Stochastic Kuramoto model: the Sakaguchi model

AI

1 The model

We consider a system of N coupled oscillators, originally introduced by Sakaguchi [1], described by the Langevin equation

$$\dot{\phi}_i(t) = f_i - \frac{K}{N}\sum_j \sin(\phi_i(t) - \phi_j(t)) + \eta_i(t), \tag{1}$$

where f_i is an external constant force, and the Gaussian noise η_i obeys the fluctuation–dissipation relation

$$\langle \eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta(t-t'). \tag{2}$$

Notice that we have chosen the system units such that the external force f_i has dimension of frequency, which corresponds to taking the friction coefficient in eq. (2) equal to one.

By introducing the complex order parameter

$$\sigma(t)\exp(\mathbb{i}\psi(t)) = \frac{1}{N}\sum_j \exp(\mathbb{i}\phi_j(t)), \tag{3}$$

where $0 \le \sigma(t) \le 1$ measures the system coherence and $\psi(t)$ is the common average phase, equation (1) becomes

$$\dot{\phi}_i(t) = f_i - K\sigma(t)\sin(\phi_i(t) - \psi(t)) + \eta_i(t), \tag{4}$$

this set of equations if formally equivalent to the set of equations (1). Let f_0 be the mean deterministic force, calculated over the N oscillator sample $f_0 =$

$\sum_j f_j/N$, we expect that the center of mass will oscillate with the frequency f_0, so we can use the mean field approximation $\psi(t) = f_0 t + \psi_0$ in eq. (4) and thus we can redefine the dynamical variables as $\theta_i(t) = \phi_i(t) - \psi(t)$, so as eq. (4) reads

$$\dot{\theta}_i(t) = \omega_i - K\sigma \sin(\theta_i(t)) + \eta_i(t), \tag{5}$$

where we have redefined the external force as $\omega_i = f_i - f_0$, and $\sigma(t)$ has been replaced with its mean field value σ.

Eq. (5) represents now a set of N uncoupled equations for the variables θ_i, and the mean field values of σ and ψ_0 can be obtained self consistently as discussed below. Eq. (5) corresponds to a Brownian particle moving in a periodic potential under the effect of a constant drift force ω_i. Here and in the following we assume that the system reaches a steady state in the long time limit. In the course of this paper, we will discuss this assumption where relevant. The Langevin equation can be reformulated in terms of a Fokker-Planck (FP) equation for the probability distribution function (PDF) of finding the particle i at position θ at time t

$$\partial_t p(\theta, \omega_i, t) = \partial_\theta \left[(K\sigma \sin\theta - \omega_i)p + T\partial_\theta p \right]. \tag{6}$$

Thus, the stationary probability distribution function (PDF) of the position of such a particle reads

$$p(\theta, \omega_i) = \mathcal{N}\beta \mathrm{e}^{\beta(K\sigma\cos\theta + \omega_i\theta)} \left[\frac{I(2\pi)}{1 - \exp(-\beta 2\pi\omega_i)} - I(\theta) \right], \tag{7}$$

where $I(x) = \int_0^x \mathrm{d}y \exp\left[-\beta(K\sigma\cos y + \omega_i y)\right]$, and $\mathcal{N}$ is a normalization constant depending implicitly on $\beta = 1/T$, $K \cdot \sigma$ and ω_i. The steady–state probability current thus reads $J_{ss} = \mathcal{N}$, and the particle steady–state velocity reads

$$\begin{aligned} &v_\theta(K\sigma, \omega_i, T) = 2\pi\mathcal{N} \\ &= 2\pi \left\{ \beta \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{e}^{\beta(K\sigma\cos\theta + \omega_i\theta)} \left[\frac{I(2\pi)}{1 - \exp(-\beta 2\pi\omega_i)} - I(\theta) \right] \right\}^{-1}. \end{aligned} \tag{8}$$

As $N \to \infty$, we can adopt a continuous description, where the constant forces acting on the oscillators are distributed according to the probability distribution $g(f)$ with mean value f_0. By introducing the shifted force distribution

$$g_0(\omega) = g(f_0 + \omega), \tag{9}$$

the self-consistent equation for the modulus σ of the complex order parameter, characterizing the degree of order or coherence in the configuration of the variables θ_i is then given by

$$\sigma e^{\mathbb{i}\psi_0} = \int d\omega \, g_0(\omega) \int_0^{2\pi} d\theta \, p(\theta,\omega) \exp(\mathbb{i}\theta) \tag{10}$$

which can be decomposed into its real and imaginary part

$$\sigma \cos(\psi_0) = \int d\omega \, g_0(\omega) \int_0^{2\pi} d\theta \, p(\theta,\omega) \cos\theta \, , \tag{11}$$

$$\sigma \sin(\psi_0) = \int d\omega \, g_0(\omega) \int_0^{2\pi} d\theta \, p(\theta,\omega) \sin\theta \, . \tag{12}$$

