15 May $\qquad u \in C^\infty(\mathbb{R}^d)$

$$\mathrm{Re}\left\{\Delta u\left(\bar u_r + \frac{d-1}{r}\bar u\right)\right\} =$$

$$= \nabla\cdot \mathrm{Re}\left\{\nabla u\left(\bar u_r + \frac{d-1}{r}\bar u\right)\right\} - \nabla\cdot\left(\frac{x}{2r}|\nabla u|^2\right)$$

$$+ \nabla\left(\frac{d-1}{4}\frac{x}{r^3}|u|^2\right)$$

$$- \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right)$$

$$- \frac{(d-1)(d-3)}{4r^3}|u|^2$$

$$\nabla\cdot \mathrm{Re}\left\{\nabla u\left(\bar u_r + \frac{d-1}{r}\bar u\right)\right\} = \mathrm{Re}\left\{\Delta u\left(\bar u_r + \frac{d-1}{r}\bar u\right)\right\}$$

$$+ \mathrm{Re}\left\{\partial_j u\,\partial_j\left(\bar u_r + \frac{d-1}{r}\bar u\right)\right\}$$

$$\mathrm{Re}\left\{\partial_j u\,\partial_j\left(\bar u_r + \frac{d-1}{r}\bar u\right)\right\} = \mathrm{Re}\left\{\partial_j u\,\partial_j\left(\frac{x_k}{r}\partial_k \bar u\right)\right\}$$

$$+ (d-1)\,\mathrm{Re}\left\{\partial_j u\,\partial_j\left(\frac{1}{r}\bar u\right)\right\}$$

$$= \mathrm{Re}\left\{\frac{x_k}{r}\frac{1}{2}\partial_k|\nabla u|^2\right\} + \frac{|\nabla u|^2}{r}$$

$$- \frac{1}{r}\cancel{\mathrm{Re}}\left\{\overbrace{\partial_j u\,\frac{x_j}{r}}^{\partial_r u|^2}\,\frac{x_k}{r}\partial_k\bar u\right\}$$

$$+ \frac{(d-1)}{r}|\nabla u|^2 - (d-1)\frac{x_j}{r}\frac{1}{2}\partial_j|u|^2$$

$$= \frac{1}{2} \partial_k \left(\frac{x_k}{r} |\nabla u|^2 \right) - \frac{1}{2} \underbrace{\nabla \cdot \frac{x}{r}}_{\frac{d-1}{r}} |\nabla u|^2$$

$$+ \frac{1}{r} \left(|\nabla u| - |\partial_r u|^2 \right)$$

$$+ \frac{d-1}{r} |\nabla u|^2 - \frac{d-1}{2} \partial_j \left(\frac{x_j}{r} |u|^2 \right)$$

$$+ \frac{d-1}{2} \underbrace{\nabla \cdot \frac{x}{r}}_{\frac{d-1}{r}} |u|^2$$

$$\nabla \cdot \frac{x}{r^q} = (d-q) \frac{1}{r}$$

<u>Lemma</u> $\displaystyle \lim_{t \to +\infty} \int_{|x| \geq t \lg t} |u|^{p+1} dx = 0$

<u>Lemma</u> $\forall \varepsilon > 0$ and $\forall t > 2$ and $\tau > 0$

$\exists$ $t_0 > \max\{t, 2\tau\}$ s.t.

$$\int_{t_0 - 2\tau}^{t_0} \int_{|x| \leq s \lg s} |u|^{p+1} dx \, ds \leq \varepsilon$$

Pf $\quad \infty > \int_{\mathbb{R}} ds \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} dx \geq$

$$\geq \int_t^{+\infty} \frac{ds}{s \lg s} \int_{|x| \leq s \lg s} |u|^{p+1} dx =$$

$$= \sum_{k=0}^{\infty} \int_{t+2k\tau}^{t+2(k+1)\tau} \frac{ds}{s \lg s} \int_{|x| \leq s \lg s} |u|^{p+1} ds$$

$$\geq \sum_{k=0}^{\infty} \frac{1}{(t+2(k+1)\tau) \lg(t+2(k+1)\tau)} \int_{t+2k\tau}^{t+2(k+1)\tau} \int_{|x| \leq s \lg s} |u|^{p+1} ds$$

$$\sum_{k=0}^{\infty} \frac{1}{(t+2(k+1)\tau) \lg(t+2(k+1)\tau)} = +\infty$$

$$\Rightarrow \liminf_{k \to +\infty} \int_{t+2k\tau}^{t+2(k+1)\tau} \int_{|x| \leq s \lg s} |u|^{p+1} ds = 0$$

