$$2\epsilon \geq d(p, Q) + d(Q, q) \geq d(p, q) \geq \delta,$$

which contradicts the definition of ϵ. **Q.E.D.**

Putting together Props. 1, 2, and 3, we obtain the following theorem, which is the main goal of this section.

THEOREM. *Let* $S \subset R^3$ *be a regular compact orientable surface. Then there exists a differentiable function* $g\colon V \to R$, *defined in an open set* $V \subset R^3$, *with* $V \supset S$ *(precisely a tubular neighborhood of* S*), which has zero as a regular value and is such that* $S = g^{-1}(0)$.

Remark 1. It is possible to prove the existence of a tubular neighborhood of an orientable surface, even if the surface is not compact; the theorem is true, therefore, without the restriction of compactness. The proof is, however, more technical. In this general case, the $\epsilon(p) > 0$ is not constant as in the compact case but may vary with p.

Remark 2. It is possible to prove that a regular compact surface in R^3 is orientable; the hypothesis of orientability in the theorem (the compact case) is therefore unnecessary. A proof of this fact can be found in H. Samelson, "Orientability of Hypersurfaces in R^n," *Proc. A.M.S.* 22 (1969), 301–302.

2-8. A Geometric Definition of Area†

In this section we shall present a geometric justification for the definition of area given in Sec. 2-5. More precisely, we shall give a geometric definition of area and shall prove that in the case of a bounded region of a regular surface such a definition leads to the formula given for the area in Sec. 2-5.

To define the area of a region $R \subset S$ we shall start with a *partition* $\mathcal{P}$ of R into a finite number of regions R_i, that is, we write $R = \bigcup_i R_i$, where the intersection of two such regions R_i is either empty or made up of boundary points of both regions (Fig. 2-33). The *diameter* of R_i is the supremum of the distances (in R^3) of any two points in R_i; the largest diameter of the R_i's of a given partition $\mathcal{P}$ is called the *norm* μ of $\mathcal{P}$. If we now take a partition of each R_i, we obtain a second partition of R, which is said to *refine* $\mathcal{P}$.

Given a partition

$$R = \bigcup_i R_i$$

of R, we choose arbitrarily points $p_i \in R_i$ and project R_i onto the tangent

†This section may be omitted on a first reading.

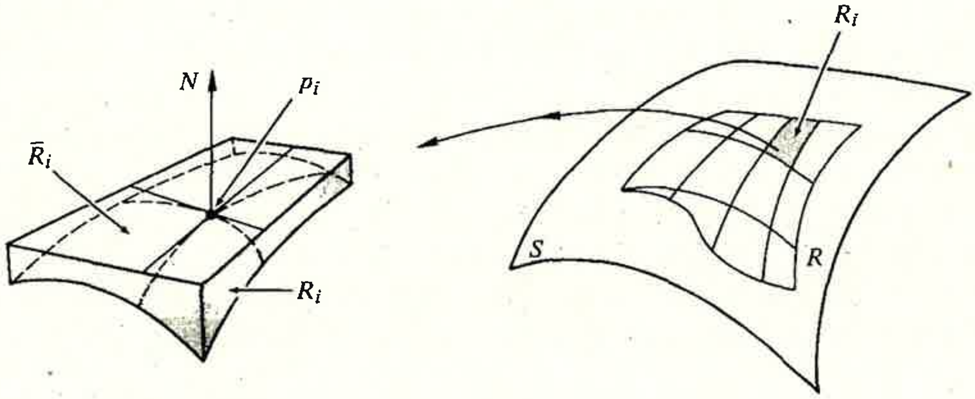

Figure 2-33

plane at p_i in the direction of the normal line at p_i; this projection is denoted by $\bar{R}_i$ and its area by $A(\bar{R}_i)$. The sum $\sum_i A(\bar{R}_i)$ is an approximation of what we understand intuitively by the area of R.

If, by choosing partitions $\mathcal{P}_1, \ldots, \mathcal{P}_n, \ldots$ more and more refined and such that the norm μ_n of $\mathcal{P}_n$ converges to zero, there exists a limit of $\sum_i A(\bar{R}_i)$ and this limit is independent of all choices, then we say that R has an *area* $A(R)$ defined by

$$A(R) = \lim_{\mu_n \to 0} \sum_i A(\bar{R}_i).$$

An instructive discussion of this definition can be found in R. Courant, *Differential and Integral Calculus*, Vol. II, Wiley-Interscience, New York, 1936, p. 311.

We shall show that a bounded region of a regular surface does have an area. We shall restrict ourselves to bounded regions contained in a coordinate neighborhood and shall obtain an expression for the area in terms of the coefficients of the first fundamental form in the corresponding coordinate system.

PROPOSITION. *Let* $\mathbf{x}: U \to S$ *be a coordinate system in a regular surface* S *and let* $R = \mathbf{x}(Q)$ *be a bounded region of* S *contained in* $\mathbf{x}(U)$. *Then* R *has an area given by*

$$A(R) = \iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v| \, du \, dv.$$

Proof. Consider a partition, $R = \bigcup_i R_i$, of R. Since R is bounded and closed (hence, compact), we can assume that the partition is sufficiently refined so that any two normal lines of R_i are never orthogonal. In fact, because the normal lines vary continuously in S, there exists for each $p \in R$ a neighborhood of p in S where any two normals are never orthogonal; these neighborhoods constitute a family of open sets covering R, and considering a partition of R the norm of which is smaller than the Lebesgue number of the covering (Sec. 2-7, Property 3 of compact sets), we shall satisfy the required condition.