By assuming that the force distribution $g(f)$ is symmetric around f_0, and noticing that $p(\theta,-\omega) = p(-\theta,\omega)$, the imaginary part on the right–hand side of eq.(10) vanishes, and so one is left with

$$\sigma = \int d\omega \, g_0(\omega) \int_0^{2\pi} d\theta \, p(\theta,\omega) \cos(\theta), \tag{13}$$

whose solution provides the mean field value for σ.

As discussed in [1], for $N \to \infty$ this model exhibits a critical coupling strength K_c, such that for $K > K_c$ the systems exhibits a dynamical phase transition with synchronization $\sigma > 0$, while the system is incoherent for $K < K_c$, and each particle described by the coordinate θ_i oscillates with its proper frequency ω_i. Thus, for $K \gtrsim K_c$ we expect σ to be positive but small, and we can expand eq. (13) in powers of $\epsilon = K\sigma/T$, obtaining

$$\sigma = \frac{K\sigma T}{2} \int_{-\infty}^{+\infty} d\omega \, \frac{g_0(\omega)}{(T^2+\omega^2)} \left[1 - \frac{K^2\sigma^2 \, (T^2 - 2\omega^2)}{2\,(T^2+\omega^2)\,(4T^2+\omega^2)} \right] + O\left(\epsilon^5\right) \tag{14}$$

while expanding eq. (8) the average velocity of the dynamical variable θ reads

$$v_\theta(\omega) = \omega \left[1 - \frac{K^2\sigma^2}{2(T^2+\omega^2)} + O\left(\epsilon^4\right) \right] . \tag{15}$$

Inspection of eq. (14) provides the critical coupling strength for which a non-vanishing solution to that equation appears

$$K_c = 2 \left[\int_{-\infty}^{+\infty} d\omega \, g_0(\omega) \frac{T}{(T^2+\omega^2)} \right]^{-1} . \tag{16}$$

The value of the order parameter σ as a function of K and T, for $K > K_c$ can be obtained by solving eq. (14), which gives

$$\sigma \simeq \sqrt{\frac{K-K_c}{K_c K^3 I_3}} \simeq \sqrt{\frac{K-K_c}{K_c^4 I_3}}, \tag{17}$$

where

$$I_3 = \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \frac{g_0(\omega) T \left(T^2 - 2\omega^2\right)}{4\left(T^2+\omega^2\right)^2 \left(4T^2+\omega^2\right)}, \tag{18}$$

$$\tag{19}$$

Notice that the equations (6), (7), (8) are exact for any value of $K\sigma$, and in general the value of σ as a function of K can be evaluated by solving the self-consistent equation (14) numerically. A few comments on the velocity $v_\theta(\omega)$ are now in order. Such a quantity represents the velocity of the dynamical variable θ and is thus the velocity deviation of a particle under the effect of a force ω with respect to the center of mass velocity f_0. Inspection of eq. (5) suggests that for $K < K_c$ (thus for $\sigma = 0$) $v_\theta(\omega) = \omega$. On the other hand $v_\theta(\omega)$ goes to zero as K increases above K_c: the higher K the higher are the barriers of the periodic force in eq. (5), while σ is also an increasing function of K.

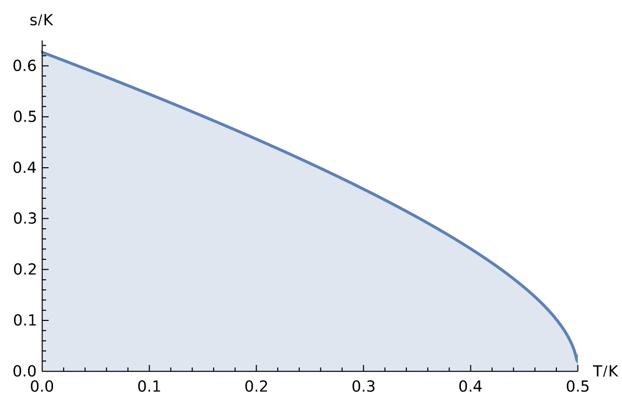

Figure 1: Phase diagram with the critical curve as given by eq. (16) for a gaussian force distribution $g_0(\omega)$ with variance s^2. In the shaded area the model exhibits an ordered phase with $\sigma > 0$.

