$$\begin{cases} i\partial_t u = -\Delta u + \lambda |u|^{p-1} u \\ u_{t=0} = u_0 \in H^1(\mathbb{R}^d) \end{cases}$$

$$1 < p < d^* = \begin{cases} +\infty & d = 1,2 \\ \frac{d+2}{d-2} & d > 2 \end{cases}$$

$\lambda = 1, -1$

$$u(t) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds$$

$\forall u_0$ $\exists$ $(-S_*, T_*)$ maximal and a solution

$$u \in C^0((-S, T), H^1(\mathbb{R}^d))$$

and for any $I \subset\subset (-S, T)$

$$u \in L^\alpha(I, W^{1,\beta}(\mathbb{R}^d)) \quad \forall$$

(α, β) admissible

$$\frac{2}{\alpha} + \frac{d}{\beta} = \frac{d}{2}$$

$\alpha, \beta \geq 2$

for $d = 2$ the endpoint case is excluded

$\alpha = 2$ $\quad \beta = \dfrac{d}{\frac{d}{2} - 1}$ $\quad (d = 3 \quad \beta = 6)$

$$E(u(t))$$
$$Q(u(t))$$
$$P_j(u(t))$$
are constant.

$$P_n = \chi_{[0,1]}\left(\frac{\sqrt{-\Delta}}{2\pi n}\right)$$

$$\begin{cases} i\partial_t (1-P_n)u_n = 0 \\ (1-P_n)u_n|_{t=0} = 0 \end{cases} \qquad \chi_{\mathbb{R}^d \setminus D(0,n)}(\xi) \quad \varphi\left(\frac{\xi}{n}\right)$$

$$\widehat{P_n f}(\xi) = \chi_{[0,1]}\left(\frac{|\xi|}{n}\right)\hat{f}(\xi)$$

$$\widehat{Q_n f}(\xi) = \varphi\left(\frac{\xi}{n}\right)\hat{f}(\xi)$$

$$\begin{cases} i\partial_t u_n = -\cancel{P_n}\,\Delta u_n + \lambda Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right) \\ u_n|_{t=0} = Q_n u_0 \end{cases}$$

$$P_n u_n = u_n \qquad (1-P_n)u_n = 0$$

$$\begin{cases} E_n(v) = \frac{1}{2}\|P_n \nabla v\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n v|^{p+1}\,dx \\ P_j(v) \\ Q(v) = \frac{1}{2}\|v\|_{L^2}^2 \end{cases}$$

We fixed $M > \|u_0\|_{H^1}$ and we found that $\exists\ T = T(M) > 0$ s.t. $\|u_n(t)\|_{H^1} \leq 2M \quad \forall\, t \in [0, T(M)]$

Let $I = [0,T] \subseteq [0, T(M)] \cap [0, T_*)$

we will show that

$$u_n \to u \in C^0(I, H^1) \cap L^q(I, W^{1,p+1}(\mathbb{R}^d))$$

$$u(t) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds$$

$$u_n(t) = e^{it\Delta} Q_n u_0 - i \int_0^t e^{i(t-s)\Delta} Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right) ds$$

$$u(t) - u_n(t) = e^{it\Delta}(1-Q_n)u_0 - i\int_0^t e^{i(t-s)\Delta}(1-Q_n)|u(s)|^{p-1}u(s)\, ds$$

$$- i \int_0^t e^{i(t-s)\Delta} Q_n\left(|u(s)|^{p-1}u(s) - |Q_n u(s)|^{p-1} Q_n u(s)\right) ds$$

$$- i \int_0^t e^{i(t-s)\Delta} Q_n\left(|Q_n u|^{p-1} Q_n u - |Q_n u_n|^{p-1} Q_n u_n\right) ds$$

$$st(I) = L^p(I, W^{1,p+1}) \cap L^\infty(I, H^1)$$

$$|u - u_n|_{st(I)} \leq c_0 |(1-Q_n)u_0|_{H^1} + c_0 \left|(1-Q_n)|u|^{p-1}u\right|_{L^{q'}(I, W^{1,\frac{p+1}{p}})}$$

$$+ c_0 \left|\cancel{Q_n}\left(|u|^{p-1}u - |Q_n u|^{p-1} Q_n u\right)\right|_{L^{q'}(I, W^{1,\frac{p+1}{p}})}$$

$$+ c_0 \left|\cancel{Q_n}\left(|Q_n u|^{p-1} Q_n u - |Q_n u_n|^{p-1} Q_n u_n\right)\right|_{L^{q'}(I, W^{1,\frac{p+1}{p}})}$$

$$\leq c_0 |(1-Q_n)u_0|_{H^1} + c_0 \left|(1-Q_n)|u|^{p-1}u\right|_{L^{q'}(I, W^{1,\frac{p+1}{p}})}$$

$$+ c_0 C|I|^{\theta}\left(|u|^{p-1}_{L^\infty(I,H^1)} + |\cancel{Q_n} u|^{p-1}_{L^\infty(I,H^1)}\right) |(1-Q_n)u|_{L^{p+1}(I, W^{1,p+1})}$$

