5 May

$$\begin{cases} i\partial_t u = -\Delta u + |u|^{p-1}u & t \geq 0 \quad x \in \mathbb{R}^d \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}^d) \end{cases}$$

$$1 + \frac{4}{d} < p < d^* \qquad d^* = \begin{cases} +\infty & d = 1,2 \\ \frac{d+2}{d-2} & d \geq 3 \end{cases}$$

<u>Theorem</u> $\exists$ a unique $u \in C^0(\mathbb{R}, H^1(\mathbb{R}^d))$.
We have $u \in L^a(\mathbb{R}, W^{1,b}(\mathbb{R}^d))$ with (a,b) admissible and $\exists$ $u_\pm \in H^1(\mathbb{R}^d)$ s.t.

$$\lim_{t \to \pm\infty} \| u(t) - e^{it\Delta} u_\pm \|_{H^1(\mathbb{R}^d)} = 0$$

$e^{it\Delta}u_\pm$ solves $i\partial_t u = -\Delta u$

<u>Remark</u> we will prove only $d \geq 3$ — Ginibre–Velo

$d = 1,2$ — Nakanishi

For $1 < p \leq 1 + \frac{2}{d}$ the result is false

$$i\partial_t u = -\Delta u + V u \qquad \| u(t) - e^{it\Delta} u_+ \| \xrightarrow{t \to +\infty} 0$$

$$1 + \frac{2}{d} < p \leq 1 + \frac{4}{d}$$

$$u \in L^q(\mathbb{R}_+, W^{1,p+1}(\mathbb{R}^d))$$

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$$

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds \qquad e^{-it\Delta}$$

$$e^{-it\Delta} u(t) = u_0 - \int_0^t e^{-is\Delta} |u(s)|^{p-1} u(s)\, ds$$

We want to prove that $\exists\, u_+ \in H^1(\mathbb{R}^d)$

s.t.

$$\begin{cases} \lim\limits_{t \to +\infty} \left| e^{-it\Delta} u(t) - u_+ \right|_{H^1} = 0 \\ = \lim\limits_{t \to +\infty} \left| u(t) - e^{it\Delta} u_+ \right|_{H^1} = 0 \end{cases}$$

$$e^{-it|\xi|^2} \qquad H^1$$

$$L^2(\mathbb{R}^d, \langle \xi \rangle^2 dx)$$

$$e^{-it\Delta} u(t) = u_0 - i \int_0^t e^{-is\Delta} |u(s)|^{p-1} u(s)\, ds$$

$t_1 < t_2$

$$e^{-it_2\Delta} u(t_2) - e^{-it_1\Delta} u(t_1) = -i \int_{t_1}^{t_2} e^{-is\Delta} |u(s)|^{p-1} u(s)\, ds$$

$$\left| e^{-it_2\Delta} u(t_2) - e^{-it_1\Delta} u(t_1) \right|_{H^1} \leq$$

$$\leq C_0 \left\| |u|^{p-1} u \right\|_{L^{q'}\left((t_1,t_2),\, W^{1,\frac{p+1}{p}}(\mathbb{R}^d)\right)}$$

$$= C_0 \left\| \left\| |u|^{p-1} \nabla u \right\|_{L^{\frac{p+1}{p}}(\mathbb{R}^d)} \right\|_{L^{q'}(t_1,t_2)}$$

$$\leq C_0 \left\| \| u \|^{p-1}_{L^{p+1}} \| \nabla u \|_{L^{p+1}} \right\|_{L^{q'}(t_1,t_2)}$$

$$\frac{p}{p+1} = \frac{p-1}{p+1} + \frac{1}{p+1}$$

$$\frac{1}{q'} = \frac{p-1}{\alpha} + \frac{1}{q}$$

$$\leq C_0 \| u \|^{p-1}_{L^{\cancel{(p-1)}\frac{\alpha}{\cancel{p-1}}}\left((t_1,t_2),\, L^{p+1}(\mathbb{R}^d)\right)} \| u \|_{L^{q}\left((t_1,t_2),\, W^{1,p+1}(\mathbb{R}^d)\right)}$$