2 Stochastic Thermodynamics of the microscopic model

In section 1 we have discussed the dynamical properties of the model system we will use in the present paper. We can now turn our attention to its thermodynamic properties, namely the input and delivered power, and the system efficiency as a global motor.

One can consider two possible scenarios as far as the forces applied on each single particle are concerned.

In the first case the forces acting on the motor system can be either positive or negative, thus resembling the macroscopic power grids of power plants and consumers considered, e.g., in [4, 5, 6, 7]. Differently from those works, we consider here microscopic oscillators, in the over-damped regime, and with white noise acting on them. In this scenario, taking inspiration from the macroscopic realm, one may call *users* those oscillators with a negative force acting on them $f_i < 0$, and *producers* those oscillators with a positive force $f_i > 0$, and a single force distribution characterizes the system.

The second possible scenario resembles the case of *molecular motors*, where both a negative ($f_i^- < 0$) and a positive force ($f_i^+ > 0$) are applied on the same particle i. This is the case in, e.g., biological molecular motors such as kinesin and myosin [8, 9] where the energy extracted by ATP hydrolysis drives the motor forward (corresponding to $f_i^+ > 0$) while the motor does work to carry a cargo, modelled by a negative load (corresponding to $f_i^- < 0$) In this case one deals with two different distributions of forces, $g_+(f_+)$ and $g_-(f_-)$.

In both scenarios, in order for the system to behave globally as a motor, and to perform work against the negative forces, we must require the center of mass to have an average positive velocity, and thus $f_0 > 0$.

In the following, we will consider only the first scenario and study the delivered power P_{out} and the input power P_{in}, and, where possible, optimize P_{out} wrt different parameters. We will also discuss the efficiency at maximum power (EMP) $\eta^* = P^*_{\text{out}}/P^*_{\text{in}}$.

2.1 Single force distribution

We consider here the stochastic thermodynamics of a system with either negative or positive forces applied on each oscillators, and distributed according to the single PDF $g(f)$.

We can thus introduce the relevant thermodynamic quantities, namely the average input power, absorbed by the producers, and the average output power released by the users. Recalling that v_θ as given by eq. (8) gives the deviation of the i-th particle's average velocity from the center of mass velocity f_0, the average output and input power read

$$\begin{aligned} P_{\rm out} &= -\int_{-\infty}^{0} \mathrm{d}f\, g(f)\, [v_\theta(f-f_0)+f_0]\, f \\ &= -\int_{-\infty}^{-f_0} \mathrm{d}\omega\, g_0(\omega)\, [v_\theta(\omega)+f_0]\, (\omega+f_0)\,, \end{aligned} \tag{20}$$

$$\begin{aligned} P_{\rm in} &= \int_{0}^{+\infty} \mathrm{d}f\, g(f)\, [v_\theta(f-f_0)+f_0]\, f \\ &= \int_{-f_0}^{+\infty} \mathrm{d}\omega\, g_0(\omega)\, [v_\theta(\omega)+f_0]\, (\omega+f_0)\,. \end{aligned} \tag{21}$$

while the thermodynamic efficiency of the system reads

$$\eta = \frac{P_{\rm out}}{P_{\rm in}}. \tag{22}$$

Substituting eqs. (15) and (17) into (20) and (21) the output and input power becomes, up to the first order in $K-K_c$,

$$P_{\rm out} = P_0^< + \frac{K-K_c}{K_c^2 I_3} I_2^<, \tag{23}$$

$$P_{\rm in} = P_0^> + \frac{K-K_c}{K_c^2 I_3} I_2^>, \tag{24}$$

where

$$P_0^< = -\int_{-\infty}^{-f_0} \mathrm{d}\omega g_0(\omega)(\omega+f_0)^2 < 0, \tag{25}$$

$$I_2^< = \int_{-\infty}^{-f_0} \mathrm{d}\omega g_0(\omega) \frac{(\omega+f_0)\omega}{2(T^2+\omega^2)} \geq 0, \tag{26}$$

with analogous definitions for $P_0^>$, $I_2^>$. We notice that, in absence of partial synchronization ($K < K_c$, $\sigma = 0$), i.e., when the users and the producers are decoupled, eq. (20), (23), and (25) predict that the delivered power is negative. Since $v_\theta(\omega) = \omega$ for $K < K_c$, the *users* oscillates with their proper frequency (force) which is negative, and so the product of the applied forces times the average velocity is positive. The term $v_\theta(\omega)$ in eq. (20) is always negative, as the integration variable runs over negative value. Thus by increasing K above K_c the modulus of $v_\theta(\omega)$ decreases and tends to zero for very large K. This implies that for some value of K the rhs of eq. (20) becomes positive, such values depending on f_0 and T, and on the details of the distribution $g_0(\omega)$, e.g. its width.

