Geophysical Fluid Dynamics

Vorticity Dynamics

Vorticity

- A vortex line is a curve in the fluid such that its tangent at any point gives the direction of the local vorticity.
- A vortex line is related to the vorticity the same way a streamline is related to the velocity vector.
- A vortex line satisfies the equations:

$$\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z} \qquad (1)$$

Vorticity

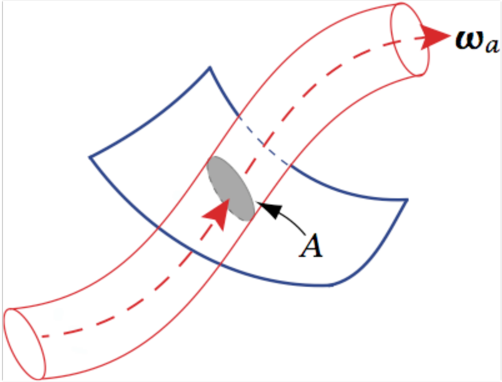
- A group of vortex lines bound a vortex tube.
- The circulation around a narrow vortex tube is

$$d\Gamma = \boldsymbol{\omega} \cdot dA \tag{2}$$

- The strength of a vortex tube is defined as the circulation around a closed circuit taken on the surface of the tube.

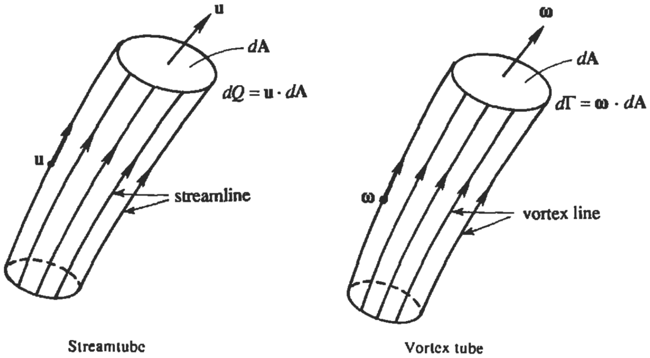
$$\Gamma = \oint_c u_i dx_i \tag{3}$$

Vorticity



A vortex tube passing through a material sheet. The circulation is the integral of the velocity around the boundary of A, and is equal to the integral of the normal component of vorticity over A.

Vorticity


u
dA
$dQ = \mathbf{u} \cdot d\mathbf{A}$
u
streamline
Streamtube
ω
dA
$d\Gamma = \boldsymbol{\omega} \cdot d\mathbf{A}$
ω
vortex line
Vortex tube

Vorticity

- IRROTATIONALITY does not imply the absence of viscous stresses.
- In fact, viscous stresses must always exist in irrotational flows for real fluids, because the fluid elements deform in such a flow.
- An irrotational fluid is define as being curl-free; hence, $\omega = 0$ and therefore $\Gamma_C = 0$ for any C.

Vorticity

- If the flow is irrotational then the net viscous forces vanish on the element.
- The only example of vorticity and no viscous stresses is that of solid-body rotation.
- In solid-body rotation the fluid elements do not deform.
- Viscous stresses are proportional to the deformation rate, and hence they are zero for this flow.

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0 \tag{4}$$

Kelvin's Circulation Theorem

In an inviscid, barotropic flow with conservative body forces, the circulation around a closed curve C moving with the fluid remains constant in time (if the motion is observed from a nonrotating frame).

$$\frac{D\Gamma}{Dt} = 0 \tag{5}$$

Assumptions

- inviscid flow: $\mu = 0$
- conservative body forces: $f_i = \nabla(-gz)$
- Barotropic flow, Density is a function of pressure only: $\rho = \rho(p)$

Reversible forms of compressibility are OK (pressure) but mixing is irreversible and therefore $\rho \neq \rho(T, S)$.