$$+ C_0 |I|^{\vartheta} C \Big(\underbrace{|u|^{p-1}_{L^\infty(I,H^1)}} + |u_n|^{p-1}_{L^\infty(I,H^1)} \Big) \, |\cancel{Q_n} u - \cancel{Q_n} u_n|_{L^{p+1}(I, W^{1,p+1})}$$

$$|u - u_n|_{St(I)} \leq$$

$$\leq C_0 |(1-Q_n) u_0|_{H^1} + C_0 \big| (1-Q_n) |u|^{p-1} u \big|_{L^{q'}(I, W^{1,\frac{p+1}{p}})}$$

$$+ C(M)\, |I|^{\vartheta} \; |(1-Q_n) u|_{L^{p+1}(I, W^{1,p+1})}$$

$$+ \underbrace{C(M)\, |I|^{\vartheta}}_{< \frac{1}{2}} \; |u - u_n|_{St(I)}$$

$$\frac{1}{2} |u - u_n|_{St(I)} \leq \cancel{(1 - C(M)|I|^{\vartheta})} \, |u - u_n|_{St(I)}$$

$$\leq \underbrace{C_0 |(1-Q_n) u_0|_{H^1}}_{\to 0} + \underbrace{C_0 \big| (1-Q_n) |u|^{p-1} u \big|_{L^{q'}(I, W^{1,\frac{p+1}{p}})}}_{\to 0}$$

$$+ C(M)\, |I|^{\vartheta} \, |\underbrace{(1-Q_n) u}_{\to 0}|_{L^{p+1}(I, W^{1,p+1})}$$

$$Q_n u_0 \xrightarrow{n \to +\infty} u_0 \qquad \forall\, u_0 \in H^1(\mathbb{R}^d)$$

$$(1-Q_n) f \xrightarrow{n \to +\infty} 0 \qquad \forall\, f \in L^{q'}(I, W^{1,\frac{p+1}{p}})$$

If I is small

$$u - u_n \longrightarrow 0 \quad \text{in} \quad St(I) \qquad C^0(I, H^1)$$

$$E_n(u_n(t)) = \frac{1}{2} |\nabla u_n(t)|^2_{L^2} + \frac{\lambda}{p+1} \int_{\mathbb{R}^d} |Q_n u_n(t)|^{p+1}\, dx$$

$$= \frac{1}{2} |\nabla Q_n u_0|^2_{L^2} + \frac{\lambda}{p+1} \int_{\mathbb{R}^d} |Q_n u_0|^{p+1} \, dx$$

For $n \to +\infty$

u_n ~~in~~ $C^0(I, H^1)$

$\downarrow$

$$u \implies |\nabla u_n(t)|^2_{L^2} \longrightarrow |\nabla u(t)|^2_{L^2}$$

$$Q_n u_n = Q_n u + Q_n(u_n - u)$$

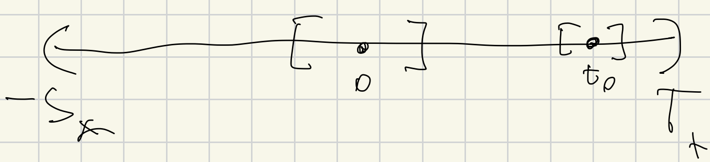
$$|Q_n u_n(t) - u(t)|_{H} \leq |(Q_n - 1) u(t)|_{H^1} + |u_n(t) - u(t)|_{H^1}$$

$$|Q_n u_n - u|_{C^0(I, H^1)} \leq \underbrace{|(Q_n - 1) u|_{C^0(I, H^1)}}_{\downarrow \, 0} + \underbrace{|u_n - u|_{C^0(I, H^1)}}_{\downarrow \, 0}$$

$$\begin{array}{ccc} E_n(u_n(t)) & \xrightarrow{n \to +\infty} & E(u(t)) \\ \parallel & & \parallel \\ E_n(Q_n u_0) & \xrightarrow{n \to +\infty} & E(u_0) \end{array}$$

We conclude that $E(u(t)) = E(u_0)$ in I

So we have $u \in C^0((-S_*, T_*), H^1)$



and we found a $T > 0$ s.t.

in $[-T, T]$ where $E(u(t))$ is constant

$t \to E(u(t)) \in C^0((-S_*, T_*), \mathbb{R})$
is locally constant $\Rightarrow$ $E(u(t)) \equiv E(u_0)$

