$$H = -\partial_x^2 + q(x)$$

26 maggio

$$m(x,k) = 1 + \int_x^{+\infty} \overbrace{\frac{e^{2ik(y-x)} - 1}{2ik}}^{D_k(y-x)} q(y)\, m(y,k)\, dy$$

$$\left(\begin{aligned} f_1(x,k) &= e^{ikx} m_1(x,k) \qquad & m_1(x,k) \xrightarrow{x\to+\infty} 1 \\ f_2(x,k) &= e^{-ikx} m_2(x,k) \qquad & m_2(x,k) \xrightarrow{x\to-\infty} 1 \\ \operatorname{Im} k &\geq 0 \end{aligned} \right)$$

1) $$|m(x,k) - 1| \leq e^{\frac{\int_x^{+\infty} |q(y)|\,dy}{|k|}} \qquad \frac{\int_x^{+\infty} |q(y)|\,dy}{|k|}$$

2) $\exists$ K t.c.

$$|m(x,k) - 1| \leq K \frac{(1 + \max(-x, 0)) \int_x^{+\infty} (1+|y|)|q(y)|\,dy}{1 + |k|}$$

$$\langle x \rangle q \in L^1(\mathbb{R})$$

$$m(x,k) = \overset{m_0(x,k)}{1} + \sum_{n=1}^{+\infty} m_n(x,k)$$

$$m_n(x,k) = \int_x^{+\infty} D_k(y-x)\, q(y)\, m_{n-1}(y,k)\, dy$$

$$|m_n(x,k)| \leq \frac{1}{|k|^n} \frac{\left(\int_x^{+\infty} |q(y)|\,dy\right)^n}{n!}$$

$$|m(x,k)-1| \leq \sum_{n=1}^{\infty} |m_n(x,k)| \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{|k|^n} \frac{\left(\int_x^{+\infty} |q(y)|\,dy\right)^n}{n!}$$

$$\leq \frac{1}{|k|} \int_x^{+\infty} |q(y)|\,dy \underbrace{\sum_{n=0}^{+\infty} \frac{1}{|k|^n} \frac{\left(\int_x^{+\infty} |q(y)|\,dy\right)^n}{n}}_{\exp\left(\frac{1}{|k|} \int_x^{+\infty} |q(y)|\,dy\right)}$$

$$m(x,k) = \overset{m_0(x,k)}{1} + \sum_{n=1}^{+\infty} m_n(x,k)$$

$$|m_n(x,k)| = \left| \int_x^{+\infty} D_k(y-x)\, q(y)\, m_{n-1}(y,k)\,dy \right|$$

$$\leq \int_{x \leq x_1 \leq \dots \leq x_n} |D_k(x_1-x)\, D_k(x_2-x_1) \cdots D_k(x_n-x_{n-1})|\, |q(x_1) \cdots q(x_n)|\, dx_1 \dots dx_n$$

$$y \geq 0 \qquad |D_k(y)| \leq \begin{cases} \frac{1}{|k|} \\ y \end{cases}$$

$$\leq \int_{x \leq x_1 \leq \dots \leq x_n} (x_1-x)(x_2-x_1) \cdots (x_n-x_{n-1})\, |q(x_1) \cdots q(x_n)|\, dx_1 \dots dx_n$$

$$\leq \int_{x \leq x_1 \leq \dots \leq x_n} (x_1-x)(x_2-x) \cdots (x_n-x)\, |q(x_1)| \cdots |q(x_n)|\, dx_1 \dots dx_n$$

$$= \frac{\left(\int_x^{+\infty} (y-x)\, |q(y)|\, dy\right)^n}{n!}$$

$$\left| m(x,k) - 1 \right| \le \underbrace{\int_x^{+\infty} (y-x)\,|q(y)|\,dy}_{\gamma(x)} \sum_{n=0}^{\infty} \frac{\gamma^n(x)}{(n+1)!}$$

$$= \gamma(x)\, e^{\gamma(x)}$$

$$|m(x,k)-1| \le \int_x^{+\infty} (y-x)\,|q(y)|\,dy \quad e^{\int_x^{+\infty} (y-x)\,|q(y)|\,dy}$$

$$= \int_x^{+\infty} (y-x)\,|q(y)|\,dy \quad e^{-x \int_x^{+\infty} |q(y)|\,dy} \quad e^{\int_x^{+\infty} y\,|q(y)|}$$