Kelvin's Circulation Theorem

Integrating around a closed contour:

$$\frac{D\Gamma}{Dt} = \oint_c [-dP + dg + 1/2d(u_i u_i)] = 0 \tag{6}$$

- P and g are single valued since they are reversible forms of work
- $u_i u_i$ is single valued since u_i is continuous

Kelvin's Circulation Theorem

Circulation is the surface integral of vorticity. Integrating around a closed contour:

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_A \omega_i dA = 0 \tag{7}$$

or $\int_A \omega_i dA$ is constant as we follow the flow.
If the flow is irrotational $\omega_i = 0$ the flow will remain irrotational under the four assumptions:

- Inviscid flow
- Conservative body forces
- Barotropic flow
- Nonrotaing frame

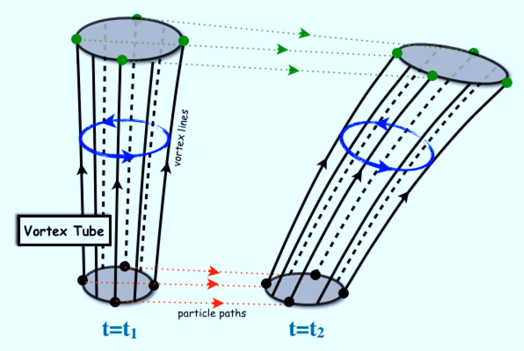
Kelvin's Circulation Theorem

- **Inviscid flow:** Circulation is preserved if there are no net viscous forces along the path followed by C. If C moves into viscous regions such as boundary layers along solid surface, then circulation changes. Viscous effects cause diffusion of vorticity in or out of the fluid circuit, thereby changing the circulation.
- **Conservative body forces:** gravity acts through the centre of mass of a fluid particle and therefore does not rotate it.

Kelvin's Circulation Theorem

- **Barotropic flow:** in a baroclinic flow, lines of constant p and ρ are not parallel, and the net pressure force does not pass through the centre of mass creating a torque which changes the vorticity and circulation. Geophysical flows, which are dominated by baroclinicity, are full of vorticity. Examples: heating from below creates a buoyant force generating a plume ($\rho = \rho(T)$), cooling from above will generate rolls and vorticity.
- **Nonrotating frame:** motions observed with respect to a rotating frame of reference can developed vorticity through Coriolis (shown later).

Vorticity



Due to shear in the velocity field, the vortex tube is stretched and tilted. As long as the fluid is inviscid and barotropic, incompressible, Kelvin's circularity theorem assures that the circularity is conserved with time. Since vorticity is divergent-free, the circularity along different cross sections is the same.

Vorticity Equation in a Nonrotating frame

- The flow is barotropic
- we retain viscous effects
- baroclinicity and the effect of a rotating frame of reference will be dealt in the next derivation.

Vorticity is $\omega = \nabla x \mathbf{u}$ and its curl is zero $\nabla \cdot \omega = 0$. The rate of change of vorticity is:

$$\boxed{\frac{D}{Dt}\omega = (\omega \cdot \nabla)\mathbf{u} + \nu \nabla^2 \omega} \tag{8}$$

where the first term on the r.h.s. is the rate of change of vorticity due to stretching and tilting of vortex lines, and the second term is the rate of change of vorticity due to diffusion of vorticity.

Vorticity Equation in a Rotating frame

- The flow is nonbarotropic
- We use a rotating frame of reference.
- We still approximate to a nearly incompressible Boussinesq fluid so that the contonuity equation is $\nabla \cdot \mathbf{u} = o$
- Continuity is $u_{i,i} = 0$
- The momentum equation is

$$\frac{\partial u_i}{\partial t} + u_j u_{i,j} + 2\varepsilon_{ijk}\Omega_j u_k = -(1/\rho)p_{,i} + g_i + \nu u_{i,jj} \qquad (9)$$

- after some manipulation we get to

$$\boxed{\frac{\partial u_i}{\partial t} + (1/2u_j^2 + \Pi)_{,i} - \varepsilon_{ijk} u_j(\omega_k + 2\Omega_k) = -(1/\rho)p_{,i} - \nu\varepsilon_{ijk}\omega_{k,j}} \qquad (10)$$

This is a form of the N-S equation, and we now take its curl.

