i^2$$

$$\{F, G\} = \Omega(X_F, X_G)$$

$$\frac{d}{dt} F(u_n(t)) = \langle \nabla F(u_n(t)), X_{E_n}(u_n(t)) \rangle$$

$$= - \langle i \; \underbrace{i \nabla F(u_n(t))}_{X_F(u_n(t))}, X_{E_n}(u_n(t)) \rangle$$

$$= \Omega(X_{E_n}(u_n(t)), X_F(u_n(t))) =$$

$$= \{E_n, F\}(u_n(t))$$

$$\{E_n, Q\} = 0$$

$E_n($

$$E_n(u) = \frac{1}{2}\|P_n \nabla u\|^2_{L^2} + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |P_n u|^{p+1} dx$$

$$\shortparallel$$

$$E_n(e^{i\vartheta} u)$$

S^1 azione su $H^1(\mathbb{R}^d)$

$$0 = \frac{d}{d\vartheta}\Big|_0 E_n(e^{i\vartheta} u) = \langle \nabla E_n(u), \frac{d}{d\vartheta} e^{i\vartheta} u|_{\vartheta=0} \rangle$$

$$= \langle \nabla E_n(u), i u \rangle = -\langle i^2 \nabla E_n, i \nabla Q \rangle$$

$$\nabla Q(u) = u \qquad = -\langle i X_{E_n}, X_Q \rangle$$

$$\nabla P_j = i\partial_j u$$

$$P_j(u) = \frac{1}{2}\langle i\partial_j u, u\rangle \qquad = -\{E_n, Q\}$$

$$\{E_n, P_j\} = 0$$

$$\tau_D u(\cdot) = u(\cdot - D)$$

$$E_n(\tau_D u) = E_n(u)$$