May 19 $i\dot{u} = (-\Delta + V)u$

$$i\partial_t u = -\Delta u + |u|^{p-1} u \qquad u\big|_{t=0} = u_0 \in H^1$$

$$d^* > p > 1 + \frac{4}{d} \qquad d \geq 3$$

$$d^* = \frac{d+2}{d-2}$$

$$u \in C^0(\mathbb{R}, H^1)$$

$$u_0 \in H^2(\mathbb{R}^d, \mathbb{C})$$

$$\Rightarrow \quad u \in C^0(\mathbb{R}, H^2)$$

$$|u|_{L^\infty(\mathbb{R}, H^1)} \leq C(u_0)$$

$$|u(t)|_{H^1} = \underbrace{|u(t)|_{L^2}}_{= |u_0|_{L^2}} + |\nabla u(t)|_{L^2}$$

$$E(u(t)) = \frac{|\nabla u(t)|_{L^2}^2}{2} + \frac{|u(t)|_{L^{p+1}}^{p+1}}{p+1} =$$

$$= E(u_0)$$

$$|\nabla u(t)|_{L^2}^2 \leq 2\, E(u_0)$$

$\mathcal{O}_{u_1}$ aim is $\lim_{t \to +\infty} |u(t)|_{L^{p+1}} = 0$

Lemma $\forall \varepsilon > a, b \in \mathbb{R}_+$ $\exists$ $t_0 > \max\{a, b\}$ st

$$\sup_{s \in [t_0 - b, t_0]} |u(s)|_{L^{p+1}} \leq \varepsilon$$

Pf

$$u(t) = e^{it\Delta} u_0 \underbrace{- i \int_0^{t-\tau} e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds}_{w(t,\tau)}$$

$$\underbrace{- i \int_{t-s}^{t} e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds}_{z(t,\tau)}$$

$$q = \begin{cases} +\infty & \text{if } p \geq 2 \\ \frac{2}{2-p} & \text{if } p < 2 \end{cases}$$

$$|w(t,\tau)|_{L^q(\mathbb{R}^d)} \leq C_{u_0} \tau^{\overbrace{-d\left(\frac{1}{2} - \frac{1}{q}\right)}^{<-1} + 1}$$

$$2 < p+1 < q$$

$$\frac{1}{p+1} = \frac{1-\alpha}{2} + \frac{\alpha}{q} \qquad \alpha \in (0,1)$$

$$\alpha = \frac{\frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{q}}$$

$$|w(t,\tau)|_{L^{p+1}} \leq |w(t,\tau)|_{L^2}^{1-\alpha} \; |w(t,\tau)|_{L^{p+1}}^{\alpha}$$

$$\leq \left(2\, |u_0|_{L^2}\right)^{1-\alpha} \; C' \; \tau^{\left(-d\left(\frac{1}{2}-\frac{1}{q}\right)+1\right) \frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}}$$

$$= C_{u_0} \; \tau^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right) + \frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}}$$

There

$$= C_{u_0} \; \tau^{-\frac{d(p-1) - 2\max\{1, p-1\}}{2(p+1)}}$$

$$-d\left(\frac{1}{2}-\frac{1}{p+1}\right) + \frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}$$

$$q = \begin{cases} +\infty & \text{if } p \geq 2 \\ \frac{2}{2-p} & \text{if } p < 2 \end{cases}$$

If $q = \infty$

$$= -d\left(\frac{1}{2}-\frac{1}{p+1}\right)+2\left(\frac{1}{2}-\frac{1}{p+1}\right)$$

$$= -(d-2)\left(\frac{1}{2}-\frac{1}{p+1}\right) =$$

$$= -(d-2)\,\frac{p-1}{2(p+1)} =$$

$$= -\,\frac{d(p-1)-2(p-1)}{2(p+1)} = -\,\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}$$

$$-d\left(\frac{1}{2}-\frac{1}{p+1}\right)+\frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}$$

$$q=\begin{cases} +\infty & \text{if } p\geq 2\\ \frac{2}{2-p} & \text{if } p<2\end{cases}$$