here

$\alpha > q$. Otherwise $\alpha \leq q$

$$\frac{1}{q'} = \frac{p-1}{\alpha} + \frac{1}{q} \geq \frac{p}{q}$$

$$\frac{p}{q} \leq \frac{1}{q'} = 1 - \frac{1}{q}$$

$$p \leq q - 1 \qquad p+1 \leq q$$

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$$

Notice that $\left(2 + \frac{4}{d},\, 2 + \frac{4}{d}\right)$ is admissible

$$\frac{2}{\alpha} + \frac{d}{\beta} = \frac{d}{2} \qquad \alpha = \beta$$

$$\frac{1}{\alpha}\,(d+2) = \frac{d}{2}$$

$$\alpha = \frac{d+2}{\frac{d}{2}} = \frac{2d+4}{d} = 2 + \frac{4}{d}$$

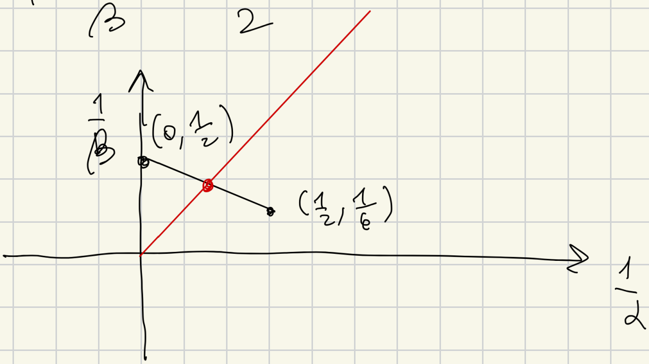
By hypothesis $\quad p > 1 + \frac{4}{d}$

$$p+1 > 2 + \frac{4}{d}$$

$$q > 2 + \frac{4}{d}$$

$$\frac{2}{q} + \frac{d}{p+1} < \frac{d+2}{2+\frac{4}{d}} = \frac{d}{2}$$

$$\frac{2}{\alpha} + \frac{d}{\beta} = \frac{d}{2} \qquad d = 3$$



$$\alpha = 2 \quad \beta = 6$$

$$W^{k,p}(\mathbb{R}^d) \qquad \langle\sqrt{-\Delta}\rangle^k f$$

$$\| e^{-it_2\Delta} u(t_2) - e^{-it_1\Delta} u(t_1) \|_{H^1} \leq$$

$$\leq c_0 \quad \|u\|^{p-1}_{L^{\cancel{(p-1)}\frac{\alpha}{\cancel{p-1}}}((t_1,t_2), L^{p+1}(\mathbb{R}^d))} \quad \|u\|_{L^q((t_1,t_2), W^{1,p+1}(\mathbb{R}^d))}$$

$$(q, p+1)$$

$$\alpha > q \qquad (\alpha, \beta) \quad \text{admissible}$$

$$\beta < p+1$$

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} \qquad \tau \in (0,1]$$

$$W^{1,\beta}(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$$

$$\| e^{-it_2\Delta} u(t_2) - e^{-it_1\Delta} u(t_1) \|_{H^1} \leq$$

$$\leq c_0^{\alpha} \quad \|u\|^{p-1}_{L^{\alpha}((t_1,t_2), W^{1,\beta}(\mathbb{R}^d))} \quad \|u\|_{L^q((t_1,t_2), W^{1,p+1}(\mathbb{R}^d))}$$

$$\xrightarrow{t_1 \to +\infty} 0$$

$$\Rightarrow \forall \lim_{t \to +\infty} e^{-it\Delta} u(t) = u_+ \in H^1(\mathbb{R}^d)$$

$$\forall \varepsilon > 0 \quad \exists \; M_\varepsilon \quad \text{s.t.} \quad \text{if}$$

