Riassunto

Formulazione Lagrangiana (coord. $q, \dot{q}$)

- eq. del moto ricavate dalla funz. Lagrangiana $L(q,\dot{q},t)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q_h} = 0 \qquad h=1,\dots,n$$

Formulazione Hamiltoniana (coord. p, q)

- eq. del moto ricavate dalla funz. Hamiltoniana $H(p,q,t)$

$$\begin{cases} \dot{p}_h = -\dfrac{\partial H}{\partial q_h} \\ \dot{q}_h = \dfrac{\partial H}{\partial p_h} \end{cases} \quad h=1,\dots,n \qquad \longleftrightarrow \qquad \dot{x}_i = \sum_{j=1}^{2n} E_{ij} \frac{\partial H}{\partial x_j}$$

$$\bar{x} = \begin{pmatrix} \bar{p} \\ \bar{q} \end{pmatrix}$$

Riassumiamo i vari spazi entrati in gioco finora.

SPAZIO DELLE CONFIGURAZIONI : Q con coord. $(q_1,\dots,q_n)$

Formalismo Lagrangiano: Q è completato a uno spazio 2n-dimensionale detto

SPAZIO DEGLI STATI $\underbrace{TQ}_{\text{"FIBRATO TANGENTE"}}$ con coord. $(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_n)$

Formalismo Hamiltoniano: Q è completato a uno spazio 2n-dimensionale detto

SPAZIO DELLE FASI $\underbrace{T^*Q}_{\text{"FIBRATO COTANGENTE"}}$ con coord. $(p_1,\dots,p_n,q_1,\dots,q_n)$

PARENTESI DI POISSON

Def. PARENTESI DI POISSON è un'applicazione bilineare che prende come input due funzioni di (p,q,t) definite sullo sp. delle fasi e restituisce un'altra funzione di (p,q,t):

$$f(p,q,t)\,,\quad g(p,q,t) \longmapsto \{f,g\}(p,q,t)$$

dove

$$\{f,g\}(p,q,t) = \sum_{h=1}^{n}\left(\frac{\partial f}{\partial q_h}\cdot\frac{\partial g}{\partial p_h} - \frac{\partial g}{\partial q_h}\cdot\frac{\partial f}{\partial p_h}\right)$$

Nel FORMALISMO COMPATTO:

Parentesi di Poisson:

$$\{f,g\} = \sum_{i,j=1}^{2n}\frac{\partial f}{\partial x_i}E_{ij}\frac{\partial g}{\partial x_j}$$

Infatti possiamo riscrivere il termine di destra come

$$\bar\nabla_x f\cdot E\bar\nabla_x g = (\bar\nabla_p f, \bar\nabla_q f)\begin{pmatrix}0 & -\mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\bar\nabla_p g\\ \bar\nabla_q g\end{pmatrix} =$$

$$= (\bar\nabla_q f, -\bar\nabla_p f)\begin{pmatrix}\bar\nabla_p g\\ \bar\nabla_q g\end{pmatrix} = \bar\nabla_q f\cdot\bar\nabla_p g - \bar\nabla_p f\cdot\bar\nabla_q g =$$

$$= \sum_{h=1}^{n}\left(\frac{\partial f}{\partial q_h}\frac{\partial g}{\partial p_h} - \frac{\partial f}{\partial p_h}\frac{\partial g}{\partial q_h}\right)$$

in componenti

Proprietà delle Parentesi di Poisson

a) $\{f, g\} = -\{g, f\} \Rightarrow \{f,f\}=0$ ANTISIM.

b) $\{f, \alpha_1 g_1 + \alpha_2 g_2\} = \alpha_1 \{f, g_1\} + \alpha_2 \{f, g_2\}$ BILINEARE

$\alpha_1, \alpha_2 \in \mathbb{R}$

c) $\{f, g_1 \cdot g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\}$

d) Identità di Jacobi

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

La Parentesi di Poisson è un'applicazione BILIN. ANTISIM. che soddisfa l'ID. DI JACOBI (ci permette di definire un'ALGEBRA DI LIE)

$$\{f, g\} = \sum_h \left(\frac{\partial f}{\partial q_h} \frac{\partial g}{\partial p_h} - \frac{\partial f}{\partial p_h} \frac{\partial g}{\partial q_h} \right) = \sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g}{\partial x_j}$$

