

FUNZIONI GENERATRICI di transf. canoniche

Consideriamo la transf.

$$\begin{cases} p = \sqrt{2\tilde{p}} \cos \tilde{q} \\ q = \sqrt{2\tilde{p}} \sin \tilde{q} \end{cases}$$

Vediamo che possiamo invertire la prima eq. in ricavare $\tilde{p}$ in funzione di p . Otteniamo

$$\tilde{p} = \frac{p^2}{2 \cos^2 \tilde{q}}$$

Se poi sostituiamo nella seconda eq. abbiamo

$$q = p \tan \tilde{q}$$

Quindi un modo equivalente di dare la transf. è fornire le relazioni

$$\begin{cases} \tilde{p} = \frac{p^2}{2 \cos^2 \tilde{q}} \\ q = p \tan \tilde{q} \end{cases}$$

$$\text{cioè} \quad \begin{cases} \tilde{p} = \tilde{p}(p, \tilde{q}) \\ \tilde{q} = \tilde{q}(p, \tilde{q}) \end{cases}$$

In questo modo posso usare $(p, \tilde{q})$ come coord. indipendenti sulla spina delle fasi (anche se non sono coord. canoniche, cioè le eq. del moto per i moti $(p(t), \tilde{q}(t))$ espressi in qte coord. non sono nella forma delle eq. di Hamilton).

Qta cose ho generalizzato a un caso generico e scegliendo come coord. indep. tra le diverse coppie $(q, \tilde{q})$ $(\tilde{p}, q)$ $(p, \tilde{q})$ $(p, \tilde{p})$!

Facciamo la scelta $(\tilde{p}, q)$ come coordinate indipendenti e diamo la transf. di coord. nella forma

$$\begin{cases} p_h = p_h(\tilde{p}, q, t) \\ \tilde{q}_k = \tilde{\mu}_k(\tilde{p}, q, t) \end{cases} \rightarrow \begin{cases} p_h = u_h(\tilde{p}, \tilde{q}, t) \equiv p_h(\tilde{p}, v(\tilde{p}, \tilde{q}, t), t) \\ q_k = v_k(\tilde{p}, \tilde{q}, t) \end{cases}$$

$\rightsquigarrow$
inversione

Possiamo pensare di definire le funzioni p_h e $\tilde{\mu}_k$ a partire da una funzione $F_2(\tilde{p}, q, t)$ come

$$p_h(\tilde{p}, q, t) \equiv \frac{\partial F_2(\tilde{p}, q, t)}{\partial q_h}$$

$$\tilde{\mu}_k(\tilde{p}, q, t) \equiv \frac{\partial F_2(\tilde{p}, q, t)}{\partial \tilde{p}_k}$$

Per invertire $\tilde{q} = \frac{\partial F_2(\tilde{p}, q, t)}{\partial \tilde{p}}$ nelle q , bisogna che $\det\left(\frac{\partial^2 F_2}{\partial \tilde{p}_h \partial q_k}\right) \neq 0$.

Prop. Quando una transf. di coord. sullo sp. delle fasi è definita in maniera implicita da

$$(b) \begin{cases} p_h = \frac{\partial F_2(\tilde{p}, q, t)}{\partial q_h} \\ \tilde{q}_k = \frac{\partial F_2(\tilde{p}, q, t)}{\partial \tilde{p}_k} \end{cases} \quad \text{con } \det\left(\frac{\partial^2 F_2}{\partial \tilde{p} \partial q}\right) \neq 0$$

allora tale transf. è CANONICA con $K = \tilde{H} + \frac{\partial F_2}{\partial t}$.

La funzione $F_2(\tilde{p}, q, t)$ è detta **FUNZIONE GENERATRICE** delle transf. canoniche.