$$t_2 > t_1 > M_\varepsilon \Rightarrow$$

$$\| e^{-it_2\Delta} u(t_2) - e^{-it_1\Delta} u(t_1) \|_{H^1} < \varepsilon$$

$$\Rightarrow \cancel{\forall}$$

$$e^{-it\Delta} u(t) = u_0 - i \int_0^t e^{-is\Delta} |u(s)|^{p-1} u(s)\, ds$$

$$u_+ = u_0 - i \int_0^{+\infty} e^{-is\Delta} |u(s)|^{p-1} u(s)\, ds$$

$$\lim_{t \to +\infty} e^{-it\Delta} u(t) = u_+ \in H^1(\mathbb{R}^n)$$

$$\Updownarrow$$

$$\lim_{t \to +\infty} \| e^{-it\Delta} u(t) - u_+ \|_{H^1} = 0$$

$$\Updownarrow$$

$$\lim_{t \to +\infty} \| u(t) - e^{it\Delta} u_+ \|_{H^1}$$

$$\alpha > q \qquad \overset{(q,\ p+1)}{(\alpha, \beta)} \quad \text{admissible}$$

$$\beta < p+1$$

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} \qquad \tau \in [0, 1]$$

$$\dot{W}^{\tau,\beta}(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$$

$$\alpha > q \implies \beta < p+1$$

$$2 \leq \beta$$

$$2 < \beta < p+1 < d^* + 1 = \frac{d+2}{d-2} + 1 = \frac{d+2+d-2}{d-2} =$$

$$= \frac{2d}{d-2}$$

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} > \frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d}$$

$$\frac{1}{\beta} - \frac{\tau}{d} > \frac{1}{2} - \frac{1}{d}$$

$$\frac{1-\tau}{d} > \frac{1}{2} - \frac{1}{\beta} > 0$$

$$\implies \qquad \tau \in (0, 1)$$

<u>Theorem</u> $d \geq 3$ $u \in C^0(\mathbb{R}, H^1(\mathbb{R}^d))$

Then $2 < r < \frac{2d}{d-2}$

$$\lim_{t \to \infty} \| u(t) \|_{L^r(\mathbb{R}^d)} = 0$$

This implies global Strichartz estimates.

<u>Remark</u> enough $\boxed{r = p+1}$

$2 < r < p+1$

$$0 < \| u(t) \|_{L^r(\mathbb{R}^d)} \leq \| u(t) \|_{L^{p+1}}^{\alpha} \| u_0 \|_{L^2}^{1-\alpha}$$

$$\frac{1}{p+1} < \frac{1}{r} < \frac{1}{2} \qquad \frac{1}{r} = \frac{\alpha}{p+1} + \frac{1-\alpha}{2}$$

$\alpha \in (0,1)$

$p+1 < r < d^* + 1$

$$\frac{1}{d^*+1} < \frac{1}{r} < \frac{1}{p+1} \qquad \frac{1}{r} = \frac{\alpha}{p+1} + \frac{1-\alpha}{d^*+1}$$

$$0 \leq \| u(t) \|_{L^r} \leq \| u(t) \|_{L^{p+1}}^{\alpha} \| u(t) \|_{L^{d^*+1}(\mathbb{R}^d)}^{1-\alpha}$$

$$\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{d^*+1}(\mathbb{R}^d)$$

$$\leq C_S \| u(t) \|_{L^{p+1}}^{\alpha} \| \nabla u(t) \|_{L^2(\mathbb{R}^d)}^{1-\alpha}$$

$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|^2_{L^2} + \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}}$$

$$\|\nabla u(t)\| \le 2\, E(u_0)$$

$$\le C_S \|u(t)\|^{\alpha}_{L^{p+1}} \left(2E(u_0)\right)^{1-\alpha}$$

$$\downarrow$$
$$0$$

Lemma $f(x) = a - x + b x^{\alpha}$ $\quad x \ge 0$

$\alpha > 1$ $\quad a, b > 0$. Assume

$\exists \; 0 < x_0 < x_1$ s.t. $f(x_0) = f(x_1) = 0$

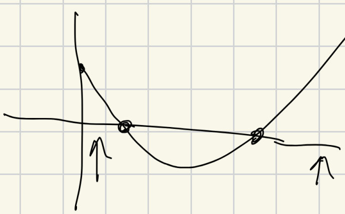
$\phi \in C^0(I, [0, +\infty))$ s.t.

