2 1 Morawetz

$$\begin{cases} i\partial_t u = -\Delta u + |u|^{p-1}u \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}^d) \end{cases} \qquad 1+\frac{4}{d} < p < d^* = \frac{d+2}{d-2} \qquad d \geq 3$$

$$\lim_{t\to+\infty} \|u(t)\|_{L^{p+1}} = 0$$

1) $$\int_0^{+\infty} dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r}\,dx < +\infty$$

$$u(t) = e^{i\Delta t}u_0 - i\int_0^t e^{i(t-s)\Delta}|u(s)|^{p-1}u(s)\,ds$$

2) $$\lim_{t\to+\infty} \|u(t)\|_{L^{p+1}(|x| \geq t \lg t)} = 0$$

$$u(t) \xrightarrow{t\to+\infty} 0 \quad \text{in } L^{p+1}(\mathbb{R}^d)$$

3) $\forall \varepsilon > 0\ \forall a, b \in \mathbb{R}_+\ \exists\ t_0 > \max\{a, b\}$ t.c.

$$\sup_{t\in[t_0-b,\,t_0]} \|u(t)\|_{L^{p+1}(\mathbb{R}^d)} < \varepsilon$$

$$u(t) = e^{it\Delta}u_0 - i\int_0^{t-\tau} e^{i(t-s)\Delta}|u(s)|^{p-1}u(s)\,ds$$

$$-i\int_{t-\tau}^{t} e^{i(t-s)\Delta}|u(s)|^{p-1}u(s)\,ds$$

$$= e^{it\Delta}u_0 + W(t,\tau) + Z(t,\tau)$$

$$|u(t)|_{L^{p+1}} \leq |e^{it\Delta}u_0|_{L^{p+1}} + C\,\tau^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}}$$

$$+ C\,\tau^{1-\frac{d(p-1)}{2(p+1)}} \sup_{s\in[t-\tau,t]} |u(s)|^{p}_{L^{p+1}}$$

sia $\varepsilon > 0$ $\qquad \tau_\varepsilon$ è definito in modo che

$$C\,\tau_\varepsilon^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}} = \frac{\varepsilon}{4}$$

$\exists$ t_0 arbitrariamente grande t.c.

$$\sup_{s\in[t_0-\tau_\varepsilon,t_0]} |u(s)|_{L^{p+1}} \leq \frac{\varepsilon}{4}$$

$$t_\varepsilon = \sup\{\tilde{t} \geq t_0 \text{ t.c. } \sup_{s\in[t-\tau_\varepsilon,t]} |u(s)|_{L^{p+1}} \leq \varepsilon \quad \forall\, t\in[t_0,\tilde{t}]\}$$

$\forall\, t \in [t_0, t_\varepsilon)$ $\qquad |u(t)|_{L^{p+1}} \leq \varepsilon$

sia $t_\varepsilon < +\infty \implies |u(t_\varepsilon)|_{L^{p+1}} \leq \varepsilon$

Infatti $|u(t_\varepsilon)| = \varepsilon$

$$|u(t)|_{L^{p+1}} \leq |e^{it\Delta}u_0|_{L^{p+1}} + C\,\tau_\varepsilon^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}}$$

$$+\,C\,\tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}} \sup_{s\in[t-\tau_\varepsilon,t]} |u(s)|^p_{L^{p+1}}$$

$t = t_\varepsilon$

$$\varepsilon < \frac{\varepsilon}{2} + \left(C\,\tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}}\,\varepsilon^{p-1}\right)\varepsilon$$

supponiamo che si abbia

$$\boxed{C\,\tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}}\,\varepsilon^{p-1} < \frac{1}{2}} \quad \text{falso}$$

Allora $\quad \varepsilon < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \varepsilon < \varepsilon$ assurdo