Dim. Riscriviamo (o) in termini delle funz. $u(\tilde{p}, \tilde{q}, t)$ e $v(\tilde{p}, \tilde{q}, t)$ che def. la transf. canonica $p = u(\tilde{p}, \tilde{q}, t)$ e $q = v(\tilde{p}, \tilde{q}, t)$

$$(o)' \begin{cases} u_n(\tilde{p}, \tilde{q}, t) = \frac{\partial F_2}{\partial q_n}(\tilde{p}, v(\tilde{p}, \tilde{q}, t), t) \\ \tilde{q}_n = \frac{\partial F_2}{\partial \tilde{p}_n}(\tilde{p}, v(\tilde{p}, \tilde{q}, t), t) \end{cases}$$

Affinché sia canonica, dobbiamo dimostrare che le par. di Poisson fondam. sono preservate, cioè che

$$\{v_n, u_n\} = \delta_{nn} \quad \{v_n, v_k\} = 0 \quad \{u_n, u_k\} = 0$$

Per far ciò deriviamo prima le (o)', ottenendo altre relazioni che le funz. u e v devono soddisfare:

$$(I) \quad \frac{\partial u_n}{\partial \tilde{p}_l} = \frac{\partial^2 F_2}{\partial q_n \partial \tilde{p}_l} + \sum_m \frac{\partial^2 F_2}{\partial q_n \partial q_m} \frac{\partial v_m}{\partial \tilde{p}_l}$$

$$(II) \quad \frac{\partial u_n}{\partial \tilde{q}_l} = \sum_m \frac{\partial^2 F_2}{\partial q_n \partial q_m} \frac{\partial v_m}{\partial \tilde{q}_l}$$

$$(III) \quad \delta_{kl} = \sum_m \frac{\partial F_2}{\partial \tilde{p}_k} \frac{\partial v_m}{\partial \tilde{q}_l}$$

$$(IV) \quad 0 = \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial \tilde{p}_l} + \sum_m \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial q_m} \frac{\partial v_m}{\partial \tilde{p}_l}$$

$$\begin{cases} u_n(\tilde{p}, \tilde{q}, t) = \frac{\partial F_2}{\partial q_n}(\tilde{p}, v(\tilde{p}, \tilde{q}, t), t) \\ \tilde{q}_n = \frac{\partial F_2}{\partial \tilde{p}_n}(\tilde{p}, v(\tilde{p}, \tilde{q}, t), t) \end{cases}$$

Definiamo

$$A_{km} \equiv \frac{\partial^2 F_2}{\partial q_k \partial q_m}$$

$$B_{km} \equiv \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial q_m}$$

$$C_{ke} \equiv \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial \tilde{p}_e}$$

Allora:

$$(III) \Rightarrow \frac{\partial v_m}{\partial \tilde{q}_e} = (\bar{B}^{-1})_{me}$$

$$(IV) \Rightarrow \frac{\partial v_m}{\partial \tilde{p}_e} = -(\bar{B}^{-1}C)_{me}$$

$$(II) \Rightarrow \frac{\partial u_h}{\partial \tilde{q}_e} = (A\bar{B}^{-1})_{he}$$

$$(I) \Rightarrow \frac{\partial u_h}{\partial \tilde{p}_e} = (B^T)_{he} - (A\bar{B}^{-1}C)_{he}$$

In particolare, da qk vedo
possiamo ricavare lo Jacobiano
della transf. can. $J_{ij} = \frac{\partial x_i}{\partial \tilde{x}_j}$

$$J = \begin{pmatrix} B^T - A\bar{B}^{-1}C & A\bar{B}^{-1} \\ -\bar{B}^{-1}C & \bar{B}^{-1} \end{pmatrix}$$