$$|x| \qquad e^{|x| \int_{\mathbb{R}} |q(y)|\,dy}$$

$$m(x,k) = 1 + \int_x^{+\infty} D_k(y-x)\; q(y)\; m(y,k)\, dy$$

$$|m(x,k)| \le 1 + \int_x^{+\infty} (y-x)\; |q(y)|\,|m(y,k)|\, dy$$

$$\le 1 + \int_0^{+\infty} y\; |q(y)|\; |m(y,k)|\, dy$$

$$- x \int_x^{+\infty} |q(y)|\; |m(y,k)|\, dy$$

$$1 + \int_0^{+\infty} y\; |q(y)|\; |m(y,k)|\, dy \le$$

$\gamma \ge 0$

$$\gamma(y) = \int_y^{+\infty} (s-y)\,|q(s)|\,ds \le \gamma(0)$$

$$|m(y,k) - 1| \le \gamma(y)\, e^{\gamma(y)} \le \gamma(0)\, e^{\gamma(0)}$$

$$|m(y,k)| \le 1 + \gamma(0)\, e^{\gamma(0)}$$

$$\leq 1 + \underbrace{\int_0^{+\infty} y\,|q(y)|\,dy}_{\gamma(0)} \quad \gamma(0)\, e^{\gamma(0)}$$

$$= 1 + \gamma^2(0)\, e^{\gamma(0)} =: K$$

$$|m(x,k)| \leq K$$

$$- x \int_x^{+\infty} |q(y)|\,|m(y,k)|\,dy$$

$$|m(x,k)| \leq K - x \int_x^{+\infty} |q(y)|\,|m(y,k)|\,dy$$

$$M(x,k) = \frac{m(x,k)}{K(1+|x|)}$$

$$\frac{|m(x,k)|}{K(1+|x|)} \leq 1 - \frac{x}{\cancel{K}(1+|x|)} \int_x^{+\infty} |q(y)|\, \cancel{K}(1+|y|)\, \frac{|m(y,k)|}{K(1+|y|)}$$

$$|M(x,k)| \leq 1 + \int_x^{+\infty} (1+|y|)\,|q(y)|\,|M(y,k)|\,dy$$

Per Gronwall

$$|M(x,k)| \leq e^{\int_x^{+\infty} (1+|y|)\,|q(y)|\,dy}$$

$$|M(x,k)| \leq 1 \qquad k \qquad x \geq 0$$

Nel minimo caso

Per $x \geq 0$ $|m(x,k)| \leq K$

per $x < 0$

$$|m(x,k)| \leq (1+|x|)\, K\, e^{\int_x^{+\infty}(1+|y|)\,|q(y)|\,dy}$$

$$= (1 + \max(-x,0))\, K\, e^{\int_x^{+\infty}(1+|y|)\,|q(y)|\,dy}$$

$$f_1(x,k) = e^{ikx}\, m_1(x,k)$$
$$f_2(x,k) = e^{-ikx}\, m_2(x,k)$$

funzioni di Jost

$$[f_1(x,k), f_2(x,k)] = \partial_x f_1\, f_2 - f_1\, \partial_x f_2 = W(k)$$

Gli autovalori di $H = -\partial_x^2 + q(x)$ corrispondono

agli zeri di $W(k) = 0$ $\quad \operatorname{Im} k \geq 0$

$z = k^2$

Se $W(0) = 0$ come succede per $q \equiv 0$

$q \equiv 0$

$$W(k) = [e^{ikx}, e^{-ikx}] = 2ik$$

non si ha un autovalore bensì una risonanza. 0 è una risonanza per H

se $\exists$ $u \in L^\infty(\mathbb{R})$ $u \not\equiv 0$

$$H u = 0$$

$$-\partial_x^2 1 = 0$$

$W(k) = [f_1(x,k), f_2(x,k)] =$

$= [e^{ikx} m_1(x,k), e^{-ikx} m_2(x,k)]$

$$m_1(x,k) = 1 + \int_x^{+\infty} \frac{e^{2i(y-x)} - 1}{2ik} m_1(y,k) q(y)\, dy$$