Vorticity Equation in a Rotating frame

$$\boxed{\frac{D}{Dt}\omega = (\omega + 2\Omega)\cdot\nabla\mathbf{u} + (1/\rho^2)\nabla\rho\wedge\nabla p + \nu\nabla^2\omega} \tag{11}$$

This is the Vorticity equation for a nearly incompressible fluid (Boussinesq) in rotating coordinates.

1. 1st term is the rate of change of relative vorticity following a fluid particle
2. 2nd term is Absolute Vorticity
3. 3rd term is the rate of change due to baroclinicity of the flow
4. 4th term is the rate of change due to diffusion

$(\omega + 2\Omega)\cdot\nabla\mathbf{u}$ is a crucial term in the vorticity dynamics.

Vorticity Equation in a Rotating frame

$$\boxed{(\omega \cdot \nabla)\mathbf{u} = \omega \frac{\partial \mathbf{u}}{\partial s}} \tag{12}$$

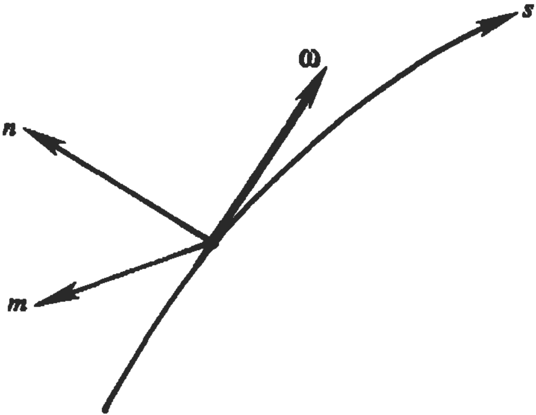
$=$ the magnitude of ω times the derivative of $\mathbf{u}$ in the direction of ω

$$\boxed{(\omega \cdot \nabla)\mathbf{u} = \omega \frac{\partial u_s}{\partial s} + \omega \frac{\partial u_n}{\partial s} + \omega \frac{\partial u_m}{\partial s}} \tag{13}$$

.

The first is the increase of u_s along the vortex line s (stretching of vortex line). The second and third represent the change of normal velocity components along s: rate of turning and tilting of vortex lines about the m and n axes.

Vorticity

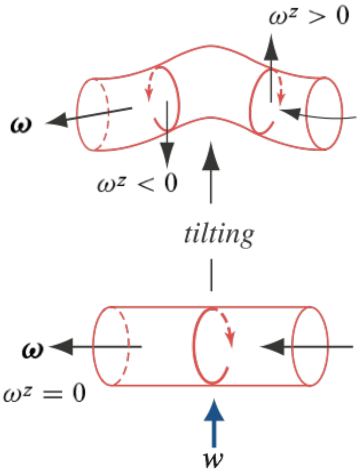


Vorticity Equation in a Rotating frame

$$\boxed{2(\Omega \cdot \nabla)\mathbf{u} = 2\Omega\left(\frac{\partial \mathbf{u}}{\partial z}\right)} \tag{14}$$

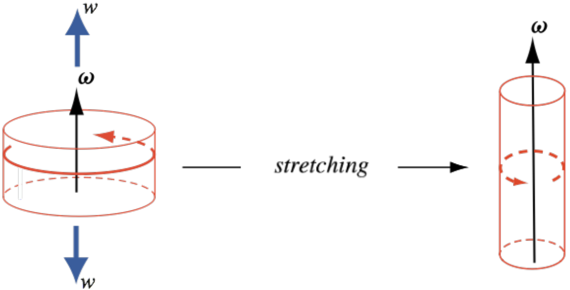
The third term says that stretching of fluid lines in the z direction increases ω_z.
The first and second say that turning and tilting of fluid lines increase the relative vorticity along x and y. Here only fluid lines have to tilt/turn. This is because vertical fluid lines contain planetary vorticity 2Ω.

