$$E^2(T, M) =$$

$$= \{ v \in L^\infty([0,T], H^2) \cap W^{1,q}([0,T], L^{p+1}) \cap E^1(T, a) \}$$

$$|v|_T^{(2)} := |v|_{L^\infty([0,T], H^2)} + |\partial_t v|_{L^q([0,T], L^{p+1})} \leq M$$

$$E^1(T, a) = \{ v \in L^\infty([0,T], H^1) \cap L^q([0,T], W^{1,p+1})$$

s.t.

$$|v|_T^{(1)} = |v|_{L^\infty([0,T], H^1)} +$$

$$+ |v|_{L^q([0,T], W^{1,p+1})} \leq a \}$$

$$\begin{cases} i\partial_t u = -\Delta u + \lambda |u|^{p-1} u \\ u_0 \in H^2 \end{cases}$$

$$u = e^{it\Delta} u_0 - i\lambda \underbrace{\int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds}_{\phi(u)}$$

t

$$\nabla u = e^{it\Delta} \nabla u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} \nabla\left(|u|^{p-1} u\right) ds$$

$|u|^{p-1} \nabla u$

$$\phi(u) \;\cancel{u(t)} = e^{it\Delta} u_0 - i\lambda \underbrace{\int_0^t e^{is\Delta} |u(t-s)|^{p-1} u(t-s)\, ds}_{\phi(u)}$$

$$\partial_t \phi(u)\;\cancel{u(t)} = ie^{it\Delta} \Delta u_0 - i\lambda e^{it\Delta} |u(0)|^{p-1} u(0)$$

$$- i\lambda \int_0^t e^{is\Delta} \partial_t\left(|u(t-s)|^{p-1} u(t-s)\right) ds$$

$$\phi : E^{(2)}(T, M) \circlearrowleft$$

(a, b) admissible $\quad a = \infty \quad b = 2 \quad |u(0)|^{p}$

$$|\partial_t \phi(u)|_{L^a([0,T]), L^b)} \leq C_0 |u_0|_{H^2} + C_0 \left| |u(0)|^{p-1} u(0) \right|_{L^2}$$

$$+ \left| \partial_s |u(s)|^{p-1} u(s) \right|_{L^{q'}([0,T], L^{\frac{p+1}{p}})}$$

$$|u(0)|^{p}_{L^{2p}} \leq C_0^{p} |u_0|^{p}_{\dot{H}^{1}}$$

$$\frac{1}{2p} = \frac{1}{2} - \frac{s}{d} \qquad 0<s<2$$

$$s = \frac{d}{2} - \frac{d}{2p} = \frac{d}{2}\,\frac{1}{p'}$$

$$\frac{1}{2d^*} = \frac{1}{2}\,\frac{d-2}{d+2} = \frac{1}{2} - \frac{\sigma}{d} \qquad 2>\sigma>s$$

$$\frac{\sigma}{d} = \frac{1}{2}\left(1 - \frac{d-2}{d+2}\right) = \frac{1}{2}\,\frac{d+2-(d-2)}{d+2} =$$

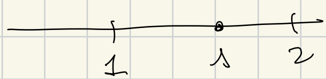
$$= \frac{2}{d+2}$$

$$\frac{\sigma}{d} = \frac{2}{d+2} \qquad \sigma = \frac{2d}{d+2} < 2$$

$$|u(0)|^p_{L^{2p}} \le C_d\, |u(0)|^p_{\dot H^s} \qquad 0<s<2.$$

If $s \le 1 \Rightarrow |u(0)|^p_{L^{2p}} \le C_d\, |u(0)|^p_{H^1} \le C_d\, a^p$

If $1 < s < 2$ $\qquad s = (1-\alpha) + 2\alpha \quad$ for $0<\alpha<1$



$$|u(0)|^p_{L^{2p}} \le C_d\, |u(0)|^{(1-\alpha)p}_{\dot H^1}\, |u(0)|^{\alpha p}_{\dot H^2} \qquad \alpha p < 1$$

$$\le C_d\, a^{(1-\alpha)p}\, M$$

$$p' = \frac{p}{p-1}$$

$$s = 1+\alpha \Rightarrow \alpha = s-1$$

$$\alpha p = (s-1)p = \left(\frac{d}{2}\,\frac{1}{p'} - 1\right)p = \left(\frac{d}{2}\,\frac{p-1}{p} - 1\right)p =$$

$$= \frac{d}{2}(p-1) - p = p\left(\frac{d}{2}-1\right) - \frac{d}{2} = p\,\frac{d-2}{2} - \frac{d}{2}$$

$$< d^{*}\frac{d-2}{2} - \frac{d}{2} = \cancel{\frac{d+2}{d-2}}\,\cancel{\frac{d-2}{2}} - \frac{d}{2} =$$

$$= \frac{\cancel{d}+2}{2} - \frac{\cancel{d}}{2} = 1$$

$$\left\| \partial_s |u(s)|^{p-1} u(s) \right\|_{L^{q'}\left((0,T], L^{\frac{p+1}{p}}\right)} \lesssim$$

$$H^1 \hookrightarrow L^{d^*+1}$$

$$\left\| |u|^{p-1} \partial_t u \right\|_{L^{q'}\left((0,T], L^{\frac{p+1}{p}}\right)} \leq \qquad \hookrightarrow L^{p+1}$$