Possiamo ora calcolare le par. di Poisson fondamentali:

$$\begin{aligned} \{v_m, u_h\} &= \sum_e \left(\frac{\partial v_m}{\partial \tilde{q}_e} \frac{\partial u_h}{\partial \tilde{p}_e} - \frac{\partial v_m}{\partial \tilde{p}_e} \frac{\partial u_h}{\partial \tilde{q}_e} \right) = \\ &= \left[\underbrace{\bar{B}^{-1}}_{=I} (B - \underbrace{C^T}_{\bar{C}} \bar{B}^{-1} \underbrace{A^T}_{\bar{A}}) + \cancel{\bar{B}^{-1}C} \bar{B}^{-1} \cancel{A^T} \right]_{mh} \end{aligned}$$

$$= \delta_{mh} \quad \leftarrow \text{In particolare, la transf. can. \u00e8 UNIVALENTE}$$

$$\{v_m, v_h\} = \left[-\bar{B}^{-1}C^T\bar{B}^{-1} + \bar{B}^{-1}C\bar{B}^{-1} \right]_{mh} = 0$$

$$\{u_m, u_h\} = \left[A\bar{B}^{-1}(B - C^T\bar{B}^{-1}A^T) - (B^T - A\bar{B}^{-1}C)\bar{B}^{-1}A^T \right]_{mh} = 0$$

Le nuove Hamiltoniane \u00e8 $K(\tilde{p}, \tilde{q}, t) = \tilde{H}(\tilde{p}, \tilde{q}, t) + K_0(\tilde{p}, \tilde{q}, t)$.

Dove $K_0(\tilde{p}, \tilde{q}, t)$ \u00e8 t.c. $E \nabla_{\tilde{x}} K_0 = \frac{\partial \tilde{w}}{\partial t}(w(\tilde{x}, t), t)$ (*).

Dimostriamo che $K_0(\tilde{p}, \tilde{q}) = \frac{\partial F_2}{\partial t}(\tilde{p}, v(\tilde{p}, \tilde{q}, t))$ soddisfa (*)

- Calcoliamo il termine di destra in (*):

$$w_i(\tilde{w}(x,t)|_t) = x_i \Rightarrow \sum_j \frac{\partial w_i}{\partial x_j} \overset{\substack{\text{deriviamo} \\ \partial/\partial t}}{\frac{\partial \tilde{w}_j}{\partial t}} + \frac{\partial w_i}{\partial t} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial w}{\partial t} = -J \frac{\partial \tilde{w}}{\partial t}$$

- Quindi dimostrare (*) corrisponde a dimostrare che

$$\frac{\partial w}{\partial t} = -JE \nabla_{\tilde{x}} \frac{\partial F_2}{\partial t} \quad (*)'$$

- Deriviamo (*)' in t e definiamo $a_k = \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial t}$ $b_h = \frac{\partial^2 F_2}{\partial q_h \partial t}$

$$\begin{cases} \frac{\partial u_h}{\partial t} = \frac{\partial^2 F_2}{\partial q_h \partial t} + \sum_m \frac{\partial^2 F_2}{\partial q_h \partial q_m} \frac{\partial v_m}{\partial t} \\ 0 = \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial t} + \sum_m \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial q_m} \frac{\partial v_m}{\partial t} \end{cases} \Rightarrow \begin{cases} \frac{\partial u}{\partial t} = b - A\tilde{B}'a \\ \frac{\partial v}{\partial t} = -\tilde{B}'a \end{cases}$$

$$\frac{\partial w}{\partial t} = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} b - A\tilde{B}'a \\ -\tilde{B}'a \end{pmatrix}$$

$$\frac{\partial}{\partial \tilde{p}_k} \left(\frac{\partial F_2}{\partial t} \right) = \frac{\partial^2 F_2}{\partial t \partial \tilde{p}_k} + \sum_m \frac{\partial^2 F_2}{\partial t \partial q_m} \frac{\partial v_m}{\partial \tilde{p}_k} = a_k - (\tilde{B}'c)^T b$$

$$\frac{\partial}{\partial \tilde{q}_h} \left(\frac{\partial F_2}{\partial t} \right) = \sum_m \frac{\partial^2 F_2}{\partial t \partial q_m} \frac{\partial v_m}{\partial \tilde{q}_h} = (\tilde{B}^T b)_h$$