The tilting of Vorticity



Suppose that the vorticity ω is initially directed horizontally, so that ω^z is zero. The vertical material lines and also the vortex lines are tilted by the positive vertical velocity, creating a vertically oriented vorticity.

The tilting of Vorticity



If the fluid is incompressible, circularity is conserved. Vorticity is tied to material lines, and so amplified in the direction of stretching. Because the volume of fluid is conserved, the end surfaces shrink, the material lines through the cylinder ends converge and the integral of vorticity over a material surface (the circulation) remains constant.

Equations governing Geophysical flows

1. Importance of Rotation
2. Importance of stratification
3. Momentum Eqs.
4. Energy Eq.
5. Equation of State
6. Boussinesq Approximation
7. GFD equations

Importance of rotation

- Geophysical Fluid Dynamics is different from traditional fluid mechanics because a geophysical flow is affected by **rotation** of the Earth and by **stratification**

Importance of rotation

Rotation of the Earth is

$$\Omega = \frac{2\pi \text{radians}}{\text{time of revolution}} = \frac{2\pi}{24hr} = 7.29 \times 10^{-5} s^{-1}. \tag{15}$$

If the fluid motion evolves on a time scale comparable to or longer than Ω then the fluid will 'feel' the effect of rotation:

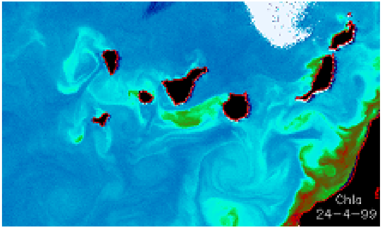
$$\varepsilon = \frac{\text{time of revolution}}{\text{motion time scale}} = \frac{2\pi/\Omega}{T}. \tag{16}$$

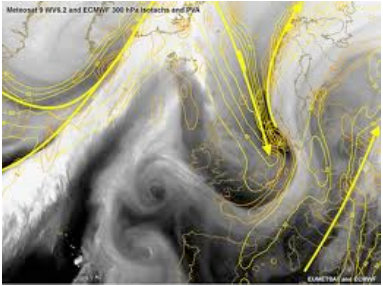
Or better:

$$\boxed{\varepsilon = \frac{2\pi/\Omega}{L/U}.} \tag{17}$$

If $\varepsilon \leq 1 \rightarrow$ rotation is important.

Importance of rotation

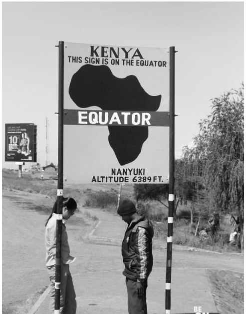




An ocean current meandering over $10km$ with $v = 10cm/s$
$\rightarrow \varepsilon \ll 1$
A wind blowing over a $1000km$ anticyclone with $v = 10m/s$
$\rightarrow \varepsilon \ll 1$

Coriolis effect at very short time/space scales? ...

What about your bathroom sink/toilet? [just conservation of energy/angular momentum]



An equatorial entrepreneur (Nanyuki, Kenya)

Can you really see the Coriolis effect on the spiral of water? WHY?

Because time scale is much shorter than 1 day, and the pressure gradient force is balanced by acceleration, not Coriolis!