Dim

a) b) } ovvie

c) $\sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \cdot \underbrace{\frac{\partial}{\partial x_j}(g_1 \cdot g_2)}_{g_1 \frac{\partial g_2}{\partial x_j} + g_2 \frac{\partial g_1}{\partial x_j}} = \left(\sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g_1}{\partial x_j} \right) g_2 + \left(\sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g_2}{\partial x_j} \right) g_1$ //

d) $\underbrace{\{f,\{g,h\}\}+\{g,\{h,f\}\}}+\{h,\{f,g\}\}=0$

Notazione : $\partial_i f \equiv \dfrac{\partial f}{\partial x_i}$

$$\{f,\{g,h\}\}-\{g,\{f,h\}\}=$$

$$=\sum_{i,j=1}^{2n}\partial_i f\,E_{ij}\,\partial_j\left[\sum_{k,m=1}^{2n}\partial_k g\,E_{km}\,\partial_m h\right]-\sum_{a,b=1}^{2n}\partial_a g\,E_{ab}\,\partial_b\left[\sum_{c,d=1}^{2n}\partial_c f\,E_{cd}\,\partial_d h\right]$$

$$=\sum_{ijkm}\left[\partial_i f\,E_{ij}\,\partial_j\partial_k g\,E_{km}\,\partial_m h+\partial_i f\,E_{ij}\,\partial_k g\,E_{km}\,\partial_j\partial_m h\right]$$

$$-\sum_{abcd}\left[\partial_a g\,E_{ab}\,\partial_b\partial_c f\,E_{cd}\,\partial_d h+\underset{\substack{\uparrow\\ k}}{\partial_a} g\,E_{ab}\,\underset{\substack{\uparrow\\ i}}{\partial_c} f\,E_{cd}\,\underset{\substack{\uparrow\\ m}}{\partial_b}\underset{\substack{\uparrow\\ j}}{\partial_d} h\right]$$

$$=\sum_{ijkm}\left[\partial_i f\,E_{ij}\,\partial_j\partial_k g\,E_{km}\,\partial_m h\right]+\sum_{ijkm}\cancel{\partial_i f\,E_{ij}\,\partial_k g\,E_{km}}\,\partial_j\partial_m h$$

$$-\sum_{abcd}\left[\underset{\substack{\uparrow\\ j}}{\partial_a} g\,E_{ab}\,\underset{\substack{\uparrow\\ i}}{\partial_b}\underset{\substack{\uparrow\\ k}}{\partial_c} f\,E_{cd}\,\underset{\substack{\uparrow\\ m}}{\partial_d} h\right]-\sum_{ijkm}\cancel{\partial_k g\,E_{km}\,\partial_i f}\,E_{ij}\,\underbrace{\partial_m\partial_j h}_{=\partial_j\partial_m h}$$

$$=\sum_{ijkm}\left[\partial_i f\,E_{ij}\,\partial_j\partial_k g\,E_{km}\,\partial_m h-\partial_j g\,\underbrace{E_{ji}}_{=-E_{ij}}\,\partial_i\partial_k f\,E_{km}\,\partial_m h\right]$$

$$=\sum_{km}\left(\sum_{ij}\left(\partial_i f\,E_{ij}\,\partial_k\partial_j g+\partial_k\partial_i f\,E_{ij}\,\partial_j g\right)\right)E_{km}\,\partial_m h$$

$$=\sum_{km}\partial_k\left(\sum_{ij}\partial_i f\,E_{ij}\,\partial_j g\right)E_{km}\,\partial_m h=\{\{f,g\},h\}=$$

$$=-\{h,\{f,g\}\}\quad //$$

Parentesi di Poisson FONDAMENTALI

Scegliamo come f e g le funzioni $\bar{p} \mapsto p_h \leftarrow p_h$

$\bar{q} \mapsto q_h \leftarrow q_h$

$$\{p_h, p_k\} = \sum_\ell \Big(\underbrace{\frac{\partial p_h}{\partial q_\ell}}_{0} \underbrace{\frac{\partial p_k}{\partial p_\ell}}_{\delta_{k\ell}} - \frac{\partial p_h}{\partial p_\ell} \underbrace{\frac{\partial p_k}{\partial q_\ell}}_{0} \Big) = 0 \qquad \forall h,k = 1,\dots,n$$

$[F(x,y) = x \quad \partial_x F = 1 \quad \partial_y F = 0]$

$$\{q_h, q_k\} = 0 \qquad \forall h,k = 1,\dots,n$$

$$\{q_h, p_k\} = \sum_\ell \Big(\underbrace{\frac{\partial q_h}{\partial q_\ell}}_{\delta_{h\ell}} \underbrace{\frac{\partial p_k}{\partial p_\ell}}_{\delta_{k\ell}} - \underbrace{\frac{\partial q_h}{\partial p_\ell}}_{0} \underbrace{\frac{\partial p_k}{\partial q_\ell}}_{0} \Big) = \sum_\ell \delta_{h\ell}\,\delta_{k\ell} = \delta_{hk}$$

$$\{x_i, x_j\} = \sum_{r,s=1}^{2n} \underbrace{\frac{\partial x_i}{\partial x_r}}_{\delta_{ir}} E_{rs} \underbrace{\frac{\partial x_j}{\partial x_s}}_{\delta_{js}} = E_{ij}$$