$$\Rightarrow \nabla_{\tilde{x}} \frac{\partial F_2}{\partial t} = \begin{pmatrix} a - C\tilde{B}^T b \\ \tilde{B}^T b \end{pmatrix}$$

$$-JE \nabla_{\tilde{x}} \frac{\partial F_2}{\partial t} = - \begin{pmatrix} \tilde{B}^T - A\tilde{B}'c & A\tilde{B}' \\ -\tilde{B}'c & \tilde{B}' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a - C\tilde{B}^T b \\ \tilde{B}^T b \end{pmatrix}$$

$$= \begin{pmatrix} -B^T + A\bar{B}^T C & -A\bar{B}^T \\ B^{-1}C & -\bar{B}^T \end{pmatrix} \begin{pmatrix} -\bar{B}^T b \\ a - C\bar{B}^T b \end{pmatrix}$$

$$= \begin{pmatrix} b - A\bar{B}^T C \bar{B}^{-T} b - A\bar{B}^T a + A\bar{B}^T C \bar{B}^{-T} b \\ -\bar{B}^T C \bar{B}^{-T} b - \bar{B}^T a + \bar{B}^T C \bar{B}^{-T} b \end{pmatrix} = \begin{pmatrix} b - A\bar{B}^T a \\ -\bar{B}^T a \end{pmatrix}$$

Il che verifica (*). //

Funzioni tipo F_2 generano la maggior parte delle transf. interessanti:

• Identità: $F_2(\tilde{p}, q, t) = \sum_k \tilde{p}_k q_k$: $p_h = \frac{\partial F_2}{\partial q_h} = \tilde{p}_h$ $\tilde{q}_h = \frac{\partial F_2}{\partial \tilde{p}_h} = q_h$

• Trasf. puntuali estese: $F_2 = \sum_k \tilde{p}_k \hat{v}_k(q, t)$:
 $p_h = \frac{\partial F_2}{\partial q_h} = \sum_k \tilde{p}_k \frac{\partial \hat{v}_k(q, t)}{\partial q_h}$ $\tilde{q}_h = \frac{\partial F_2}{\partial \tilde{p}_h} = \hat{v}_h(q, t)$

Con dim. analoghe a F_2 , si può dim. che transf. canoniche in forma implicita si possono ottenere anche con funz. di altro tp:

$$F_1(q, \tilde{q}, t) \rightarrow \begin{cases} p_h = \frac{\partial F_1}{\partial q_h} \\ \tilde{p}_k = -\frac{\partial F_1}{\partial \tilde{q}_k} \end{cases} \quad K = \tilde{H} + \frac{\partial F_1}{\partial t}$$

$$F_3(p, \tilde{q}, t) \rightarrow \begin{cases} q_h = -\frac{\partial F_3}{\partial p_h} \\ \tilde{p}_k = -\frac{\partial F_3}{\partial \tilde{q}_k} \end{cases} \quad K = \tilde{H} + \frac{\partial F_3}{\partial t}$$

$$F_4(p, \tilde{p}, t) \rightarrow \begin{cases} q_h = -\frac{\partial F_4}{\partial p_h} \\ \tilde{q}_k = \frac{\partial F_4}{\partial \tilde{p}_k} \end{cases} \quad K = \tilde{H} + \frac{\partial F_4}{\partial t}$$

APPENDICE: Funzione $F_1(q, \tilde{q}, t)$

Consideriamo una funzione $F_1(q, \tilde{q}, t)$; essa definisce una transf. di coord. in maniera implicita con

$$\begin{cases} p_h(q, \tilde{q}, t) = \frac{\partial F_1}{\partial q_h}(q, \tilde{q}, t) \\ \tilde{p}_k(q, \tilde{q}, t) = -\frac{\partial F_1}{\partial \tilde{q}_k}(q, \tilde{q}, t) \end{cases} \quad (*) \quad \text{con} \quad \det\left(\frac{\partial^2 F_1}{\partial q_h \partial \tilde{q}_k}\right) \neq 0$$

Verifichiamo che una transf. così definita è una transf. CANONICA.