Importance of stratification

- Geophysical Fluid Dynamics is different from traditional fluid mechanics because a geophysical flow is affected by rotation of the Earth and by **stratification**

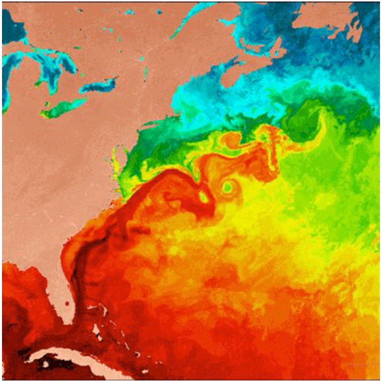

Importance of stratification

Geophysical fluids tend to arrange themselves towards a minimum of Potential Energy under gravity. But the fluid is moving ...

$$\boxed{\sigma = \frac{\rho_o U^2}{2\delta\rho g H}} \tag{18}$$

1. If $\sigma \sim 1 \rightarrow$ a significant perturbation to the stratification can consume a major part of the available kinetic energy, thereby modifying the flow field significantly.
2. If $\sigma \ll 1 \rightarrow$ there is insufficient kinetic energy to perturb the stratification in any significant way, and the latter greatly constrains the flow.
3. If $\sigma \gg 1 \rightarrow$ potential-energy variations created by vertical excursions of the fluid against their buoyancy forces cause a negligible drop in kinetic energy, and the stratification is easily erased by vertical mixing.

Rotation and Stratification



Natural length and velocity scales:
In the Ocean: $L \sim 60km$ and $U \sim 4m/s$
In the Atmosphere: $L \sim 500km$ and $U \sim 30m/s$
This has large implications for the numerical resolution of the flow, grid spacing, parameterizations of eddy fluxes, etc...

Boussinesq Approximation: Continuity

In Geophysical Fluids ρ' is very small. We thus write

$$\rho = \rho_o + \rho'(x, y, z, t)$$

Boussinesq Approximation: Continuity

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$$\rho = \rho_o + \rho'(x, y, z, t)$$

Continuity Equation is now

$$\boxed{\nabla \cdot \mathbf{u} = 0}. \tag{19}$$

This does not mean that density ρ is regarded as constant along the direction of motion, but simply that the magnitude $1/\rho \frac{D\rho}{Dt}$ is small in comparison to the magnitudes of the velocity gradients in $\nabla \cdot \mathbf{u}$.

Boussinesq Approximation: Thermal Energy Equation

$$\rho \frac{De}{Dt} = -\nabla \cdot \mathbf{q} - p(\nabla \cdot \mathbf{u}) + \phi$$

Neglecting viscous dissipation of energy ϕ and assuming Fourier's law of heat conduction:

Boussinesq Approximation: Thermal Energy Equation

$$\rho \frac{De}{Dt} = -\nabla \cdot \mathbf{q} - p(\nabla \cdot \mathbf{u}) + \phi$$

Neglecting viscous dissipation of energy ϕ and assuming Fourier's law of heat conduction:

$$\rho C_v \frac{DT}{Dt} = k \nabla^2 T$$

or

$$\boxed{\frac{DT}{Dt} = \kappa_T \nabla^2 T}$$

Where κ_T is the thermal diffusivity.
We have now introduced temperature T.

Equation of state

for Air: $\rho = \frac{P}{RT}$
for seawater, $\rho = \rho(p, T, S)$, linear EOS:

$$\rho = \rho\left[1 - \alpha(T - T_o) + \beta(S - S_o)\right]$$

so there is also Salinity ... diffusion of salt (isomorphic to Temperature equation)

$$\boxed{\frac{DS}{Dt} = \kappa_s \nabla^2 S}$$

Boussinesq Approximation: Thermal and Salt Eq.

In a turbulent fluid, diffusion is accomplished by eddies which mix HEAT and SALT at the same rate. Not the same for molecular scales. In Geophysical fluids we talk of **eddy diffusivity** for the turbulent diffusion coefficient.