$t_\varepsilon < +\infty$

$$\tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}}\,\varepsilon^{p-1} \geq \frac{1}{2C}$$

$$(4C)^{p-1}\,\tau_\varepsilon^{-\frac{d(p-1)-2\max\{p-1,1\}}{2(p+1)}(p-1)} = \varepsilon^{p-1}$$

$$\tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}-\frac{d(p-1)^2-2(p-1)\max\{p-1,1\}}{2(p+1)}} \geq C_1 > 0$$

$$2(p+1) - d(p-1) - d(p-1)^2 + 2(p-1)\max\{p-1,1\}$$

$$= (p-1)\left(2\max\{p-1,1\} - d - d(p-1)\right) + 2(p+1)$$

$$= (p-1)\left(2\max\{p-1,1\} + 2 - d(p-1)\right) - d(p-1) \underbrace{-2(p-1) + 2(p+1)}_{4}$$

$$= (p-1)\left(2\max\{p-1,1\} + 2 - d(p-1)\right) - d(p-1) + 4$$

se $\quad p-1 \leq 1$

$$(p-1)\left(4 - d(p-1)\right) - d(p-1) + 4 =$$

$$= 4p - \cancel{4} - d(p^2 - 2p + \cancel{1}) - dp + \cancel{d} \cancel{+4}$$

$$= p\left[4 - dp + 2d - d\right] = p\left[4 - dp + d\right] =$$

$$= p\left(4 + d - dp\right) = 4d\left(1 + \frac{4}{d} - p\right) < 0$$

$$p-1 \geqslant 1$$

$$(p-1)\left(2\max\{p-1,1\}+2-d(p-1)\right)-d(p-1)+4$$

$$=(p-1)\left(2(p-1)+2-d(p-1)\right)-d(p-1)+4$$

$$=(p-1)\left(2-(d-2)(p-1)\right)-d(p-1)+4 \qquad d\geq 4$$

$$\leq (p-1)\left(2-2(p-1)\right)-4(p-1)+4$$

$$=-2(p-1)(p-2)-4(p-1)+4$$

$$=-2(p-1)p+\cancel{4(p-1)}-\cancel{4(p-1)}+4<0$$

$$p-1>1$$

$$p(p-1)>2$$

$$2p(p-1)>4$$

$d=3$ $\quad$ $\cancel{p \geq 1}$ $\quad p-1>1$ $\quad \alpha = p-1$

$$(p-1)\left(2\max\{p-1,1\}+2-d(p-1)\right)-d(p-1)+4$$

$$(p-1)\left(2(p-1)+2-3(p-1)\right)-3(p-1)+4$$

$$= \alpha\,(2-\alpha)-3\alpha+4$$

$$= -\alpha^2-\alpha+4 = -q(\alpha) < 0$$

$$\alpha^2+\alpha-4=0 \qquad \alpha_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{17}}{2}$$

$\alpha_- < 0 < \alpha_+$ $\quad$ Per $\alpha > \alpha_+$ ho $q(\alpha)>0$

Se $p-1 > \alpha_+ = \frac{\sqrt{17}}{2} - \frac{1}{2}$ allora ho il <0

Cioè per $d=3$ $\quad p > \frac{\sqrt{17}+1}{2} > \frac{7}{3}$

ottengo il risultato $\quad \underline{2 \geq p >} \cancel{1} + \frac{4}{\cancel{d}\,3} = \frac{7}{3}$

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$$\frac{d^2}{dx^2} u = -k^2 u \qquad u = e^{\pm ikx}$$

$$\underline{\left(-\frac{d^2}{dx^2} + q(x)\right) u = k^2 u}$$

$$q(x) \in L^1(\mathbb{R})$$

$$\langle x \rangle q \in L^1(\mathbb{R})$$

$$\eta(x) = \int_x^{+\infty} |q(y)| \, dy,$$

$$\gamma(x) = \int_x^{+\infty} (y-x) \, |q(y)| \, dy$$

$k \in \mathbb{C}$ $\operatorname{Im} k \geq 0$ $\exists$

$$(-\partial_x^2 + V) f_j(x,k) = k f_j(x,k) \qquad j = 1,2$$

$$f_1(x,k) = e^{ixk} m_1(x,k) \qquad m_1(x,k) \xrightarrow{x \to +\infty} 1$$

$$f_2(x,k) = e^{-ixk} m_2(x,k) \qquad m_2(x,k) \xrightarrow{x \to -\infty} 1$$

$$f(x,k) = f_1(x,k) \qquad m(x,k) = m_1(x,k)$$

Then $\exists$ $m(x,k)$ come sopra t.c.