Fix a region R_i of the partition and choose a point $p_i \in R_i = \mathbf{x}(Q_i)$. We want to compute the area of the normal projection $\bar{R}_i$ of R_i onto the tangent plane at p_i. To do this, consider a new system of axes $p_i\bar{x}\bar{y}\bar{z}$ in R^3, obtained from $Oxyz$ by a translation Op_i, followed by a rotation which takes the z axis into the normal line at p_i in such a way that both systems have the same orientation (Fig. 2-34). In the new axes, the parametrization can be written

$$\bar{\mathbf{x}}(u, v) = (\bar{x}(u, v), \bar{y}(u, v), \bar{z}(u, v)),$$

where the explicit form of $\bar{\mathbf{x}}(u, v)$ does not interest us; it is enough to know that the vector $\bar{\mathbf{x}}(u, v)$ is obtained from the vector $\mathbf{x}(u, v)$ by a translation followed by an orthogonal linear map.

We observe that $\partial(\bar{x}, \bar{y})/\partial(u, v) \neq 0$ in Q_i; otherwise, the $\bar{z}$ component of some normal vector in R_i is zero and there are two orthogonal normal lines in R_i, a contradiction of our assumptions.

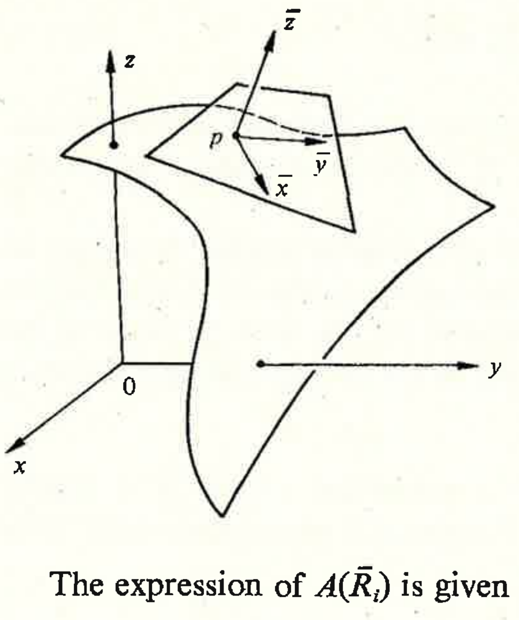

Figure 2-34

The expression of $A(\bar{R}_i)$ is given by

$$A(\bar{R}_i) = \iint_{\bar{R}_i} d\bar{x}\, d\bar{y}.$$

Since $\partial(\bar{x}, \bar{y})/\partial(u, v) \neq 0$, we can consider the change of coordinates $\bar{x} = \bar{x}(u, v)$, $\bar{y} = \bar{y}(u, v)$ and transform the above expression into

$$A(\bar{R}_i) = \iint_{Q_i} \frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)}\, du\, dv.$$

We remark now that, at p_i, the vectors $\bar{\mathbf{x}}_u$ and $\bar{\mathbf{x}}_v$ belong to the $\bar{x}\bar{y}$ plane; therefore,

$$\frac{\partial \bar{z}}{\partial u} = \frac{\partial \bar{z}}{\partial v} = 0 \qquad \text{at } p_i;$$

hence,

$$\left|\frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)}\right| = \left|\frac{\partial \bar{\mathbf{x}}}{\partial u} \wedge \frac{\partial \bar{\mathbf{x}}}{\partial v}\right| \qquad \text{at } p_i.$$

It follows that

$$\left|\frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)}\right| - \left|\frac{\partial \bar{\mathbf{x}}}{\partial u} \wedge \frac{\partial \bar{\mathbf{x}}}{\partial v}\right| = \epsilon_i(u, v), \qquad (u, v) \in Q_i,$$

where $\epsilon_i(u, v)$ is a continuous function in Q_i with $\epsilon_i(\mathbf{x}^{-1}(p_i)) = 0$. Since the length of a vector is preserved by translations and orthogonal linear maps, we obtain

$$\left|\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v}\right| = \left|\frac{\partial \bar{\mathbf{x}}}{\partial u} \wedge \frac{\partial \bar{\mathbf{x}}}{\partial v}\right| = \left|\frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)}\right| - \epsilon_i(x, y).$$

Now let M_i and m_i be the maximum and the minimum of the continuous function $\epsilon_i(u, v)$ in the compact region Q_i; thus,

$$m_i \leq \left|\frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)}\right| - \left|\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v}\right| \leq M_i;$$

hence,

$$m_i \iint_{Q_i} du\, dv \leq A(\bar{R}_i) - \iint_{Q_i} \left|\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v}\right| du\, dv \leq M_i \iint_{Q_i} du\, dv.$$

Doing the same for all R_i, we obtain

$$\sum_i m_i A(Q_i) \leq \sum_i A(\bar{R}_i) - \iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v|\, du\, dv \leq \sum_i M_i A(Q_i).$$

Now, refine more and more the given partition in such a way that the norm $\mu \longrightarrow 0$. Then $M_i \longrightarrow m_i$. Therefore, there exists the limit of $\sum_i A(\bar{R}_i)$, given by

$$A(R) = \iint_Q \left|\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v}\right| du\, dv,$$

which is clearly independent of the choice of the partitions and of the point p_i in each partition. **Q.E.D.**