May 8

$i\partial_t u = -\Delta u + |u|^{p-1}u \qquad u_{|t=0} = u_0 \in H^1(\mathbb{R}^d)$

$d^{*} > p > 1 + \frac{4}{d} \qquad u \in C^0(\mathbb{R}, H^1(\mathbb{R}^d))$

$d \geq 3 \qquad \lim_{t\to+\infty} |u(t)|_{L^{p+1}} = 0$

Lemma $\infty > p \geq 1 \qquad q < d \qquad 0 \leq q \leq p.$

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^q} \leq \left(\frac{p}{d-q}\right)^q |u|^{p-q}_{L^p(\mathbb{R}^d)} |\nabla u|^q_{L^p(\mathbb{R}^d)}$$

Pf $u \in W^{1,p}(\mathbb{R}^d)$. It is enough to consider $u \in C_c^\infty(\mathbb{R}^d)$. Indeed if $\{u_n\}$ is in $C_c^\infty(\mathbb{R}^d)$ with $u_n \longrightarrow u$ in $W^{1,p}(\mathbb{R}^d)$

up to a subsequence $u_n(x) \longrightarrow u(x)$ a.e.

$$\frac{|u_n(x)|^p}{|x|^q} \longrightarrow \frac{|u(x)|^p}{|x|^q} \quad \text{a.e}$$

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^q} dx \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^d} \frac{|u_n(x)|^p}{|x|^q} dx$$

$$\leq \lim_{n\to+\infty} C_{p,q} |u_n|^{p-q}_{L^p} |\nabla u_n|^q_{L^p} = C_{pq} |u|^{p-q}_{L^p} |\nabla u|^q_{L^p}$$

$$z = \frac{x}{|x|^q}$$

$$\nabla\cdot z = (d-q)\,|x|^{-q}$$

$$\nabla\cdot z = \nabla\cdot\frac{x}{|x|^q} = \nabla\frac{1}{|x|^q}\cdot x + \frac{1}{|x|^q}\overbrace{\nabla\cdot x}^{d} =$$

$$= -q\,\frac{1}{|x|^{q+1}}\overbrace{\frac{x}{|x|}\cdot x}^{|x|^2} + \frac{d}{|x|^q} =$$

$$= -\frac{q}{|x|^q} + \frac{d}{|x|^q}$$

$$|u|^p\,\nabla\cdot z = \nabla\cdot(|u|^p z) - p|u|^{p-1}\nabla u\cdot z$$

$$= \frac{|u|^p}{|x|^q}(d-q)$$

$$(d-q)\int_{|x|\geq r}\frac{|u(x)|^p}{|x|^q}\,dx = \int_{|x|\geq r}\nabla\cdot(|u|^p z) - p\int_{|x|\geq r}|u|^{p-1}\nabla u\cdot z$$

$$z = \frac{x}{|x|^q}$$

$$= \underbrace{(-)\int_{|x|=r}|u|^p\, z\cdot\frac{x}{|x|}\,ds}_{\geq 0} - p\int_{|x|\geq r}|u|^{p-1}\nabla u\cdot z$$

$$\frac{|x|^2}{|x|^{q+1}}$$

$$\leq p\int_{|x|\geq r}|u|^{p-1}\frac{|\nabla u|}{|x|^{q-1}}\,dx$$

$$(d-q)\int_{|x|\geq r}\frac{|u(x)|^p}{|x|^q} \leq p\int_{|x|\geq r}\frac{|u|^{p\frac{(q-1)}{q}}}{|x|^{q-1}}\,|u|^{\frac{p-q}{q}}\,|\nabla u|\,dx \qquad \frac{q-1}{q} = \frac{1}{q'}$$

$$\frac{p}{q'} + \frac{p}{q} - 1 = p-1$$

$$1 = \frac{1}{q'} + \frac{p-q}{pq} + \frac{1}{p}$$

$$= \frac{1}{q'} + \frac{1}{q} - \frac{1}{p} + \frac{1}{p}$$

$$\leq p\,\left|\frac{|u|^{\frac{p}{q'}}}{|x|^{q-1}}\right|_{L^{q'}}\,\left|\,|u|^{\frac{p-q}{q}}\right|_{L^{\frac{pq}{p-q}}}\,|\nabla u|_{L^p}$$

$$= p\,\left|\frac{|u|^{\frac{p}{q'}}}{|x|^{\frac{\cancel{q-1}}{q'}\frac{q}{\cancel{q-1}}}}\right|_{L^{q'}}\,|u|^{\frac{p-q}{q}}_{L^p}\,|\nabla u|_{L^p}$$