$$\leq \| u \|^{p-1}_{L^{\beta}\left((0,T), L^{p+1}\right)} \| \partial_t u \|_{L^{q}\left((0,T), L^{p+1}\right)}$$

$$\frac{p-1}{\beta} + \frac{1}{q} = \frac{1}{q'}$$

$$\leq C_d T^{\frac{p-1}{\beta}} \| u \|^{p-1}_{L^{\infty}\left((0,T), H^1\right)} \| \partial_t u \|_{L^{q}\left((0,T), L^{p+1}\right)}$$

$$\leq C_d T^{\frac{p-1}{\beta}} a^{p-1} M$$

$$\| \partial_t \phi(u) \|_{L^{q}\left((0,T), L^{p+1}\right)} \leq C_0 \| u_0 \|_{H^2} + C_d a +$$

$$+ C_d a^{(1-\alpha)p} M^{\alpha p}$$

$$+ C_d T^{\frac{p-1}{\beta}} a^{p-1} M$$

I take $C_d \, T^{\frac{\beta-1}{\beta}} a^{p-1} < \frac{1}{8}$

I choose M s.t.

$$\frac{M}{8} > C_0 |u_0|_{H^2} + C_d a + C_d \, a^{(1-d)p} \, M^{\alpha p}$$

$$|\partial_t \phi(u)|_{L^q((0,T), L^{p+1})} < \frac{M}{4}$$

$$\phi(u) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)} |u|^{p-1} u \, ds$$

$$\begin{cases} (i\partial_t - \Delta)\Phi(u) = i\lambda |u|^{p-1} u \\ \Phi(u)|_{t=0} = u_0 \end{cases}$$

$$\Delta \Phi(u) = i\partial_t \Phi(u) - i\lambda |u|^{p-1} u$$

$$|\Delta \phi(u)|_{L^\infty([0,T], L^2)} \simeq |\phi(u)|_{L^\infty([0,T], \dot{H}^2)}$$

$$\leq |i\partial_t \phi(u)|_{L^\infty([0,T], L^2)} + ||u|^{p-1} u|_{L^\infty([0,T], L^2)}$$

$$\leq \frac{M}{4} + \underbrace{C a^p + C a^{(1-\alpha)p} M^{\alpha p}}_{< \frac{M}{8}}$$

$$|\Delta \phi(u)|_{L^\infty([0,T], L^2)} < M \frac{3}{8}$$

$$|\phi|_T^{(2)} \leq \frac{3}{8}M + \frac{M}{4} < \frac{M}{2}$$

$$\Phi : E^{(2)}(M, T, a) \circlearrowleft$$

Theorem Let $u \in C^0((-S, T), H^1(\mathbb{R}^d))$ be a maximal solution and suppose that $u_0 \in H^2(\mathbb{R}^d)$. Then $u \in C^0((-S, T), H^2(\mathbb{R}^d))$.

Pf Let T_* be the maximal forward lifetime of $u \in C^0([0, T^*), H^2(\mathbb{R}^d))$.

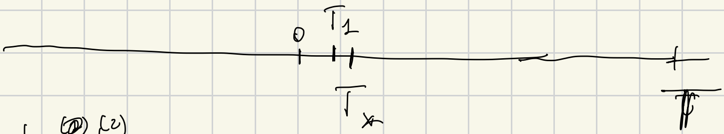
We want to show $T_* = T$. It is obvious that $T_* \leq T$.

Let us suppose $T_* < T \leq +\infty$

$$\Rightarrow \lim_{t \to T_*^-} |u(t)|_{\dot{H}^2} = +\infty$$

Let $a \in \mathbb{R}_+$

$$\begin{cases} |u|_{St(\mathbb{I}, H^1)} \leq a \\ |u|_{L^\infty([0, T_*], H^1)} \leq a \end{cases}$$



$$\overbrace{|\partial_t u|_{L^q([0,T_1],L^{p+1})} + |\Delta u|_{L^\infty([0,T_1],L^2)}}^{|u|^{(2)}_{T_1}} \leq$$

$$\leq C\,|u_0|_{H^2} + C_\alpha + C_\alpha \left(|u|^{(2)}_{T_1}\right)^{p\alpha}$$

$$+ C_\alpha\, T_1^{\frac{p-1}{\beta}}\, |u|^{(2)}_{T_1}$$

$$|u|^{(2)}_{T_1} \leq C_0 |u_0|_{H^2} + C_\alpha + C_\alpha \left(|u|^{(2)}_{T_1}\right)^{p\alpha}$$

$$+ \underbrace{C_\alpha\, T_1^{\frac{p-1}{\beta}}}_{<\frac{1}{2}}\, |u|^{(2)}_{T_1}$$

$$|u|^{(2)}_{T_1} \leq 2C_0|u_0|_{H^2} + 2C_\alpha + 2C_\alpha\left(|u|^{(2)}_{T_1}\right)^{p\alpha}$$

$$\underset{\downarrow\ +\infty}{|u|^{(2)}_{T_1}} \left(1 - 2C_\alpha \underbrace{\left(|u|^{(2)}_{T_1}\right)^{p\alpha-1}}_{\xrightarrow{T_1 \to T_*}\ 0}\right) \leq 2C_0|u_0|_{H^2} + 2C_\alpha$$

$$0 < T_1 < T_*$$

$$T_1 \longrightarrow T_*$$

This means $T_* < T$ is false