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Computational Methods for Fluid Dynamics and Heat Transfer

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Description of the problem

In the fully developed[1] laminar channel flow for straight pipes of constant cross-section, constraints from the continuity equation and the no-slip condition, require the existence of only the streamwise velocity component $u = u(y, z)$, while the other two velocity components vanish, i.e. $v = w = 0$.

In this situation, we are left with only the $x-$momentum equation, which reduces to the following Poisson equation:

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{1}$$

where, in eq. (1), μ is the dynamic viscosity of the fluid and $\mathrm{d}p/\,\mathrm{d}x$ is the streamwise pressure gradient (please note that, in this situation, $p = p(x)$). A schematic diagram of a fully developed laminar channel flow is shown in figure 1.

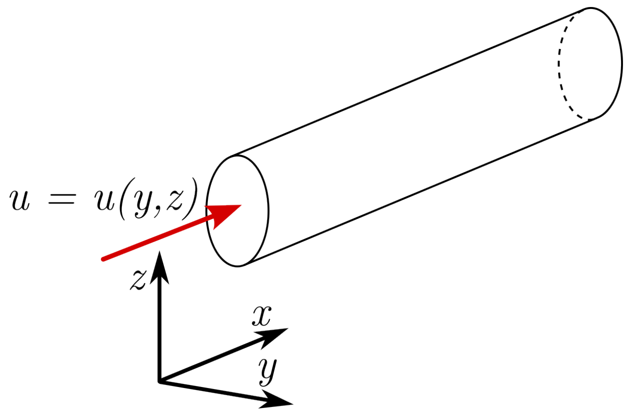

Figure 1: Fully developed laminar channel flow in a straight pipe.

Fluid flow parameters

For these type of problems, the fluid flow characteristics are expressed in terms of certain hydrodynamic parameters which are define here.

Hydraulic diameter The *Hydraulic Diameter* D_h, for a pipe of generic cross-section, is defined as

$$D_h = \frac{4\,A}{P} \tag{2}$$

where A is the cross-section area and P is the wetted perimeter.

[1] We recall that with *fully developed laminar flow* we mean that region where both the velocity components v and w lying in the cross-section plane of the pipe *and* the gradient of the axial velocity component $\partial u/\partial x$ are everywhere zero.

Reynolds number The *Reynolds number* is defined as

$$Re = \frac{\rho\, D_h\, U}{\mu} \tag{3}$$

where ρ is the density and U is the bulk, i.e. mean, streamwise velocity value.

Fanning friction factor The *Fanning friction factor*[2] f is defined as the ratio of the *average* wall shear stress $\overline{\tau}_w$ to the flow kinetic energy per unit volume $\rho U^2/2$

$$f = \frac{\overline{\tau}_w}{\rho\, U^2/2} \tag{4}$$

Since for a single phase, fully developed flow in a pipe, the shear stress at the fluid-solid boundary is balanced by the pressure drop, a one-dimensional force balance can be written as:

$$A\,\Delta p = P\, L \overline{\tau}_w$$

where L is the pipe length.
Therefore, equation (4) can be written also as

$$f = \frac{(\,\mathrm{d}p/\,\mathrm{d}x)\, A}{\rho\, U^2/2\, P} \tag{5}$$

Proposed problems

Develop, in MATLAB or other language of choice, a Finite Volume (FV) program/script that computes the flow field in a straight duct of *rectangular* cross-section, assuming a steady laminar flow of a Newtonian fluid of constant thermophysical properties.
Consider the two following cases:

1. Square channel with $L_y = L_z = 25$ [mm].
   $\mathrm{d}p/\,\mathrm{d}x = 1.0$ [Pa/m].
   $\rho = 997$ [kg/m^3].
   $\mu = 8.9 \times 10^{-4}$ [kg/m s].

2. Rectangular channel with $L_y = 10$ [mm] and $L_z = 25$ [mm].
   $\mathrm{d}p/\,\mathrm{d}x = 1.0$ [Pa/m].
   $\rho = 997$ [kg/m^3].
   $\mu = 8.9 \times 10^{-4}$ [kg/m s].

[2] This quantity is not to be confused with the *Moody (or Darcy) friction factor*, which is a dimensionless parameter defined as $f_D = -(\,\mathrm{d}p/\,\mathrm{d}x) D_h / \rho\, U^2/2$.
For circular pipes the relation between the two parameters is

$$f = \frac{f_D}{4}$$

Using and *adequate* number of cells, for both cases:

a. Plot a *contour* of the velocity field.

b. Compute the *Fanning friction factor* and compare it with the following approximation

$$f\,\mathrm{Re} = 24\left(1 - 1.3553\alpha + 1.9467\alpha^2 - 1.7012\alpha^3 + 0.9564\alpha^4 - 0.2537\alpha^5\right) \quad (6)$$

where

$$\alpha = min(L_z/L_y, L_y/L_z)$$

The equation (6) closely approximates, within $+0.05\%$, the following exact analytical expression for the fully developed *Fanning friction factor* for ducts of rectangular cross-section

$$f\,\mathrm{Re} = \frac{24}{\left(1+\dfrac{1}{\alpha}\right)^2\left(1 - \dfrac{192}{\pi^4\alpha}\displaystyle\sum_{n=1,3,\ldots}^{\infty}\dfrac{\tanh\left(n\pi\alpha/2\right)}{n^5}\right)} \quad (7)$$

TIP

After the (discrete) solution of equation (1), the *Fanning friction factor* can be computed using either eq. (4), once the *wall-averaged* shear stress and mean velocity U have been computed, or eq. (5) if the mean velocity U has been obtained.

References

[1] F. P. Incropera, D. P. Dewitt, T. L. Bergman, A. S. Lavine, *Fundamentals of Heat and Mass Transfer*, 6th Ed., Wiley, (2007).

[2] R.K. Shah, A.L. London, *Laminar Flow Forced Convection in Ducts*, Elsevier Inc., (1978).