$$q' = \frac{q}{q-1}$$

$$= p\,\left|\frac{|u|^p}{|x|^q}\right|^{\frac{1}{q'}}_{L^1}\,|u|^{\frac{p-q}{q}}_{L^p}\,|\nabla u|_{L^p}$$

$$(d-q)\,\left|\frac{|u|^p}{|x|^q}\right|^{\frac{1}{q}}_{L^1(|x|\geq r)} \leq p\,\cancel{\left|\frac{|u|^p}{|x|^q}\right|^{\frac{1}{q'}}_{L^1(|x|\geq r)}}\,|u|^{\frac{p-q}{q}}_{L^p(\mathbb{R}^d)}\,|\nabla u|_{L^p(\mathbb{R}^d)}$$

Lemma $d \geq 4$

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^3}\, dx \leq C_d \|u\|^2_{H^2(\mathbb{R}^d)}$$

Pf $p = 2 \quad q = 3$

$$(d-3) \int_{|x| \geq r} \frac{|u(x)|^2}{|x|^3}\, dx \leq + \cancel{2} \int_{|x| \geq r} \frac{|u|}{|x|} \frac{|\nabla u|}{|x|}\, dx$$

$$\leq 2 \left(\int_{|x| \geq r} \frac{|u|^2}{|x|^2}\, dx \right)^{\frac{1}{2}} \left(\int_{|x| \geq \cancel{R}} \frac{|\nabla u|^2}{|x|^2}\, dx \right)^{\frac{1}{2}}$$

$p = q = 2 \qquad u \qquad \nabla u$

$$\leq 2 \left(\frac{2}{d-2}\right)^2 \left(\|u\|^0_{L^2} \|\nabla u\|^2_{L^2} \right)^{\frac{1}{2}} \qquad \|\nabla^2 u\|_{L^2}$$

$$= \frac{2^8}{d-2} \qquad \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}$$

Virial like estimates

Morawetz inequality

$u_0 \in H^2(\mathbb{R}^d)$ $\quad u \in C^0(\mathbb{R}, H^2)$

$i\partial_t u = -\Delta u + |u|^{p-1}u \quad$ in $\mathcal{D}'(\mathbb{R}, L^2(\mathbb{R}^d, \mathbb{C}))$

$= \mathcal{L}(\mathcal{D}(\mathbb{R}), L^2(\mathbb{R}^d, \mathbb{C}))$

$u \in C^1(\mathbb{R}, L^2) \cap C^0(\mathbb{R}, H^2)$

$u_n \quad C^1(\mathbb{R}, H^2)$

$$\frac{d}{dt}\frac{1}{2}\left\langle i\left(\partial_r + \frac{d-1}{2r}\right)u_n, u_n\right\rangle = \frac{1}{2}\left\langle i\left(\partial_r + \frac{d-1}{2}\right)\dot{u}_n, u_n\right\rangle +$$

$i\left(\partial_r + \frac{d-1}{2}\right)$

$$+ \frac{1}{2}\left\langle i\left(\partial_r + \frac{d-1}{2}\right)u_n, \dot{u}_n\right\rangle$$

$H^1(\mathbb{R}^d)$

$$= \left\langle i\left(\partial_r + \frac{d-1}{2}\right)u_n, \dot{u}_n\right\rangle$$

$t_1 < t_2$

$$\frac{1}{2}\left\langle i\left(\partial_r + \cancel{\frac{d-1}{2r}}\right)u, u\right\rangle\Big]_{t_1}^{t_2} = \int_{t_1}^{t_2}\left\langle i\left(\partial_r + \frac{d-1}{r}\right)u, \dot{u}\right\rangle dt$$

$$\frac{d}{dt}\frac{1}{2}\left\langle i\left(\partial_r + \cancel{\frac{d-1}{2r}}\right)u, u\right\rangle = \left\langle i\left(\partial_r + \frac{d-1}{r}\right)u, \dot{u}\right\rangle$$

$$= -\left\langle\left(\partial_r + \frac{d-1}{r}\right)u, i\dot{u}\right\rangle$$

$$= -\left\langle\left(\partial_r + \frac{d-1}{r}\right)u, -\Delta u\right\rangle - \left\langle\left(\partial_r + \frac{d-1}{r}\right)u, |u|^{p-1}u\right\rangle$$