Osservazione:

$$\{x_s, g(\bar{x})\} = \sum_{ij} \underbrace{\frac{\partial x_s}{\partial x_i}}_{= \delta_{is}} E_{ij} \frac{\partial g}{\partial x_j} = \sum_j E_{sj} \frac{\partial g}{\partial x_j}$$

$$\underbrace{\{\bar{x}, g\}}_{\equiv \begin{pmatrix} \{x_1, g\} \\ \vdots \\ \{x_{2n}, g\} \end{pmatrix}} = E \bar{\nabla} g \quad \Rightarrow \quad \{p_h, g(\bar{p},\bar{q})\} = -\frac{\partial g}{\partial q_h} \qquad \{q_h, g(\bar{p},\bar{q})\} = \frac{\partial g}{\partial p_h}$$

Se $g = H$ $\quad \dot{x}_i = \sum_j E_{ij} \frac{\partial H}{\partial x_j} = \{x_i, H\}$

$\to$ Eq. Ham. $\quad \dot{x}_i = \{x_i, H\} \qquad i = 1,\dots,2n$

PARENTESI DI POISSON E COST. DEL MOTO

variabile dinamica

Cost. del moto : $f(\bar{p}, \bar{q}, t)$ t.c. se la valutiamo sulle soluzioni $(\bar{p}(t), \bar{q}(t))$ delle equazioni di Hamilton, si ha

$$\frac{d}{dt} f(\bar{p}(t), \bar{q}(t), t) = 0 .$$

Svolgiamo la derivata delle funz. composte:

$$\frac{d}{dt} f(\bar{p}(t), \bar{q}(t), t) = \frac{\partial f}{\partial t} + \sum_{\ell=1}^{n} \left(\frac{\partial f}{\partial p_\ell} \dot{p}_\ell + \frac{\partial f}{\partial q_\ell} \dot{q}_\ell \right) =$$

$$= \frac{\partial f}{\partial t} + \sum_{\ell=1}^{n} \left[\frac{\partial f}{\partial p_\ell} \left(- \frac{\partial H}{\partial q_\ell} \right) + \frac{\partial f}{\partial q_\ell} \left(\frac{\partial H}{\partial p_\ell} \right) \right]$$

prendiamo $\bar{p}(t), \bar{q}(t)$ che soddisfano eq. Ham.

$$= \frac{\partial f}{\partial t} + \underbrace{\sum_{\ell=1}^{n} \left(\frac{\partial f}{\partial q_\ell} \frac{\partial H}{\partial p_\ell} - \frac{\partial f}{\partial p_\ell} \frac{\partial H}{\partial q_\ell} \right)}_{\equiv \{f, H\}}$$

PARENTESI DI POISSON

Cioè:

$$\frac{d}{dt} f(p(t), q(t), t) = \{f, H\}(p(t), q(t), t) + \frac{\partial f}{\partial t}(p(t), q(t), t)$$

quando $p(t)$ e $q(t)$ soddisfano le eq. di Ham

Quindi:

f è una COST. DEL MOTO $\Longleftrightarrow$ $\{f, H\} + \frac{\partial f}{\partial t} \equiv 0$

identicam. come funzione

(f indip. da t) $\quad$ $(\{f, H\} = 0)$

Conservazione dell'Energia in meccanica Hamiltoniana

Vediamo quando l'Hamiltoniana è essa stessa una cost. del moto

$$\frac{dH(p(t),q(t),t)}{dt} = \underbrace{\{H,\cancel{H}\}}_{=0 \text{ per antisimmetria delle Parentesi di Poisson}} + \frac{\partial H}{\partial t}$$

$\longleftarrow$ H è cost. del moto quando è indip. esplicitam. da t (cioè quando $\frac{\partial H}{\partial t}=0$)

Extra: Parentesi di Poisson e derivata di Lie

$$\frac{d}{dt}G(\bar{x}(t),t) = \frac{\partial G}{\partial t} + \mathcal{L}_{E\bar{\nabla}H=\bar{f}}G = \frac{\partial G}{\partial t} + \sum_{i=1}^{2} f_i \frac{\partial G}{\partial x_i} =$$

$\dot{\bar{x}} = \bar{f}(\bar{x},t)$

$$= \frac{\partial G}{\partial t} - \frac{\partial H}{\partial q}\frac{\partial G}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial G}{\partial q} = \frac{\partial G}{\partial t} + \{G,H\}$$

In generale

$$\{f,g\} = \mathcal{L}_{E\nabla g} f = \sum_i \overbrace{\left(\sum_j E_{ij}\frac{\partial g}{\partial x_j}\right)}^{(E\bar{\nabla}_g)_i} \frac{\partial f}{\partial x_i} = \bar{\nabla} f \cdot E\bar{\nabla} g$$

$$\{g,f\} = \bar{\nabla} g \cdot E\,\bar{\nabla} f = \sum_{ij} \underset{\substack{\uparrow\\ \text{ANTISIM.}}}{E_{ij}} \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j} =$$

$$= -\sum_{ij} E_{ji}\frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j} = -\sum_{rs} E_{rs}\frac{\partial f}{\partial x_r}\frac{\partial g}{\partial x_s} = -\{f,g\}$$