May 12

Claim $u \in H^2(\mathbb{R}^d, \mathbb{C})$. Then

$$\langle (\partial_r + \frac{d-1}{2r})u, \Delta u \rangle \leq 0$$

Pf

$$\mathrm{Re}\{\Delta u(\bar{u}_r + \frac{d-1}{2r}\bar{u})\} = \nabla \cdot \mathrm{Re}\{\nabla u(\bar{u}_r + \frac{d-1}{2r}\bar{u})\}$$

$$- \nabla \cdot \{\frac{x}{2r}|\nabla u|^2\} +$$

$$\nabla \cdot \left(\frac{d-1}{4}\frac{x}{r^3}|u|^2\right) - \frac{1}{r}(|\nabla u|^2 - |u_r|^2) -$$

$$- \frac{(d-1)(d-3)}{4r^3}|u|^2$$

$\mathbb{R}^d \setminus D_{\mathbb{R}^d}(0, a)$ $\quad a > 0 \quad a \to 0^+$.

Lemma 5.1 $\quad q = 1 \quad p = 2$

$$\langle \Delta u, (\partial_r + \frac{d-1}{2r})u \rangle = -\int_{\mathbb{R}^d} \frac{1}{r}(|\nabla u|^2 - |u_r|^2)dx$$

$$+ \lim_{a \to 0^+} \int_{|x|=a} \frac{|\nabla u|^2}{2} dS - \lim_{a \to 0} \frac{d-1}{4} \int_{|x|=a} \frac{1}{a^2}|u|^2 dS$$

$$- \frac{(d-1)(d-3)}{4} \lim_{a \to 0} \int_{|x| \geq a} \frac{|u|^2}{r^3} dx$$

if

$u \in C^\infty(\mathbb{R}^d, \mathbb{C})$

$$\lim_{a \to 0^+} \int_{|x|=a} \frac{|\nabla u|^2}{2} \, dS = 0$$

$d \geq 3$

$$\lim_{a \to 0} \frac{d-1}{4} \int_{|x|=a} \frac{1}{a^2} |u|^2 \, dS = 0$$

$d \geq 3$ $\qquad u \in C^\infty(\mathbb{R}^d, \mathbb{C})$

$$\langle \Delta u, (\partial_r + \frac{d-1}{2r}) u \rangle = - \int_{\mathbb{R}^d} \frac{1}{r} (|\nabla u|^2 - |u_r|^2) \, dx$$

$$- \frac{(d-1)(d-3)}{4} \cdot \int_{\mathbb{R}^3} \frac{|u|^2}{r^3} \, dx$$

≤ 0

$$\langle \Delta u, (\partial_r + \frac{d-1}{2r}) u \rangle \leq 0$$

By density this remain true

$\forall u \in H^2(\mathbb{R}^d, \mathbb{C})$

For $d=3$ $\quad u \in C^\infty(\mathbb{R}^d, \mathbb{C})$

$$\langle \Delta u, (\partial_r + \tfrac{d-1}{2r}) u \rangle = -\int_{\mathbb{R}^d} \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) dx$$

$$- \lim_{a \to 0} \frac{d-1}{4} \int_{|x|=a} \frac{1}{a^2} |u|^2 dS$$

$$4\pi \frac{1}{4\pi a^2} \int_{|x|=a} |u|^2 dS \longrightarrow 4\pi \, |u(0)|^2 \geq$$

$$\langle \Delta u, (\partial_r + \tfrac{d-1}{2r}) u \rangle = -\int_{\mathbb{R}^d} \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) dx$$

$$- 2\pi \, |u(0)|^2 \leq 0$$

$$\forall u \in C^\infty(\mathbb{R}^d, \mathbb{C}) \Rightarrow \cancel{\forall}$$

$$\langle \Delta u, (\partial_r + \tfrac{d-1}{2r}) u \rangle \leq 0 \quad \forall u \in H^2$$