Corollary Go back Prop 3.4

Prop 3.4 If $1 < p < 1 + \frac{4}{d}$ and $u_0 \in L^2(\mathbb{R}^d)$

$\exists$ $T \geq T(|u_0|_{L^2}) > 0$ and

$u \in C^0([-T, T], L^2(\mathbb{R}^d)) \cap L^q([-T, T], L^{p+1}(\mathbb{R}^d))$

solution of the equation

We proved $\exists$ V neigh of u_0 in L^2 and

$v_0 \in V \longrightarrow v \in C^0([-T, T], L^2(\mathbb{R}^d))$

is Lipschitz

1) $Q(u(t)) = Q(u_0)$

2) In fact we can take T as large as we want

$u \in C^0(\mathbb{R}, L^2(\mathbb{R}^d))$

Proof $Q(u(t)) = Q(u_0)$ in $[-T, T]$

Let us take $u_0^{(n)} \in H^1(\mathbb{R}^d)$ $\quad u_0^{(n)} \xrightarrow{n \to +\infty} u_0$ in $L^2(\mathbb{R}^d)$

$$u^{(n)} \longrightarrow u \quad \text{in} \quad C^0([-T,T], L^2(\mathbb{R}^d))$$

$$u^{(n)} \in C^0([-T,T], H^1(\mathbb{R}^d))$$

$$\begin{array}{ccc} Q(u^{(n)}(t)) & \longrightarrow & Q(u(t)) \quad \text{in } C^0([-T,T]) \\ \| & & \| \\ Q(u_0^{(n)}) & \longrightarrow & Q(u_0) \end{array}$$

$Q(u(t))$ is continuous and locally constant

So if $\quad u \in C^0((-S_*, T_*), L^2(\mathbb{R}^d))$

$$Q(u(t)) = Q(u_0)$$

This implies $S_* = T_* = +\infty$

If by contradiction we had $T_* < +\infty$

would need to have

$$\lim_{t \to T_*^-} |u(t)|_{L^2} = +\infty$$

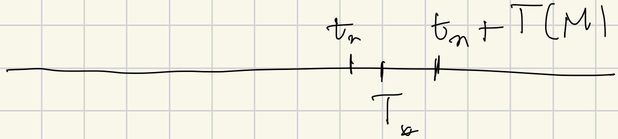
(and this is not true

$|u(t)|_{L^2} = |u_0|_{L^2}$

$\Rightarrow T_* = +\infty$)

If I had $t_n \to T_*^-$

with $\|u(t_n)\|_{L^2} \leq M < +\infty$



$$\begin{cases} i\partial_t u = -\Delta u + \lambda |u|^{p-1} u \\ u|_{t=t_n} = u(t_n) \end{cases}$$

$\exists$ $T(M) > 0$ and a solution

$$v_n \in C^0([t_n, t_n + T(M)], L^2(\mathbb{R}^d))$$

with $v_n = u$ where they are both defined

$$w = \begin{cases} u & \text{in } [0, t_n] \\ v_n & \text{in } [t_n, t_n + T(M)] \end{cases}$$

$w = u$ for $t < T_*$ $\Rightarrow$ w is a solution in a longer interval and we get a contradiction.

In a similar way it can be shown

that if $u \in C^0((-S_\infty, T_*), H^1)$ is

a maximal solution related to prop. 3.6

if $T_* < +\infty$ then

$$\lim_{t \to T_*} \|\nabla u(t)\|_{L^2} = +\infty$$

Let us check that for $p < 1 + \frac{4}{d}$

we have no blow up.

More generally there is no blow up if

1) $\lambda > 0$

2) in $p < 1 + \frac{4}{d}$

$$i \partial_t u = -\Delta u + \lambda |u|^{p-1} u$$

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} dx$$

For $\lambda = 1$

$$\frac{1}{2} \|\nabla u(t)\|_{L^2}^2 \le E(u(t)) = E(u_0)$$ This

excludes blow up.

$$p < 1 + \frac{4}{d} \qquad \lambda = -1$$

$$E(u) = \frac{1}{2} |\nabla u|^2_{L^2} - \frac{1}{p+1} |u|^{p+1}_{L^{p+1}}$$

$$\frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d} \qquad \dot{H}^{\alpha}(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$$

$$|u|_{L^{p+1}} \le C_\alpha \, |u|^{1-\alpha}_{L^2} \, |\nabla u|^{\alpha}_{L^2}$$

$$|u|^{p+1}_{L^{p+1}} \le C_\alpha^{p+1} \, |u|^{(1-\alpha)(p+1)}_{L^2} \, |\nabla u|^{\alpha(p+1)}_{L^2}$$

$$\ge \frac{1}{2} |\nabla u(t)|^2_{L^2} - \frac{C_\alpha^{p+1}}{p+1} |u_0|^{(1-\alpha)(p+1)}_{L^2} \, |\nabla u(t)|^{\alpha(p+1)}_{L^2}$$

$$|\nabla u(t)|^2_{L^2} \le 2E(u_0) + \frac{C_\alpha^{p+1}}{p+1} |u_0|^{(1-\alpha)(p+1)}_{L^2} \, |\nabla u(t)|^{\alpha(p+1)}_{L^2}$$

$$\alpha(p+1) < 2 \iff 1 < p < 1 + \frac{4}{d}$$

$$|\nabla u(t)|^2_{L^2} \left(1 - \frac{C_\alpha^{p+1} |u_0|^{(1-\alpha)(p+1)}_{L^2}}{|\nabla u(t)|^{1-\alpha(p+1)}_{L^2}} \right) \le 2E(u_0)$$

is not compatible with $|\nabla u(t)|^2_{L^2} \xrightarrow{t \to T_*} +\infty$