So $\forall \varepsilon > 0 \quad \exists k \quad$ st

$$\int_{t+2k\tau}^{t+2(k+1)\tau} \int_{|x| \leq s \lg s} |u|^{p+1} ds < \varepsilon$$

$t_0 = t + 2k\tau \quad$ obviously

$t_0 > t \qquad t_0 > 2\tau$

$$\int_{t_0}^{t_0+2\tau} \int_{|x| \le s \log s} |u|^{p+1} \, ds < \varepsilon$$

<u>Lemma</u> $\forall\, \varepsilon, a, b \in \mathbb{R}_+$ $\exists\, t_0 > \max\{a, b\}$ st

$$\sup_{s \in [t_0 - b, t_0]} \| u(s) \|_{L^{p+1}} < \varepsilon$$

<u>Pf</u> $u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s) \, ds$

Duhemel formula

$\tau > 0$

$$= e^{it\Delta} u_0 - i \int_0^{t-\tau} e^{i(t-s)\Delta} |u(s)|^{p-1} u(s) \, ds$$

$$- i \int_{t-\tau}^{t} e^{i(t-s)\Delta} |u(s)|^{p-1} u(s) \, ds$$

$$= e^{it\Delta} u_0 + w(t, \tau) + z(t, \tau)$$

<u>Claim</u> $\lim_{t \to +\infty} \| e^{it\Delta} u_0 \|_{L^{p+1}(\mathbb{R}^d)} = 0$

<u>Pf</u> If $u_0 \in H^1(\mathbb{R}^d) \cap L^{\frac{p+1}{p}}(\mathbb{R}^d)$

$$\| e^{it\Delta} u_0 \|_{L^{p+1}} \leq C \quad t^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} \quad \|u_0\|_{L^{\frac{p+1}{p}}} \xrightarrow{t\to+\infty} 0$$

if $u_0 \notin L^{\frac{p+1}{p}}(\mathbb{R}^d) \qquad \forall \varepsilon \quad \exists$

$$v_0 \in H^1(\mathbb{R}^d) \cap L^{\frac{p+1}{p}}(\mathbb{R}^d) \qquad \text{st}$$

$$C_c^\infty(\mathbb{R})$$

$$\| u_0 - v_0 \|_{H^1} < \varepsilon$$

$$\| e^{it\Delta} u_0 \|_{L^{p+1}} \leq \underbrace{\| e^{it\Delta} v_0 \|_{L^{p+1}}}_{\xrightarrow{t\to+\infty} 0} + \underbrace{\| e^{it\Delta}(u_0 - v_0) \|_{L^{p+1}}}_{\leq C \underbrace{\| \cancel{e^{it\Delta}} (u_0 - v_0) \|_{H^1}}_{<\varepsilon}}$$

<u>Claim</u> $\quad w(t,\tau) = -i \displaystyle\int_0^{t-\tau} e^{i(t-s)\Delta} |u(s)|^{p-1} u(s) \, ds$

$$\| w(t,\tau) \|_{L^{p+1}} \leq C \quad \tau^{-\frac{d(p-1) - 2\max(1,p-1)}{2(p+1)}}$$

<u>Remark</u> The exponent is negative

- If $p-1 \leq 1$

$$d(p-1) - 2 > 0 \iff p - 1 > \frac{2}{d}$$

$$p > 1 + \frac{2}{d} \quad \checkmark$$

$$p > 1 + \frac{4}{d}$$

- $p - 1 > 1 \qquad\qquad d \geq 3$

$$d(p-1)-2(p-1)=(d-2)\underbrace{(p-1)}_{\geq 1} \geq p-1>1$$

Pf

$$w(t,\tau) = -i\int_0^{t-\tau} e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds$$

$$q = \begin{cases} +\infty & \text{if } p \geq 2 \\ \frac{2}{2-p} & p<2 \end{cases}$$

$$\| w(t,\tau)\|_{L^q} \leq \int_0^{t-\tau} \| e^{i(t-s)\Delta} |u|^{p-1} u \|_{L^q} ds$$

$$\leq C\int_0^{t-\tau} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{q}\right)} \| u(s)\|^p_{L^{q'p}} ds$$

$$d\left(\frac{1}{2}-\frac{1}{q}\right)>1 \qquad q = \begin{cases} +\infty & \text{if } p \geq 2 \\ \frac{2}{2-p} & p<2 \end{cases}$$