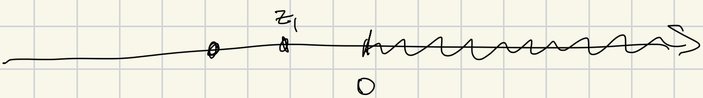
$$m_2(x,k) = 1 + \int_{-\infty}^{x} \frac{e^{2i(x-y)} - 1}{2ik} m_2(y,k) q(y)\, dy$$

$$\frac{W(k)}{2ik} \xrightarrow{k \to \infty} 1$$

Se $\quad q(y) \geq 0 \qquad q \not\equiv 0$

$$|W(k)| \geq c_0 > 0 \qquad \forall k$$

$z_1 = k_1^2$



$z \notin [0, +\infty) \qquad \exists!\ 1\ k$ con $\operatorname{Im} k > 0$

t.c. $z = k^2$

$(H - z)u = F \qquad F \in L^2(\mathbb{R})$

$$u = R_H(z) F = \int_{\mathbb{R}} R_H(x,y,z) F(y)\, dy$$

$$R_H(x,y,z) = \begin{cases} \dfrac{f_2(y,k) f_1(x,k)}{[f_1(y,k), f_2(y,k)]} & x > y \\[2ex] \dfrac{f_2(x,k) f_1(y,k)}{[f_1(y,k), f_2(y,k)]} & x < y \end{cases}$$

$$\Big((H-z) = \int_x^{+\infty} e^{-ik(x-y)} \frac{m_2(x,k)\, m_1(y,k)}{[f_1(y,k), f_2(y,k)]} F(y)\, dy$$

$$+ \int_{-\infty}^{x} e^{ik(x-y)} \frac{m_1(x,k)\, m_2(y,k)}{[f_1(y,k), f_2(y,k)]} F(y)\, dy$$

$$\mathrm{Re}\big(-i(k_R + i k_I)(x-y)\big) = k_I (x-y) < 0$$

$$|R_H(z)\, u(x)| \leq \frac{1}{c_p} \int_{\mathbb{R}} e^{-k_I |x-y|} |F(y)|\, dy$$

$$|m_1(\cdot,k)|_{L^\infty(\mathbb{R})} \; |m_2(\cdot,k)|_{L^\infty(\mathbb{R})}$$

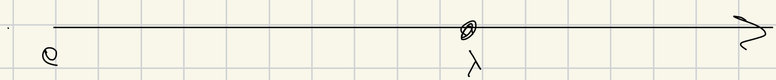
Sui $s \in \mathbb{R}$ $\quad L^{2,s}(\mathbb{R})$

$$|u|_{L^{2,s}} \doteq |\langle x\rangle^s u|_{L^2(\mathbb{R})}$$

<u>Prop</u> Per $s > \frac{3}{2}$ e $\tau > \frac{1}{2}$ $\exists\, C$ t.c.

$$\lim_{\varepsilon \to 0^+} R_H(\lambda \pm i\varepsilon) = R_H^{\pm}(\lambda) \qquad \forall\, \lambda \geq 0$$

in $\mathcal{L}\big(L^{2,\tau}(\mathbb{R}), L^{2,-s}(\mathbb{R})\big)$



$$\sup_{\varepsilon>0,\, \lambda\geq 0} \| R_H(\lambda \pm i\varepsilon)\|_{\mathcal{L}(L^{2,\tau},\, L^{2,-1})} \leq C$$

$$g(t,x)$$

Prop siano $s > \frac{3}{2}$ e $\tau > \frac{1}{2}$ $\exists$ C

$$\left\| \int_0^t e^{-i(t-s)H} g(s) \right\|_{L^2(\mathbb{R}, L^{2,-s})} \leq C \|g\|_{L^2(\mathbb{R}, L^{2,\tau}(\mathbb{R}))}$$

Dim

$\varepsilon > 0$ $\qquad$ $\lambda \geq 0$

$$\int_0^{+\infty} e^{i(\lambda+i\varepsilon)t} e^{-itH} \, dt =$$

$$= \int_0^{+\infty} e^{i\lambda t} e^{-\varepsilon t} e^{-itH} \, dt =$$

$$= \int_0^{+\infty} e^{t(i\lambda - \varepsilon - iH)} \, dt =$$

$$= \left. \frac{e^{t(i\lambda-\varepsilon-iH)}}{i\lambda - \varepsilon - iH} \right]_0^{+\infty} =$$

$$= \frac{-1}{i\lambda - \varepsilon - iH} = \frac{-1}{i\lambda + i\,i\varepsilon - iH} =$$

$$= \frac{1}{i} \frac{1}{H - \lambda - i\varepsilon} = -i\, R_H(\lambda + i\varepsilon)$$

$$\int_0^{+\infty}$$

$$\lim_{\varepsilon\to 0^+} \int_0^{+\infty} e^{i(\lambda+i\varepsilon)t}\, e^{-itH}\, dt = -iR_H^+(\lambda)$$