$$= \left(\frac{1}{2}-\frac{1}{p+1}\right)\left[-d+\frac{1}{\frac{1}{2}-\frac{1}{q}}\right] =$$

$$= \left(\frac{1}{2}-\frac{1}{p+1}\right)\left[-d+\frac{1}{\frac{1}{2}-\frac{2-p}{2}}\right] =$$

$$= \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(-d+\frac{2}{p-1}\right) =$$

$$= \frac{p-1}{2(p+1)}\left(-d + \frac{2}{p-1}\right)$$

$$\geq \frac{-d(p-1)+2}{2(p+1)} = \frac{-d(p-1)+2\max\{1,p-1\}}{2(p+1)}$$

We proved

$$\| w(t,\tau)\|_{L^{p+1}} \leq C_{u_0}\, \tau^{\frac{-d(p-1)+2\max\{1,p-1\}}{2(p+1)}}$$

$$z(t,\tau) = -i\int_{t-\tau}^{t} e^{i(t-s)\Delta}\, |u(s)|^{p-1} u(s)\, ds$$

$$\| z(t,\tau)\|_{L^{p+1}} \leq \int_{t-\tau}^{t} \| e^{i(t-s)\Delta} |u(s)|^{p-1}u(s)\|_{L^{p+1}}\, ds$$

$$\leq C_d \int_{t-\tau}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} \| |u(s)|^{p}\|_{L^{\frac{p+1}{p}}}\, ds$$

Ginibre–Velo

$$= C_d \int_{t-\tau}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} \| u(s)\|^{p}_{L^{p+1}}\, ds$$

Morawetz–Strauss

$$p < d^{*} = \frac{d+2}{d-2}$$

$$p+1 < d^* + 1 = \frac{2d}{d-2}$$

$$d\left(\frac{1}{2} - \frac{1}{p+1}\right) < 1 \quad \checkmark$$

$$\frac{1}{p+1} > \frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d}$$

$$\frac{1}{d} > \frac{1}{2} - \frac{1}{p+1}$$

$$= \frac{p-1}{2(p+1)}$$

$$q \in \left(1, \frac{2(p+1)}{d(p-1)}\right)$$

$$\frac{2(p+1)}{d(p-1)} > 1$$

$$\Updownarrow$$

$$\frac{1}{d} > \frac{p-1}{2(p+1)}$$

$$2(p+1) > d(p-1)$$

$$2p+2 > d$$

$$q\, d\left(\frac{1}{2} - \frac{1}{p+1}\right) < 1$$

$$q\, \frac{d(p-1)}{2(p+1)} < 1$$

$$|z(t,\tau)|_{L^{p+1}}$$

$$\leq C_d \int_{t-\tau}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} |u(s)|^{p}_{L^{p+1}} \, ds$$

$$\frac{1}{q}+\frac{1}{q'}=1$$

$$\leq C_d \left(\int_{t-\tau}^{t} ((t-s))^{-qd\left(\frac{1}{2}-\frac{1}{p+1}\right)} ds \right)^{\frac{1}{q}}$$

$$\left(\int_{t-\tau}^{t} |u|^{pq'}_{L^{p+1}} \, ds \right)^{\frac{1}{q'}}$$

$$\leq C \quad \tau^{\alpha} \left(\int_{t-\tau}^{t} |u|^{pq'}_{L^{p+1}} \, ds \right)^{\frac{1}{q'}}$$ for some $\alpha > 0$

$$pq' > p+1 \qquad \frac{1}{q'} < \frac{p}{p+1}$$

$$qd\left(\frac{1}{2}-\frac{1}{p+1}\right) < 1$$

$$\frac{1}{q} > \frac{d}{2} - \frac{d}{p+1}$$

$$\frac{1}{q'} = 1 - \frac{1}{q} < 1 - \frac{d}{2} + \frac{d}{p+1} = \frac{2-d}{2} + \frac{d}{p+1} =$$