If $q=\infty$ $\quad d\left(\frac{1}{2}-\frac{1}{q}\right)=\frac{d}{2}\geq\frac{3}{2}$ by $d\geq 3$

$$q=\frac{2}{2-p}$$

$$d\left(\frac{1}{2}-\frac{1}{q}\right)= d\left(\frac{1}{2}-\frac{2-p}{2}\right)=\frac{d}{2}(1-2+p)=$$

$$=\frac{d}{2}(p-1)>1$$

$$p-1>\frac{2}{d} \qquad p>1+\frac{2}{d}$$

$$p > 1 + \frac{4}{d}$$

$$\| w(t,\tau)\|_{L^q}$$

$$\leq C \int_0^{t-\tau} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{q}\right)} \| u(s)\|^p_{L^{q'p}} \, ds$$

$$\leq C \sup_{s\geq 0} \| u(s)\|^p_{L^{q'p}} \, \tau^{-d\left(\frac{1}{2}-\frac{1}{q}\right)+1}$$

Claim $\boxed{2 \leq q'p \leq p+1}$ If this is true

then $\sup_{s\geq 0} \| u(s)\|^p_{L^{q'p}} \leq C \sup_{s\geq 0} \| u(s)\|^p_{H^1} \leq C_{u_0}$

$$q = \begin{cases} +\infty & \text{if } p \geq 2 \\ \frac{2}{2-p} & p < 2 \end{cases}$$

$p \geq 2 \Rightarrow q = \infty \Rightarrow q' = 1$ $\qquad q'p = p$

$$2 \leq p < p+1$$

$1 < p < 2$ $\qquad \frac{1}{q'} = 1 - \frac{1}{q} = 1 - \frac{2-p}{2} = \frac{p}{2}$

$$q' p = \frac{2}{p} p = 2$$

$$|w(t,\tau)|_{L^q} \le C \quad \tau^{-d\left(\frac{1}{2}-\frac{1}{q}\right)+1}$$

$$\boxed{2 < p+1 \le q} \qquad q = \begin{cases} +\infty & \text{if } p \ge 2 \\ \dfrac{2}{2-p} & p < 2 \end{cases}$$

If $q = +\infty$ obvious.

$p < 2$

$$p+1 < \frac{2}{2-p}$$

$$(2-p)(p+1) < 2$$

$$2 > 2 + p - p^2$$

$$0 > p - p^2 = p(1-p) \quad ✓$$

$$2 < p+1 \le q$$

$$\frac{1}{p+1} = \frac{1-\alpha}{2} + \frac{\alpha}{q} \qquad \alpha \in (0,1)$$

$$|w(t,\tau)|_{L^{p+1}} \le |w(t,\tau)|_{L^2}^{1-\alpha} \; |w(t,\tau)|_{L^q}^{\alpha}$$

$$\alpha = \frac{\frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{q}}$$

$$\frac{1}{p+1} - \frac{1}{2} = \left(\frac{1}{q} - \frac{1}{2}\right)\alpha$$

$$W(t,\tau) = -i\int_0^{t-\tau} e^{i(t-s)\Delta}\,|u(s)|^{p-1}u(s)\,ds$$

$$= -i\,e^{i\tau\Delta}\int_0^{t-\tau} e^{i(t-\tau-s)\Delta}\,|u(s)|^{p-1}u(s)\,ds$$

$$|W(t,\tau)|_{L^2} = \left| -i\int_0^{t-\tau} e^{i(t-\tau-s)\Delta}\,|u(s)|^{p-1}u(s)\,ds\right|_{L^2}$$

$$u(t-\tau) = e^{i(t-\tau)\Delta}u_0 - i\int_0^{t-\tau} e^{i(t-\tau-s)\Delta}\,|u(s)|^{p-1}u(s)$$

$$= \left| u(t-\tau) - e^{i(t-\tau)\Delta}u_0\right|_{L^2} =$$

$$\leq |u(t-\tau)|_{L^2} + |e^{i(t-\tau)\Delta}u_0|_{L^2} = 2\,|u_0|_{L^2}$$