1) $$|m(x,k)-1| \leq e^{\frac{\eta(x)}{|k|}} \frac{\eta(x)}{|k|}$$

$$\eta(x) = \int_x^{+\infty} |q(y)|\, dy$$

2) $\exists$ K t.c.

$$|m(x,k)-1| \leq K \frac{(1+\max\{-x,0\}) \int_x^{+\infty} (1+|y|)\,|q(y)|\, dy}{1+|k|}$$

$$(-\partial_x^2 + q(x)) f(x,k) = k^2 f(x,k)$$

$$[u,v] = (u'v - uv')$$

$$(+\partial_x^2 + k^2) f(x,k) = +q(x) f(x,k)$$

$$f(x,k) = e^{ikx} \int \frac{e^{-ikx} q(x) f(x,k)\, dx}{[e^{ikx}, e^{-ikx}]} - e^{-ikx} \int \frac{e^{ikx} q(x) f(x,k)\, dx}{[e^{ikx}, e^{-ikx}]}$$

$$f(x,k) = e^{ikx} - \frac{e^{ikx}}{2ik} \int_x^{+\infty} e^{-iky} q(y) f(y,k)\, dy + \frac{e^{-ikx}}{2ik} \int_x^{+\infty} e^{iky} q(y) f(y,k)\, dy$$

$$m(x,k) = 1 - \frac{1}{2ik} \int_x^{+\infty} q(y) m(y,k)\, dy + \frac{1}{2ik} \int_x^{+\infty} e^{2ik(y-x)} q(y) m(y,k)\, dy$$

$$k = k_R + i k_I \qquad k_I \geq 0$$

$$\mathrm{Re}\{2ik(y-x)\} = -2k_I \underbrace{(y-x)}_{\geq 0} \leq 0$$

$$\boxed{m(x,k) = 1 + \int_x^{+\infty} D_k(y-x)\, q(y)\, m(y,k)\, dy}$$

$$D_k(y-x) = \frac{e^{2ik(y-x)} - 1}{2ik} = \int_0^{y-x} e^{2ikt}\, dt$$

$$|D_k(y-x)| \leq \begin{cases} \frac{1}{|k|} \\ y-x \end{cases}$$

$$m(x,k) = 1 + \int_x^{+\infty} D_k(y-x)\, q(y)\, m(y,k)\, dy$$

$$m(x,k) = \overset{m_0(x,k)}{\overset{\parallel}{1}} + \sum_{n=1}^{\infty} m_n(x,k)$$

$n \geq 1$

$$m_n(x,k) = \int_x^{+\infty} D_k(y-x)\, q(y)\, m_{n-1}(y,k)\, dy$$

$$= \int_x^{+\infty} D_k(x_1-x)\, q(x_1)\, m_{n-1}(x_1,k)\, dx_1$$

$$= \int_{x \leq x_1 \leq \cdots \leq x_n} D_k(x_1-x)\, D_k(x_2-x_1) \cdots D_k(x_n-x_{n-1})\, q(x_1) \cdots q(x_n)\, dx_1 \cdots dx_n$$

$$|m_n(x,k)|$$

$$\leq \int_{x \leq x_1 \leq \cdots \leq x_n} |D_k(x_1 - x)\, D_k(x_2 - x_1) \cdots D_k(x_n - x_{n-1})\, q(x_1) \cdots q(x_n)|\, dx_1 \cdots dx_n$$

$$\leq \int_{x \leq x_1 \leq \cdots \leq x_n} \frac{1}{|k|^n} |q(x_1)| \cdots |q(x_n)|\, dx_1 \cdots dx_n$$

$$= \frac{1}{|k|^n} \frac{\left(\int_x^{+\infty} |q(y)|\, dy \right)^n}{n!}$$

$$\frac{d}{dx} \int_x^{+\infty} |q(x_1)|$$