$$\phi(t) \le a + b\,\phi^{\alpha}(t) \quad \forall t \in I \qquad (*)$$

and if $\exists \; t_0 \in I$ s.t. $\phi(t_0) \le x_0$

then $\phi(t) \le x_0 \quad \forall \; t \in I$.

Pf $(*) \iff f(\phi(t)) \ge 0 \quad \forall t \in I$



$y = f(x) = a - x + b x^{\alpha}$

$\phi(I)$ is ~~on inter~~ connected

Let us move S t.. S $S < t$

$$u(t) = e^{i(t-S)}u(S) - i\int_S^t e^{i(t-s)\Delta}|u(s)|^{p-1}u(s)\,ds$$

$$|u|_{L^q((S,t),W^{1,p+1})} \le C_0 |u(S)|_{H^1} +$$

$$+ C_0 \Big| \, |u|^{p-1}_{L^{p+1}_x} |u|_{W^{1,p+1}_x} \Big|_{L^{q'}(S,t)}$$

$$= C_0 |u(S)|_{H^1} +$$

$$+ C_0 \left(\int_S^t |u|^{(p-\cancel{1})q' - (q-\cancel{q'})}_{L^{p+1}_x} \cancel{|u|^{q-q'}_{L^{p+1}_x}} |u|^{\cancel{q'}\, q}_{W^{1,p+1}_x} ds \right)^{\frac{1}{q'}}$$

$$|u|_{L^q((S,t),W^{1,p+1})} \le C_0 |u(S)|_{H^1} +$$

$$\le C_0 \, |u|^{p-\frac{q}{q'}}_{L^\infty((S,t),L^{p+1}_x)} |u|^{\frac{q}{q'}}_{L^q((S,t),W^{1,p+1})}$$

$$p - \frac{q}{q'} = p - q\left(1 - \frac{1}{q}\right) = p + 1 - q \ge 0 \iff p \ge 1 + \frac{4}{d}$$

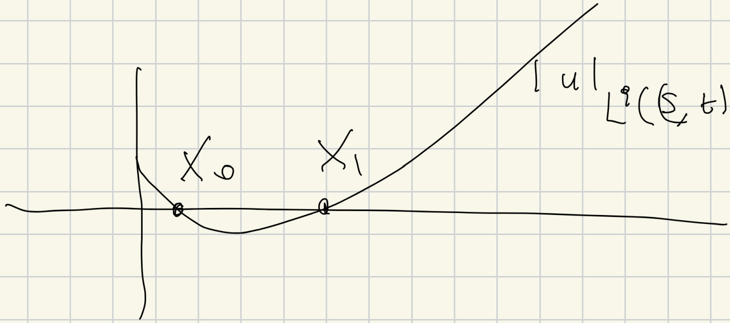
$$|u|^{p-\frac{q}{q'}}_{L^\infty((S,t),L^{p+1}_x)} \xrightarrow{S \to +\infty} 0$$

So $\forall \varepsilon > 0$ $\exists$ S st

$$|u|_{L^q((S,t),W^{1,p+1})} \le C_0 + C_0 \varepsilon |u|^{\frac{q}{q'}}_{L^q((S,t),W^{1,p+1})}$$

$$f(X) = C_0 - X + c_0 \varepsilon X^{\frac{q}{q'}}$$

$$X \geq 0$$



$$\|u\|_{L^q((S,t),W^{1,p+1})} \leq X_0$$

$$\forall\ t \geq S$$

$$\|u\|_{L^q((S,+\infty),W^{1,p+1})} \leq X_0$$

$$\Rightarrow \|u\|_{L^q(\mathbb{R}_+,W^{1,p+1})} < +\infty$$