Going back to

<u>Lemma</u> 5.4 $\quad u \in C^0(\mathbb{R}, H^2)$ a solution of $i\partial_t u = -\Delta u + |u|^{p-1} u$

$$\int_{\mathbb{R}} dt \int \frac{|u|^{p+1}}{r} dx \leq C_{p,d} \, |u_0|_{L^2} \sqrt{E(u_0)}$$

and $\quad u(t) \xrightarrow{t \to +\infty} H^1$

$$\left(\quad u(-t) = \overline{u(t)} \right)$$

$$\varphi \in C_c^\infty(\mathbb{R}^d)$$

$$|\langle u(t), \varphi \rangle| = \langle \frac{u(t)}{r^{\frac{1}{p+1}}}, r^{\frac{1}{p+1}} \varphi \rangle|$$

$$\leq \left\| \frac{u(t)}{r^{\frac{1}{p+1}}} \right\|_{L^{p+1}} \left\| r^{\frac{1}{p+1}} \varphi \right\|_{L^{\frac{p+1}{p}}}$$

$$\int_0^{+\infty} |\langle u(t), \varphi \rangle|^{p+1} dt$$

$$\leq \left\| r^{\frac{1}{p+1}} \varphi \right\|_{L^{\frac{p+1}{p}}} \int_0^{+\infty} \int_{\mathbb{R}^d} \frac{|u(t)|^{p+1}}{r} dt$$

$$< +\infty$$

$$|\langle u(t), \varphi \rangle| \in L^{p+1}(\mathbb{R}_+)$$

$$u_t \in C^0(\mathbb{R}, H^2) \quad \left(u \in C^0(\mathbb{R}, H^1) \right)$$

$$i\partial_t u = -\Delta u + |u|^{p-1}u \in C^0(\mathbb{R}, L^2(\mathbb{R}^d))$$

$$\mathcal{D}'(\mathbb{R}, L^2(\mathbb{R}^d))$$

$$u \in BC^1(\mathbb{R}, H^{-1}(\mathbb{R}^d)) = W^{1,\infty} \cap C^1$$

$$-\Delta u + |u|^{p-1}u \in L^\infty(\mathbb{R}, H^{-1}(\mathbb{R}^d))$$

$$|\Delta u|_{H^{-1}} \leq |u|_{H^1}$$

$$\langle u(t), \psi \rangle \in BC^1(\mathbb{R})$$

$$\left| \frac{d}{dt} \langle u(t), \psi \rangle \right| = |\langle \dot{u}(t), \psi \rangle| \leq$$

$$\leq |\dot{u}|_{H^{-1}} |\psi|_{H^1}$$

$2k \geq p+2$

$$|\langle u(t), \psi \rangle|^{2k} \in BC^1(\mathbb{R}) \qquad s < t$$

$$\left| \Big[|\langle u(\tau), \psi \rangle|^{2k} \Big]_s^t \right| \leq \int_s^t 2k \, |\langle u(\sigma), \psi \rangle|^{2k-1}$$

$$|\langle \dot{u}(\sigma), \psi \rangle| \, d\sigma$$

$2k - 1 \geq p+1$

$$\leq 2k \, |\langle \dot{u}(\cdot), \psi\rangle|_{L^\infty(\mathbb{R})} \, |\langle u(\cdot), \psi\rangle|_{L^\infty(\mathbb{R})}^{2k-1-(p+1)}$$

$$\int_s^t |\langle u(\sigma), \psi\rangle|^{p+1} \, d\sigma$$

$$\leq C_{u_0} \int_s^{+\infty} |\langle u(\sigma), \psi\rangle|^{p+1} \, d\sigma \xrightarrow{s \to +\infty} 0$$

$$\Rightarrow \exists \lim_{t \to +\infty} |\langle u(t), \psi\rangle|^{2k} = 0$$

$$|\langle u(t), \psi\rangle|^{2k} \in L^1(\mathbb{R})$$

$$2k \geq p+1$$

$$|\langle u(t), \psi\rangle|^{2k} \leq C_{u_0} |\langle u(t), \psi\rangle|^{p+1}$$

<u>Lemma</u> Let $u \in L^2((0,T), H^1(\mathbb{R}^d)) \cap H^1((0,T), H^{-1}(\mathbb{R}^d))$

Then $u \in C^0([0,T], L^2(\mathbb{R}^d))$ and $\exists \, C_T$

s.t.

$$|u|_{C^0([0,T], L^2(\mathbb{R}^d))}$$

$$\leq C_T \left(|u|_{L^2((0,T), H^1)} + |u|_{H^1((0,T), H^{-1})} \right)$$

Furthermore $|u(t)|^2_{L^2} \in AC([0,T))$

with $\frac{d}{dt} |u(t)|^2_{L^2} = 2 \langle u(t), \dot{u}(t) \rangle$

Pf Let us observe also $u \in C^1([0,T], L^2(\mathbb{R}^d))$

and let $t_0 \in [0,T]$.