Riscriviamo (*) in termini delle coord. $\tilde{p}, \tilde{q}$ e delle funzioni $p(\tilde{p}, \tilde{q}, t)$, $q(\tilde{p}, \tilde{q}, t)$:

$$\begin{cases} p_h(\tilde{p}, \tilde{q}, t) = \frac{\partial F_1}{\partial q_h}(q(\tilde{p}, \tilde{q}, t), \tilde{q}, t) \\ \tilde{p}_k = -\frac{\partial F_1}{\partial \tilde{q}_k}(q(\tilde{p}, \tilde{q}, t), \tilde{q}, t) \end{cases}$$

Derivando in $\tilde{p}$ e $\tilde{q}$, si ottengono le seguenti relaz. che le $p(\tilde{p}, \tilde{q}, t)$ e $q(\tilde{p}, \tilde{q}, t)$ soddisfano

$$\frac{\partial p_h}{\partial \tilde{p}_e} = \sum_m \frac{\partial^2 F_1}{\partial q_h \partial q_m} \frac{\partial q_m}{\partial \tilde{p}_e}$$

$$\delta_{ke} = - \sum_m \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial q_m} \frac{\partial q_m}{\partial \tilde{p}_e}$$

$$\frac{\partial p_h}{\partial \tilde{q}_e} = \sum_m \frac{\partial^2 F_1}{\partial q_h \partial q_m} \frac{\partial q_m}{\partial \tilde{q}_e} + \frac{\partial^2 F_1}{\partial q_h \partial \tilde{q}_e}$$

$$0 = \sum_m \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial q_m} \frac{\partial q_m}{\partial \tilde{q}_e} + \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial \tilde{q}_e}$$

Definiamo

$$A_{hm} \equiv \frac{\partial^2 F_1}{\partial q_h \partial q_m}$$

$$B_{km} \equiv \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial q_m}$$

$$C_{ke} \equiv \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial \tilde{q}_e}$$

Allora:

$$\frac{\partial q_m}{\partial \tilde{p}_e} = -(\bar{B}^{-1})_{me}$$

$$\frac{\partial q_m}{\partial \tilde{q}_e} = -(\bar{B}^{-1}C)_{me}$$

$$\rightarrow J = \begin{pmatrix} -A\bar{B}^{-1} & -A\bar{B}^{-1}C + B^T \\ -\bar{B}^{-1} & -\bar{B}^{-1}C \end{pmatrix}$$

$$\frac{\partial p_h}{\partial \tilde{p}_e} = -(A\bar{B}^{-1})_{he}$$

$$\frac{\partial p_h}{\partial \tilde{q}_e} = -(A\bar{B}^{-1}C)_{he} + B_{he}^T$$

Ora siamo pronti a dimostrare che le Parentesi di Poisson fondamentali sono PRESERVATE:

$$\{q_m, p_h\} = \sum_e \left(\frac{\partial q_m}{\partial \tilde{q}_e} \frac{\partial p_h}{\partial \tilde{p}_e} - \frac{\partial q_m}{\partial \tilde{p}_e} \frac{\partial p_h}{\partial \tilde{q}_e} \right) =$$

$$= \sum_e (\bar{B}^{-1}C)_{me} (A\bar{B}^{-1})_{he} - \sum_e (\bar{B}^{-1})_{me} ((A\bar{B}^{-1}C)_{he} - B_{he}^T)$$

$$= (A\bar{B}^{-1}C^T \bar{B}^{-T} - A\bar{B}^{-1}C\bar{B}^{-T} + \bar{B}^{-1}B)_{mh} = \delta_{mh}$$

$$\{q_m, q_h\} = \sum_e (\bar{B}^{-1}C)_{me} \bar{B}_{he}^{-1} - \sum_e (\bar{B}^{-1}C)_{he} \bar{B}_{me}^{-1} =$$