$$= \frac{2(p+1) - (p+1)d + 2d}{2(p+1)} =$$

$$= \frac{p}{p+1} + \frac{2-(p+1)d+2d}{2(p+1)} < \frac{p}{p+1}$$

$$2-(p+1)d+2d = 2-pd+d < 0$$

$$2+d < pd$$

$$\frac{2}{d}+1 < p \quad \checkmark$$

~~be~~cause $\quad p > 1 + \frac{4}{d}$

$$\left(1+\frac{2}{d} < p \leq 1+\frac{4}{d} \right)$$

$$p q' > p+1 \qquad \frac{1}{q'} < \frac{p}{p+1} \quad \text{i} \text{ punctul}$$

$$2 < p+1 < d^* + 1 \qquad \frac{1}{p+1} = \frac{\beta}{2} + \frac{1-\beta}{d^*+1}$$

$$|u|_{L^{p+1}}^{pq'} = |u|_{L^{p+1}}^{p+1} \; |u|_{L^{p+1}}^{pq'-p-1} \leq$$

$$\leq |u|_{L^{p+1}}^{p+1} \quad |u|_{L^2}^{(pq'-p-1)\alpha} \quad |u|_{L^{d^*+1}}^{(pq'-p-1)(1-\beta)}$$

$$H^1 \hookrightarrow L^{d^*+1}$$

$$= |u|_{L^{p+1}}^{p+1} \quad |u_0|_{L^2}^{(pq'-p-1)\alpha} \quad |u|_{H^1}^{(pq'-p-1)(1-\beta)}$$

$$\leq C_{u_0} \, |u|_{L^{p+1}}^{p+1}$$

$$|z(t,\tau)|_{L^{p+1}} \leq C \quad \tau^{\alpha} \left(\int_{t-\tau}^{t} |u|_{L^{p+1}}^{pq'} \, ds \right)^{\frac{1}{q'}}$$

$$\leq C_{u_0} \tau^{\alpha} \left(\int_{t-\tau}^{t} |u|_{L^{p+1}}^{p+1} ds \right)^{\frac{1}{q'}}$$

$$= C_{u_0}^{\tau^{\alpha}} \left[\left(\int_{t-\tau}^{t} \int_{|x| \geq s \lg s} |u(s)|^{p+1} dx \, ds \right)^{\frac{1}{q'}} + \left(\int_{t-\tau}^{t} \int_{|x| \leq s \lg s} |u(s)|^{p+1} dx \, ds \right)^{\frac{1}{q'}} \right]$$

$$\leq C_{u_0} \tau^{\alpha + \frac{1}{q'}} \left(\sup_{s \in [t-\tau, t]} \int_{|x| \geq s \lg s} |u(s)|^{p+1} dx \right)^{\frac{1}{q'}}$$

$$+ C_0 \tau^{\alpha} \left(\int_{t-\tau}^{t} \int_{|x| \leq s \lg s} |u(s)|^{p+1} dx \, ds \right)^{\frac{1}{q'}}$$

$$|u(t)|_{L^{p+1}} \leq |e^{it\Delta} u_0|_{L^{p+1}} + |w(t,\tau)|_{L^{p+1}} + |z(t,\tau)|_{L^{p+1}}$$

$$\leq C \tau^{-\frac{d(p-1) - 2\max\{1, p-1\}}{2(p+1)}}$$

$$+ |e^{it\Delta} u_0|_{L^{p+1}} + C_{u_0} \tau^{\alpha + \frac{1}{q'}} \left(\sup_{s \geq t-\tau} \int_{|x| \geq s \lg s} |u(s)|^{p+1} dx \right)^{\frac{1}{q'}}$$

$$+ C_0 \tau^{\alpha} \left(\int_{t-\tau}^{t} \int_{|x| \leq s \lg s} |u(s)|^{p+1} dx \, ds \right)^{\frac{1}{q'}}$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4}$$

$\tau > b$ $\qquad$ $t \geq t_1 > \max\{a, b\}$

We use lemma 5.7

$\forall \varepsilon > 0 \quad t > 2, \quad \tau > 0 \quad \exists \quad t_0 > \max\{t, 2\tau\}$

s.t. $\displaystyle\int_{t_0 - 2\tau}^{t_0} \int_{|x| \leq s \log s} |u|^{p+1} \, ds < \delta$