$$\int_0^{+\infty} e^{i\lambda t}\, e^{-itH}\, dt = -i\, R_H^+(\lambda)$$

$$\left\| \int_0^t e^{-i(t-s)H} g(s)\, ds \right\|_{L^2(\mathbb{R}_t, L^{2,-s}(\mathbb{R}_x))} =$$

$$= \left\| \int_{\mathbb{R}} e^{-i(t-s)H} \chi_{\mathbb{R}_+}(t-s)\, g(s)\, ds \right\|_{L^2(\mathbb{R}_t, L^{2,-s}(\mathbb{R}_x))} =$$

$$= \left\| R_H^+(\lambda)\, \check{g}(\lambda) \right\|_{L^2(\mathbb{R}_\lambda, L^{2,-s}(\mathbb{R}_x))}$$

$$\leq \sup_{\lambda\geq 0} \left\| R_H^+(\lambda) \right\|_{L^{2,\tau}(\mathbb{R}_x)\to L^{2,-s}(\mathbb{R}_x)} \left\| \check{g}(\lambda) \right\|_{L^2(\mathbb{R}_\lambda, L^{2,\tau}(\mathbb{R}))}$$

$$= \underbrace{\sup_{\lambda\geq 0} \left\| R_H^+(\lambda) \right\|_{L^{2,\tau}(\mathbb{R}_x)\to L^{2,-s}(\mathbb{R}_x)}}_{\leq C_{s\tau}} \left\| g \right\|_{L^2(\mathbb{R}_t, L^{2,\tau}(\mathbb{R}))}$$

$$s > \tfrac{3}{2}, \quad \tau > \tfrac{1}{2}$$

$$\| R_H(\lambda + i\varepsilon)\|_{\mathcal{L}(L^{2,\tau},\, L^{2,-1})} \le C$$

$$L^2(\mathbb{R}) \xrightarrow[\langle x\rangle^{-\tau}]{} L^{2,\tau}(\mathbb{R}) \xrightarrow{R_H(\lambda+i\varepsilon)} L^{2,-1}(\mathbb{R}) \xrightarrow[\langle x\rangle^{-1}]{} L^2(\mathbb{R})$$

$$\| \langle x\rangle^{-1} R_H(\lambda+i\varepsilon)\langle y\rangle^{-\tau}\|_{L^2\to L^2} \le C$$

$$\| \langle x\rangle^{-1} R_H(x,y,\lambda+i\varepsilon)\langle y\rangle^{-\tau}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \le C$$

$$x \le y$$

$$\int_{x<y} \langle x\rangle^{-2} |R_H(x,y,z)|^2 \langle y\rangle^{-2\tau}\, dx\, dy$$

$$k^2 = z$$

$$|R_H(x,y,z)| = \frac{|m_2(x,k)\, m_1(y,k)|}{|W(k)|}\, \underbrace{|e^{ik(y-x)}|}_{\le 1}$$

$$\int_{x<y<0} \langle x\rangle^{-2} |R_H(x,y,z)|^2 \langle y\rangle^{-2\tau}\, dx\, dy$$

$$+ \int_{0<x<y} \langle x\rangle^{-2\lambda} \, |R_H(x,y,z)|^2 \, \langle y\rangle^{-2\tau} \, dx\,dy$$

$$\underbrace{+ \int_{\cancel{x<0<y}} \langle x\rangle^{-2\lambda} \, \underbrace{|R_H(x,y,z)|^2}_{\lesssim 1} \, \langle y\rangle^{-2\tau} \, dx\,dy}_{\lesssim 1}$$

$$\int_{0<x<y} \langle x\rangle^{-2\lambda} \, \underbrace{|m_2(x,k)\, m_1(y,k)|^2}_{\lesssim \langle x\rangle^2} \, \langle y\rangle^{-2\tau} \, dx\,dy$$

$$\int_{0<\cancel{x}<y} \langle x\rangle^{-2(\lambda-1)} \, \langle y\rangle^{-2\tau} \, dx\,y$$

$$\int_{x<y<0} \langle x\rangle^{-2\lambda} \, \underbrace{|m_2(x,k)\, m_1(y,k)|}_{\langle y\rangle^2}{}^2 \, \langle y\rangle^{-2\tau} \, dx\,dy$$

$$\int_{x<y<0} \langle x\rangle^{-2\lambda} \, \langle x\rangle^2 \, \langle y\rangle^{-2\tau} \, dx\,dy$$