2.1.1 Optimization

Here we aim at optimizing the delivered power eq. (20) wrt some of the relevant parameters.

Optimization wrt the coupling strength $\partial P_{\text{out}}/\partial K = 0$ gives:

$$\frac{\partial P_{\text{out}}}{\partial K} = -\int_{-\infty}^{-f_0} \mathrm{d}\omega\, g_0(\omega)\omega\partial_K v_\theta(K,\omega,T). \tag{27}$$

Recalling that the average velocity v_θ goes to zero as K increases above K_c, we find that, for $\omega < 0$, $v_\theta(\omega, K)$ is an increasing function of K, ranging from ω for $K < K_c$ and approaching zero as $K \to \infty$. Similarly, for $\omega > 0$, $v_\theta(\omega, K)$ is a decreasing function of K, ranging from ω for $K < K_c$ and zero as $K \to \infty$. Thus, from eq. (27) it follows

$$\frac{\partial P_{\text{out}}}{\partial K} = \begin{cases} 0, & \text{if } K < K_c, \\ \geq 0 & \text{if } K \geq K_c. \end{cases} \tag{28}$$

Similarly one finds

$$\frac{\partial P_{\text{in}}}{\partial K} = \begin{cases} 0, & \text{if } K < K_c, \\ \leq 0 & \text{if } K \geq K_c. \end{cases} \tag{29}$$

In order to prove the last inequality we notice that

$$\frac{\partial P_{\text{in}}}{\partial K} = \int_{-f_0}^{+\infty} \mathrm{d}\omega\, g_0(\omega)(\omega + f_0)\partial_K v_\theta = \int_{-f_0}^{+f_0} \cdots + \int_{f_0}^{+\infty} \cdots \tag{30}$$

The second integral is negative, while for the first integral we have

$$\begin{aligned}
&\int_{-f_0}^{+f_0} \mathrm{d}\omega\, g_0(\omega)(\omega+f_0)\partial_K v_\theta = \partial_K \int_{-f_0}^{+f_0} \mathrm{d}\omega\, g_0(\omega)(\omega+f_0) v_\theta \\
= \;& \partial_K \int_{-f_0}^{+f_0} \mathrm{d}\omega\, g_0(\omega)\omega v_\theta = 2\partial_K \int_{0}^{+f_0} \mathrm{d}\omega\, g_0(\omega)\omega v_\theta \\
= \;& 2\int_{0}^{+f_0} \mathrm{d}\omega\, g_0(\omega)\omega \partial_K v_\theta \leq 0
\end{aligned} \tag{31}$$

Thus, if K is the free parameter, the optimal delivered power is achieved for $K \to \infty$, corresponding to the limit of strong coupling between users and producers, with an EMP $\eta^* = -\langle f_- \rangle / \langle f_+ \rangle$ as obtained by eqs. (20) and (21), where $\langle f_- \rangle$ and $\langle f_+ \rangle$ are the average negative and positive forces, respectively, with $f_0 = \langle f_- \rangle + \langle f_+ \rangle$.

No similar inequalities can be found when one tries to maximize P_{out} with respect to other parameters, for example f_0. So one should consider specific cases for the force distribution in order to study the relevant thermodynamic quantities.

Finally, it is worth to note that the maximal possible efficiency is also achieved for $K \to \infty$. Indeed we have

$$\partial_K \eta = \frac{(\partial_K P_{\text{out}})\, P_{\text{in}} - P_{\text{out}}\,(\partial_K P_{\text{in}})}{P_{\text{in}}^2}. \tag{32}$$

inspection of eqs. (28) and (29) suggest that $\partial_K \eta > 0$, indeed P_{in} is positive for any K, and we are interested in the regime where K is sufficiently large such that $P_{\text{out}} > 0$. In the limit of large K, η can be evaluated from eqs. (20) and (21), and we find $\eta \to -\langle f_- \rangle / \langle f_+ \rangle = 1 - f_0 / \langle f_+ \rangle$.