$$|u(t)|^2_{L^2} = |u(t_0)|^2_{L^2} + 2 \int_{t_0}^{t} \langle u(s), \dot{u}(s) \rangle ds$$

$$\leq |u(t_0)|^2_{L^2} + 2 |u|_{L^2((0,T), H^1)} |\dot{u}|_{L^2((0,T), H^{-1})}$$

$$\leq |u(t_0)|^2_{L^2} + |u|^2_{L^2((0,T), H^1)} +$$

$$+ |\dot{u}|^2_{L^2((0,T), H^{-1})}$$

Take t_0 s.t.

$$|u(t_0)|^2_{L^2} = \frac{1}{T} \int_0^T |u(s)|^2_{L^2} ds$$

$$\leq \frac{1}{T} \int_0^T |u(s)|^2_{H^1}$$

$$= \left(1 + \frac{1}{T}\right) |u|^2_{L^2((0,T), H^1)} + |\dot{u}|^2_{L^2((0,T), H^{-1})}$$

$$|u|^2_{L^\infty([0,T], L^2)} \leq$$

$$\leq \underbrace{\left(1+\frac{1}{T}\right)}_{C_T^2}\left(|u|^2_{L^2((0,T),H^1)} + |\dot u|^2_{L^2((0,T),H^{-1})}\right)$$

$$|u_n|_{C^0([0,T],L^2(\mathbb{R}^d))}$$

$$\leq C_T\left(|u_n|_{L^2((0,T),H^1)} + |u_n|_{H^1((0,T),H^{-1})}\right)$$

In the general case $\exists$ a sequence

$$u_n \in C^1([0,T], H^1(\mathbb{R}^d))$$

s.t. $u_n \longrightarrow u$ in $L^2((0,T),H^1) \cap H^1([0,T],H^{-1})$

We conclude u_n converges also in $C^0([0,T],L^2(\mathbb{R}^d))$

$u_n \longrightarrow u$. in

$$|u|_{C^0([0,T],L^2(\mathbb{R}^d))}$$

$$\leq C_T\left(|u|_{L^2((0,T),H^1)} + |u|_{H^1((0,T),H^{-1})}\right)$$

$\forall t_0 \in [0,T]$

$$|u_n(t)|^2_{L^2} = |u_n(t_0)|^2_{L^2} + 2\int_{t_0}^{t} \langle u_n(s), \dot u_n(s)\rangle\, ds$$

$\downarrow$

$$|u(t)|_{L^2}^2 = |u(t_0)|_{L^2}^2 + 2\int_{t_0}^{t} \langle u(s), \dot{u}(s) \rangle \, ds$$

$u_n \to u$ in $L^2((0,T], H^1)$

$\dot{u}_n \to \dot{u}$ in $L^2((0,T), H^{-1})$

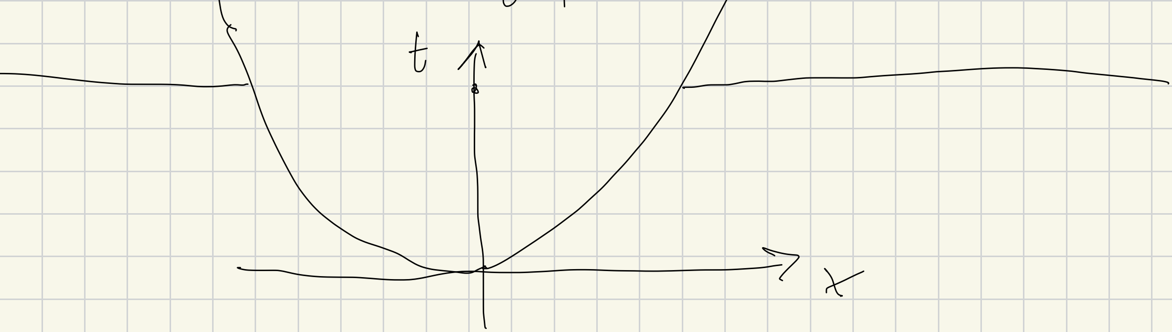
$\forall\ t, t_0 \in [0,T]$.

