Gauss and Ricci Equations in Riemannian Geometry

Introduction

- Gauss and Ricci equations describe how a submanifold is curved inside a Riemannian manifold.
- They connect intrinsic and extrinsic geometry:
  - **Intrinsic:** Curvature within the submanifold.
  - **Extrinsic:** How the submanifold bends in the ambient space.

Gauss Equation

Let:

- $M \subset \tilde{M}$: Submanifold of Riemannian manifold.
- $R, \tilde{R}$: Riemann curvature tensors of M, $\tilde{M}$.
- B: bilinear and symmetric econd fundamental form.

Gauss Equation:

$$\begin{aligned} \langle \tilde{R}(X,Y)Z,W\rangle - \langle R(X,Y)Z,W\rangle &= \langle B(X,W),B(Y,Z)\rangle + \\ &- \langle B(X,Z),B(Y,W)\rangle \end{aligned}$$

Ricci Equation

Let:

- ξ, η: Normal vector fields.
- S_η: Weingarten map.

Ricci Equation:

$$\langle \tilde{R}(X, Y)\eta, \xi \rangle - R^{\perp}(X, Y)\eta, \xi \rangle = \langle [S_\eta, S_\xi]X, Y \rangle$$

where

$$R^{\perp}(X, Y)\eta = \nabla^{\perp}_{Y}\nabla^{\perp}_{X}\eta - \nabla^{\perp}_{X}\nabla^{\perp}_{Y}\eta + \nabla^{\perp}_{[X,Y]}\eta$$

and

$$\nabla^{\perp}_{X}\eta = \tilde{\nabla}_{X}\eta + S_\eta(X)$$

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$$\nabla^\perp_X \eta = \tilde{\nabla}_X \eta + S_\eta(X)$$

This equation describes the curvature of the normal bundle in terms of the Weingarten maps.

Codazzi Equation

Codazzi Equation:

$$(\tilde{\nabla}_Y B)(X, Z, \eta) - (\tilde{\nabla}_X B)(Y, Z, \eta) = \langle \tilde{R}(X, Y)Z, \eta \rangle$$

- Describes how the second fundamental form changes.
- Involves the ambient curvature and the covariant derivative of B.

Summary Table

Equation	**Relates**	**Purpose**
Gauss	$R, \tilde{R}, B$	Intrinsic curvature via extrinsic data
Ricci	$\tilde{R}, S_\eta$	Normal curvature and shape operators
Codazzi	$\nabla B, \tilde{R}$	Derivatives of the second fundamental form