2.1.2 A specific distribution

In order to exemplify the results discussed in this section, here we consider the specific distribution

$$g(f) = \frac{1}{2}\left[\delta(f-(f_0+s)) + \delta(f-(f_0-s))\right], \tag{33}$$

where s^2 is the variance of the distribution, with $s > f_0 > 0$, i.e., there are just two types of oscillator, the *users* with an applied force $f_0 - s < 0$ and the *producers* with an applied force $f_0 + s > 0$. The shifted force distribution thus reads

$$g_0(\omega) = \frac{1}{2}\left[\delta(\omega - s) + \delta(\omega + s)\right]. \tag{34}$$

For such a distribution the critical coupling strength reads

$$K_c = 2\frac{(s^2 + T^2)}{T}. \tag{35}$$

This corresponds to the bimodal distribution considered in [10], where the linear stability of the incoherent solution $p(\theta, \omega) = 1/2\pi$ of the FP equation (6) was studied, corresponding to the non-synchronized phase $\sigma = 0$. Equations (23)-(24) thus become

$$P_{\text{out}} = \frac{1}{2}(s - f_0)(f_0 - v_\theta(s)), \tag{36}$$

$$P_{\text{in}} = \frac{1}{2}(s + f_0)(f_0 + v_\theta(s)), \tag{37}$$

We recall that for $K < K_c$ (i.e. for $\sigma = 0$), $v_\theta(s) = s$, and because of the condition $s > f_0$ we have $P_{\text{out}} < 0$, i.e. when the producers and users are not coupled, the users oscillates with their proper frequency $f_0 - s$ resulting in a negative P_{out} . The delivered power will become positive for some value of $K > K_c$, when $v_\theta(s)$ in eq. (36) becomes smaller than f_0.

2.1.3 Optimization

Here we optimize the delivered power for the force distribution (34), which corresponds to a system where on each oscillator there is either a positive $f_0 + s$ or a negative $f_0 - s$ force with probability 1/2. From eq. (15) we easily obtain the expression for the velocity deviation from the center of mass up to the fourth order in σ

$$v_\theta(s) = s\left[1 - \frac{K^2\sigma^2}{2(s^2 + T^2)}\right] \tag{38}$$

while the order parameter, as given by eq. (17), becomes

$$\sigma = \sqrt{\frac{\Delta K T(s^2 + 4T^2)}{K_c^2(T^2 - 2s^2)}} \tag{39}$$

up to the leading order in ΔK.

We can now optimize $P_{\rm out}$, as given by eq. (36), wrt to different parameters:

i) By optimizing wrt to K at fixed f_0 and s: $\partial_K P_{\rm out} = 0$, one obtains

$$\frac{\partial P_{\rm out}}{\partial K} = -1/2(s - f_0)\frac{\partial v_\theta(s)}{\partial K} > 0, \tag{40}$$

since $v_\theta(s)$ is a decreasing function of K, as already discussed above for a general force distribution $g_0(\omega)$.

ii) By optimizing $P_{\rm out}$ wrt the average force, at fixed K and s, $\partial_{f_0} P_{\rm out} = 0$, one obtains

$$f_0^*(s, K) = \frac{v_\theta(s) + s}{2} \tag{41}$$

and since the condition $s \geq v_\theta(s) > 0$ holds for any $K > 0$, we have $s > f_0^*(s, K) > s/2$.

We can thus calculate the delivered and the input power, and the efficiency at the maximum

$$P^*_{\rm out} = \frac{1}{8}(s - v_\theta(s))^2 \tag{42}$$

$$P^*_{\rm in} = \frac{1}{8}(3s + v_\theta(s))(s + 3v_\theta(s)) \tag{43}$$

$$\eta^* = \frac{(s - v_\theta(s))^2}{(3s + v_\theta(s))(s + 3v_\theta(s))} \tag{44}$$

$$\simeq \frac{\Delta K^2\left(s^2 + 4T^2\right)^2}{16K_c^2\left(T^2 - 2s^2\right)^2} \tag{45}$$

where we have used (38) and (39) to expand η^* up to the lowest order in ΔK and s/T. Plots of η^* as a function of K for different values of s are shown in fig. 2. Inspection of this figure, as well as of eqs. (42), (43) and (44) suggests that, for fixed s, when K increases above K_c, the optimal output power (42) increases, the optimal input power (43) decreases, and this results in an increase of the EMP (44). This is a consequence of the fact that $v_\theta(s) \to 0$ in the limit $K \to \infty$, where $\eta^* = 1/3$.