$\Rightarrow$ $\cancel{|u((t)|}$

$$|u(t)|_{L^2}^2 \in AC([0,T])$$

$$|u(t)|_{L^{p+1}(\mathbb{R}^d)} \xrightarrow{t \to +\infty} 0$$

<u>Lemme</u>
$$\int_{|x| \geq |t \log t|} |u|^{p+1} \, dx \xrightarrow{t \to +\infty} 0$$



<u>Pf</u> $M > 0$

$$\vartheta_M(x) = \begin{cases} \frac{|x|}{M} & \text{for } |x| \leq M \\ 1 & |x| \geq M \end{cases}$$

$\vartheta_M \in W^{1,\infty}(\mathbb{R}^d)$

$$\nabla \vartheta_M(x) = \begin{cases} \frac{1}{M}\frac{x}{|x|} & x \le M \\ 0 & x \ge M \end{cases}$$

$$|\nabla \vartheta_M|_{L^\infty} \cancel{\le} = \frac{1}{M}$$

$$t \to \frac{1}{2}\langle \vartheta_M u(t), u(t)\rangle = $$

$$= \frac{1}{2}\left|\sqrt{\vartheta_M}\, u(t)\right|^2_{L^2(\mathbb{R}^d)}$$

$$u \in BC^0(\mathbb{R}, H^1) \cap BC^1(\mathbb{R}, H^{-1})$$

$$i\partial_t u = -\Delta u + |u|^{p-1}u \qquad L^{\frac{p+1}{p}} \hookrightarrow H^{-1}$$

$$\left||u|^{p-1}u\right|_{H^{-1}} \le C_{Sob}\, |u^p|_{L^{\frac{p+1}{p}}} = C_{Sob}\, |u|^p_{L^{p+1}} \le C_{Sob}^{p+1}\, |u|^p_{H^1}$$

$$\frac{d}{dt}\frac{1}{2}\left|\sqrt{\vartheta_M}\, u(t)\right|_{L^2(\mathbb{R}^2)} = \langle \vartheta_M u(t), \dot{u}(t)\rangle$$

$$= \langle \vartheta_M u, i\Delta u\rangle_{L^2} - \cancel{\langle \vartheta_M u, i|u|^{p-1}u\rangle}$$

$$= \langle \vartheta_M u, i\nabla\cdot\nabla u\rangle_{L^2} = -\langle \nabla\vartheta_M u, i\nabla u\rangle$$

$$- \cancel{\langle \vartheta_M \nabla u, i\nabla u\rangle}$$

$$\left| \frac{d}{dt} \frac{1}{2} \langle \vartheta_M u, u \rangle \right| \le \left| \langle \nabla\vartheta_M u, i\nabla u \rangle \right|$$

$$\le \frac{1}{M} \underbrace{|u|_{L^2(\mathbb{R}^d)} \, |\nabla u|_{L^2(\mathbb{R}^d)}}_{\le C_{u_0}}$$

$$\le \frac{1}{M} C_{u_0}$$

$\forall M \gg 0$

$$\boxed{\langle \vartheta_M u(t), u(t) \rangle \le C M^{-1} t + \langle \vartheta_M u_0, u_0 \rangle}$$

$M = t \lg t \qquad t > 1$

$$\int_{|x| \ge t \lg t} |u(t)|^2 \, dx \le \langle \vartheta_{t \lg t} u(t), u(t) \rangle$$

$$\le \frac{C}{\lg t} + \int_{|x| \le t \lg t} \overbrace{\vartheta\left(\frac{|x|}{t \lg t}\right) |u_0|^2}^{\xrightarrow{t \to +\infty} 0} dx$$

$\xrightarrow{t \to +\infty} 0$

$$+ \int_{|x| \ge t \lg t} |u_0|^2 \, dx$$

$\xrightarrow{t \to +\infty} 0$

$$\| u \|_{L^{p+1}(|x| \geq t \log t)} \leq$$

$$1 < p < d^* \qquad 2 < p+1 < d^* + 1$$

$$\frac{1}{p+1} = \frac{\alpha}{2} + \frac{1-\alpha}{d^*+1} \qquad \alpha \in (0,1)$$

$$\leq \| u \|^{\alpha}_{L^2(|x| \geq t \log t)} \, \| u \|^{1-\alpha}_{L^{d^*+1}(\mathbb{R}^d)}$$

$$H^1(\mathbb{R}^d) \hookrightarrow L^{d^*+1}(\mathbb{R}^d)$$

$$\leq C_d \underbrace{\| u \|^{\alpha}_{L^2(|x| \geq t \log t)}}_{\xrightarrow{t \to +\infty} 0} \, \underbrace{\| u \|^{1-\alpha}_{H^1(\mathbb{R}^d)}}_{\leq C \, u_0}$$