$t_0 > t_1$ s.t.

s.t. for $t = t_0$

$$c_0 \tau^{\alpha} \left(\int_{t-\tau}^{t} \int_{|x| \leq s \log s} |u|^{p+1} \, dx \, ds \right)^{\frac{1}{q'}}$$

$$\leq c_0 \tau^{\alpha} (\delta)^{\frac{1}{q'}} < \frac{\varepsilon}{4}$$

Conclusion: we have proved that

$$C_0 \tau^\alpha \left(\int_{t-\tau}^{t} \int_{|x|\le s \lg s} |u(s)|^{p+1} \, dx \quad ds \right)^{\frac{1}{q'}}$$

$$t_2 \geq t_1 + 2\tau$$

$$t \in [t_2 - \tau, t_2]$$

$$t - \tau \geq t_2 - \tau - \tau = t_2 - 2\tau$$

$$\leq C_0 \tau^\alpha \left(\int_{t_2 - 2\tau}^{t_2} \int_{|x|\le s \lg s} |u(s)|^{p+1} \, dx \quad ds \right)^{\frac{1}{q'}}$$

$$\leq \frac{\varepsilon}{4}$$

$$t_0 = t_2 \qquad t_0 > \max\{a, b\}$$

$$\sup_{t \in [t_0 - b, t_0]} |u(t)|_{L^{p+1}} < \frac{3}{4} \varepsilon$$

We prove $\lim_{t \to +\infty} |u(t)|_{L^{p+1}} = 0$

$$\varepsilon > 0$$

$$t > \tau > 0$$

$$u(t) = e^{it\Delta} u_0 + W(t, \tau) + Z(t, \tau)$$

$$|u(t)|_{L^{p+1}} \le |e^{it\Delta}u_0|_{L^{p+1}} + C\,\tau_\varepsilon^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}}$$

$$+ |z(t,\tau_\varepsilon)|_{L^{p+1}} < \frac{\varepsilon}{2} + |z(t,\tau_\varepsilon)|_{L^{p+1}}$$

$$|e^{it\Delta}u_0|_{L^{p+1}} < \frac{\varepsilon}{4}$$

$$C\,\tau_\varepsilon^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}} = \frac{\varepsilon}{4}$$

$$|z(t,\tau_\varepsilon)|_{L^{p+1}} \le \int_{t-\tau_\varepsilon}^{t} \left| e^{i(t-s)\Delta} |u(s)|^{p-1}u(s)\right|_{L^{p+1}} ds$$

$$\lesssim \int_{t-\tau_\varepsilon}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} |u(s)|_{L^{p+1}}^{p}\, ds$$

$$\le C_d \int_{t-\tau_\varepsilon}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} ds \sup_{s\in[t-\tau_\varepsilon,t]} |u(s)|_{L^{p+1}}^{p}$$

$$= C_d\, \tau_\varepsilon^{1-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} \sup_{s\in[t-\tau_\varepsilon,t]} |u(s)|_{L^{p+1}}^{p}$$

$\exists\ t_0$ arbitrarily large s.t.

$$\sup_{s\in[t_0-\tau_\varepsilon,\, t_0]} |u(s)|_{L^{p+1}} < \frac{\varepsilon}{4}$$

$$t_\varepsilon = \sup\Big\{ t \geq t_0 : \sup_{[\tilde{t}-\tau_\varepsilon,\, \tilde{t}]} |u(s)|_{L^p} \leq \varepsilon \quad \forall \tilde{t} \in [t_0, t] \Big\}$$

If $\quad t_\varepsilon = +\infty \quad \forall \varepsilon > 0$

$$\Longrightarrow \quad t \geq t_0 \quad |u(t)|_{L^p} \leq \varepsilon \Longrightarrow \lim_{t\to+\infty} |u(t)|_{L^{p+1}} = 0$$

Next time we look at $t_\varepsilon < +\infty$ for some $\varepsilon > 0$.