Inspection of figure 2, as well as of eq. (45), suggests that a higher degree of quenched disorder, as parametrized by s, leads to a larger EMP close to the dynamical phase transition.

References

[1] Hidetsugu Sakaguchi. Cooperative phenomena in coupled oscillator systems under external fields. *Progress of Theoretical Physics*, 79(1):39–46, 1988.

[2] Steven H. Strogatz, Charles M. Marcus, Robert M. Westervelt, and Renato E. Mirollo. Collective dynamics of coupled oscillators with random pinning. *Physica D: Nonlinear Phenomena*, 36(1–2):23 – 50, 1989.

[3] Shin ichi Sasa. Collective dynamics from stochastic thermodynamics. *New Journal of Physics*, 17(4):045024, 2015.

[4] G. Filatrella, A. H. Nielsen, and N. F. Pedersen. Analysis of a power grid using a kuramoto-like model. *The European Physical Journal B*, 61(4):485–491, 2008.

[5] Martin Rohden, Andreas Sorge, Marc Timme, and Dirk Witthaut. Self-organized synchronization in decentralized power grids. *Phys. Rev. Lett.*, 109:064101, Aug 2012.

[6] Florian Dörfler, Michael Chertkov, and Francesco Bullo. Synchronization in complex oscillator networks and smart grids. *Proceedings of the National Academy of Sciences*, 110(6):2005–2010, 2013.

[7] Simona Olmi, Adrian Navas, Stefano Boccaletti, and Alessandro Torcini. Hysteretic transitions in the kuramoto model with inertia. *Phys. Rev. E*, 90:042905, Oct 2014.

[8] B. Alberts, A. Johnson, J. Lewis, M. Raff, K. Roberts, and P. Walter. *Molecular Biology of the Cell*. Garland, New York, 5th edition, 2007.

[9] J. Howard. *Mechanics of Motor Proteins and the Cytoskeleton*. Sinauer Associates, Sunderland, MA, 2001.

[10] Juan A. Acebrón, L. L. Bonilla, Conrad J. Pérez Vicente, Félix Ritort, and Renato Spigler. The kuramoto model: A simple paradigm for synchronization phenomena. *Rev. Mod. Phys.*, 77:137–185, Apr 2005.

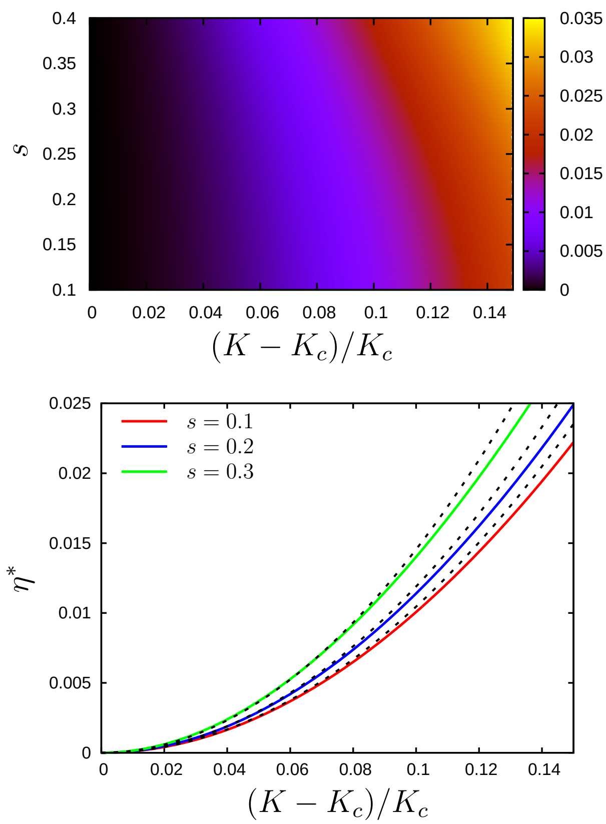

Figure 2: Top: EMP η^* as obtained by maximizing P_{out} wrt f_0, as a function of the coupling strength K and of the quenched disorder standard deviation s (eq. (44)), with $T = 1$. Bottom: Plot of η^* as a function of K for three specific values of the quenched disorder standard deviation s. The dashed lines correspond to the approximated expression (45).