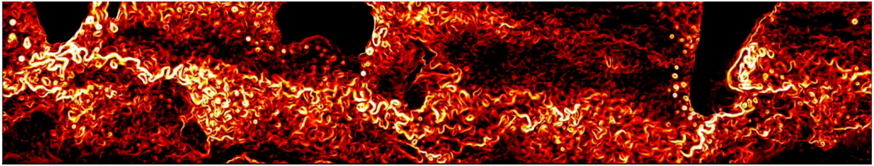
Combining with the EOS we get:

$$\boxed{\frac{D\rho'}{Dt} = \kappa \nabla^2 \rho'}$$

Boussinesq Approximation: Density

$$\boxed{\frac{D\rho'}{Dt} = \kappa \nabla^2 \rho'}$$

The Energy equation has turned into a Density equation.
We went from
7 Eqs (3×Momentum + Continuity + Energy + EOS + Salt) to
5 Eqs (3×Momentum + Continuity + Density)



Diffusivity, mixing and stirring is a fascinating topic of research in GFD.

Boussinesq Approximation: Momentum

and the momentum equations become

$$
\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho_o}\frac{\partial P'}{\partial x} + \nu\nabla^2 u && (20)\\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho_o}\frac{\partial P'}{\partial y} + \nu\nabla^2 v && (21)\\
\frac{Dw}{Dt} &= -\frac{1}{\rho_o}\frac{\partial P'}{\partial z} + \nu\nabla^2 w - \frac{\rho'}{\rho_o}g && (22)
\end{aligned}
$$

density variations are negligible in the momentum equations, except when ρ is multiplied by g.

Momentum Eqs. + Conservation of Mass

$$
\begin{aligned}
\frac{D\mathbf{u}}{Dt} + 2\Omega \wedge \mathbf{u} &= -\frac{1}{\rho}\nabla P - \frac{g\rho}{\rho_0}\mathbf{k} + \mathbf{F} && (23)\\
\nabla \cdot \mathbf{u} &= 0 && (24)\\
& && (25)
\end{aligned}
$$

+ Energy Eq. (+ Salt Eq. in the Ocean) + EOS
(6 for the Atmosphere; 7 for the Ocean)

More Approximations: Scale Analysis

Beyond the Boussinesq Approx. we can talk of orders of magnitude. So we introduce a *scale* for each variable. Scale: dimensional constant of dimensions identical to that of the variable and having a numerical value representative of the values of that same variable. Remembering that:

$$T \geq \frac{1}{\Omega}$$

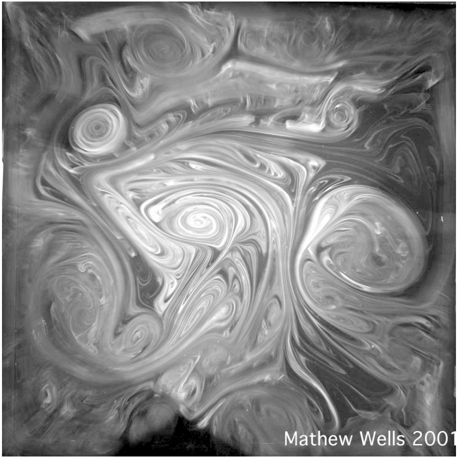
$$\frac{U}{L} \leq \Omega$$

Table: typical scales of geophysical flows

Variable	Scale	Unit	Atmos Value	Oceanic Value
(x,y)	L	m	100 km	10km
z	H	m	1 km	100 m
t	T	s	1/2 day	1 day
(u,v)	U	m/s	10 m/s	0.1 m/s

More Approximations

From the Continuity Eq. we reach the conclusion that $W \ll U$.
Large scale geophysical flows are shallow ($H \ll L$) and almost two-dimensional ($W \ll U$).



Governing Equations for Geophysical flows

$$
\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2} \\
0 &= -\frac{\partial P}{\partial z} - \rho g \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{D\rho}{Dt} &= \kappa \frac{\partial^2 \rho}{\partial z^2}
\end{aligned}
$$

5 Eqs for (u, v, w, p, ρ) that form the basis for GFD.

Modelling the General Circulation and Climate

Equations of motion are converted into a computer code (General Circulation Model; GCM), thereby simulating the climate. Climate models (coupled GCMs) use these set of equations.