We will prove

$$\frac{d}{dt}\langle\partial_r u, iu\rangle \geq (d-1)\frac{p-1}{p+1}\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}dx$$

$$\frac{d}{dt}\frac{1}{2}\langle\partial_r u, iu\rangle = -\frac{d}{dt}\frac{1}{2}\langle i\partial_r u, u\rangle = -\frac{d}{dt}\frac{1}{2}\left\langle i\left(\partial_r + \frac{d-1}{2r}\right)u, u\right\rangle$$

$$= -\left\langle i\left(\partial_r + \frac{d-1}{2r}\right)u, \dot{u}\right\rangle = \boxed{\left\langle\left(\partial_r + \frac{d-1}{r}\right)u, -\Delta u\right\rangle}$$

$$+ \left\langle\left(\partial_r + \frac{d-1}{r}\right)u, |u|^{p-1}u\right\rangle$$

Lemma $\left\langle\left(\partial_r + \frac{d-1}{r}\right)u, -\Delta u\right\rangle \geq 0$ — — —

$$\frac{d}{dt}\frac{1}{2}\langle\partial_r u, iu\rangle \geq \left\langle\left(\partial_r + \frac{d-1}{r}\right)u, |u|^{p-1}u\right\rangle$$

$$= (d-1)\int_{\mathbb{R}^d}\frac{1}{r}|u|^{p+1}dx + \int_{\mathbb{R}^d}|u|^{p-1}\underbrace{\mathrm{Re}\,\partial_r u\,\bar{u}}_{\frac{1}{2}\partial_r|u|^2}$$

$$= (d-1)\int_{\mathbb{R}^d}\frac{1}{r}|u|^{p+1}dx + \int_0^{+\infty}\int_{S^{d-1}}dS\, r^{d-1}\,\partial_r|u|^2\,(|u|^2)^{\frac{p-1}{2}}$$

$$\partial_r\frac{(|u|^2)^{\frac{p+1}{2}}}{\frac{p+1}{2}}$$

$$= (d-1)\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r} - \frac{2}{p+1}(d-1)\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}$$

$$= (d-1)\left(1 - \frac{2}{p+1}\right)\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}$$

$$= (d-1)\frac{p-1}{p+1}\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}$$

$$\partial_r = \frac{x}{|x|}\cdot\nabla$$

$$|\langle \partial_r u, iu\rangle|_{L^\infty(\mathbb{R})} \le |\partial_r u|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))} \underbrace{|u|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))}}_{|u_0|_{L^2(\mathbb{R})}}$$

$$\frac{1}{2}\frac{d}{dt}\langle \partial_r u, iu\rangle \ge \frac{d-1}{2}\,\frac{p-1}{p+1}\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}\,dx$$

<u>Lemma</u>

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r}\,dx \qquad *$$

$$\le \frac{2}{d-1}\,\frac{p+1}{p-1}\,|u_0|_{L^2(\mathbb{R}^d)}\,|\nabla u|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))}$$

$$E(u) = \frac{1}{2}|\nabla u|_{L^2}^2 + \frac{1}{p+1}|u|_{L^{p+1}}^{p+1} \ge \frac{1}{2}|\nabla u|_{L^2}^2$$

$$|\nabla u(t)|_{L^2} \le \sqrt{2E(u_0)}$$

$$\le \frac{2}{d-1}\,\frac{p+1}{p-1}\,|u_0|_{L^2(\mathbb{R}^d)}\sqrt{2}\cdot\sqrt{E(u_0)}$$

Pf of $*$. If $u_0 \in H^2$

$u_0 \in H^1 \setminus H^2$ $\qquad u_0^{(n)} \xrightarrow{n\to+\infty} u_0$ in H^1

$u_0^{(n)} \in H^2$

Then $\forall$ $T>0$ $\quad u^{(n)} \longrightarrow u$

in $C^0([-T,T], H^1)$

$$\int_{-T}^{T} dt \int_{\mathbb{R}^d} \frac{|u^{(n)}|^{p+1}}{r}\,dx \xrightarrow{n\to+\infty} \int_{-T}^{T} dt \int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}$$

$$\wedge\!| \qquad\qquad\qquad\qquad \wedge\!|$$

$$C_d\,|u_0^{(n)}|_{L^2}\,|\nabla u^{(n)}|_{L^\infty([-T,T], L^2)} \longrightarrow C_d\,|u_0|_{L^2}\,|\nabla u|_{L^\infty([-T,T], L^2)}$$

$$u(t) \xrightarrow{t\to\infty} 0 \quad \text{in } H^1(\mathbb{R}^d)$$