$$= (\bar{B}^{-1}C\bar{B}^{-T})_{mh} - (\bar{B}^{-1}C^T\bar{B}^{-T})_{mh} = 0$$

$$\{p_m, p_h\} = \sum_e [(A\bar{B}^{-1}C)_{me} - B_{me}^T] (A\bar{B}^{-1})_{he} - \sum_e [(A\bar{B}^{-1}C)_{he} - B_{he}^T] (A\bar{B}^{-1})_{me}$$

$$= (A\bar{B}^{-1}C\bar{B}^{-T}A^T)_{mh} - (B^T\bar{B}^{-T}A)_{mh} - (A\bar{B}^{-1}C^T\bar{B}^{-T}A^T)_{mh} + (A\bar{B}^{-1}B)_{mh} = 0$$

Deriviamo ora rispetto al tempo

e definiamo

$$a_h = \frac{\partial^2 F_1}{\partial q_h \partial t}$$

$$b_k = \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial t}$$

$$\begin{cases} \frac{\partial p_h}{\partial t} = \frac{\partial^2 F_1}{\partial q_h \partial t} + \sum_m \frac{\partial^2 F_1}{\partial q_h \partial q_m} \frac{\partial q_m}{\partial t} \\ 0 = \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial t} + \sum_m \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial q_m} \frac{\partial q_m}{\partial t} \end{cases}$$

$$\Rightarrow \frac{\partial q_m}{\partial t} = -(\bar{B}^{-1} b)_m$$

$$\Downarrow$$

$$\frac{\partial p_h}{\partial t} = a_h - (A \bar{B}^{-1} b)_h$$

$$\frac{\partial w}{\partial t} = \begin{pmatrix} a - A \bar{B}^{-1} b \\ -\bar{B}^{-1} b \end{pmatrix}$$

$$J = \begin{pmatrix} -A \bar{B}^{-1} & -A \bar{B}^{-1} c + B^T \\ -\bar{B}^{-1} & -\bar{B}^{-1} c \end{pmatrix} \leftarrow \frac{\partial w}{\partial x}$$

$$x = w(\tilde{w}(x, t), t) \rightarrow 0 = J \frac{\partial \tilde{w}}{\partial t} + \frac{\partial w}{\partial t} \Rightarrow \frac{\partial \tilde{w}}{\partial t} = -J^{-1} \frac{\partial w}{\partial t}$$

$$\frac{\partial F_1}{\partial t} = k_0 \leadsto E \nabla_{\tilde{x}} k_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 F_1}{\partial t \partial q} \frac{\partial q}{\partial \tilde{p}} \\ \frac{\partial^2 F_1}{\partial t \partial q} \frac{\partial q}{\partial \tilde{q}} + \frac{\partial^2 F_1}{\partial t \partial \tilde{q}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\bar{B}^{-T} a \\ -(C \bar{B}^{-T}) a + b \end{pmatrix} = \begin{pmatrix} (C \bar{B}^{-T}) a - b \\ -\bar{B}^{-T} a \end{pmatrix} = \frac{\partial \tilde{w}}{\partial t}$$

$$\frac{\partial w}{\partial t} = -J \frac{\partial \tilde{w}}{\partial t} = \begin{pmatrix} A \bar{B}^{-1} & A \bar{B}^{-1} c - B^T \\ \bar{B}^{-1} & \bar{B}^{-1} c \end{pmatrix} \begin{pmatrix} (C \bar{B}^{-T}) a - b \\ -\bar{B}^{-T} a \end{pmatrix} =$$

$$= \begin{pmatrix} A \bar{B}^{-1} c \bar{B}^{-T} a - A \bar{B}^{-1} b - A \bar{B}^{-1} c \bar{B}^{-T} a + a \\ \bar{B}^{-1} c \bar{B}^{-T} a - \bar{B}^{-1} b - \bar{B}^{-1} c \bar{B}^{-T} a \end{pmatrix} =$$

$$= \begin{pmatrix} a - A \bar{B}^{-1} b \\ -\bar{B}^{-1} b \end{pmatrix} //$$