Exercise set 1

AI

1 Fluctuating field

A fluctuating membranes is fixed on a square frame of side L and area $A_p = L^2$. Let $h(x, y)$ be its height over the reference (x, y) plane, in the limit of small fluctuations the system effective membranes reads

$$H = \sigma A_p + \frac{1}{2} \int_{A_p} \mathrm{d}x\mathrm{d}y \, \left[\sigma(\nabla h)^2 + \kappa(\nabla^2 h)^2\right], \tag{1}$$

where the surface tension σ is the energy cost for increasing the membrane total area, and the bending rigidity κ describes the resistance of the system to bending.

Introduce the noise term $\eta(x, y, t)$ with the usual fluctuation dissipation relation and temperature T and write the generalized Langevin equation for the field $h(x, y)$, then find the fluctuation spectrum defined as $\langle |h_{\mathbf{q}}|^2 \rangle$, where

$$h_{\mathbf{q}} = \int_{A_p} \mathrm{d}x\mathrm{d}y \exp\left[\mathbb{i}\mathbf{q} \cdot \mathbf{r}\right] h(x, y). \tag{2}$$

Was there a simpler way to obtain the fluctuation spectrum?

2 Stochastic heat statistics

Consider a 1D Brownian particle in a potential $U(x) = kx^2/2$ in the overdamped regime in a fluid at temperature T.

i) Consider the stochastic heat Q exchanged with the bath and evaluate its PDF $P(Q)$ in the long time limit.

ii) Obtain the same result of point i) by first calculating the statistics of Q for finite time and then take the long time limit. Hint: treat Q as a second stochastic variable and consider the joint PDF $\phi(x, Q, t)$. Write the corresponding Fokker–Planck equation and consider what you did in exercise 2, set I.

3 First exit time

Calculate the first exit time $\tau(x, y)$ of a 2D overdamped Brownian particle from a rectangular region of sides L_x and L_y.

Hint: Consider first the homogeneous differential equation associated to the differential equation for $\tau(x, y)$ and then impose the absorbing boundary conditions.

4 Kuramoto model

Find the equation for the curve separating the two phases in the s/K and T/K plane for the bimodal and the Gaussian distribution of the forces.