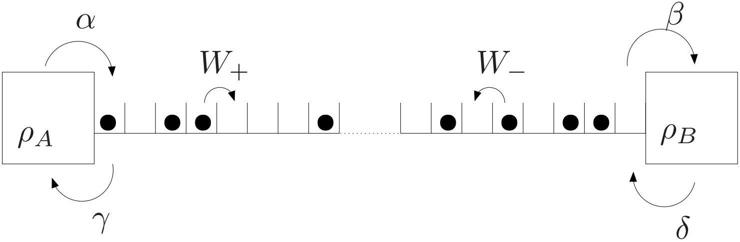
Simple exclusion process

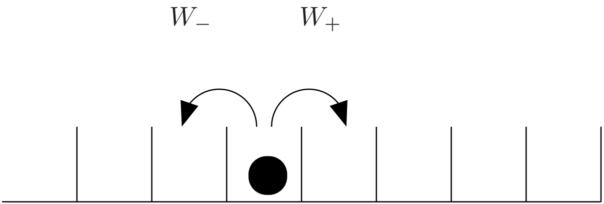


AI

Motivations

- "Simple" lattice systems
- Stochastic non–equilibrium systems
- Steady states depends on initial state, boundary conditions, and internal parameters
- Bulk and boundary perturbations may induce phase transition in steady state and dynamic behaviour
- Heat conduction, diffusion, sand pile models, avalanches, ...

Random walk on 1d-lattice



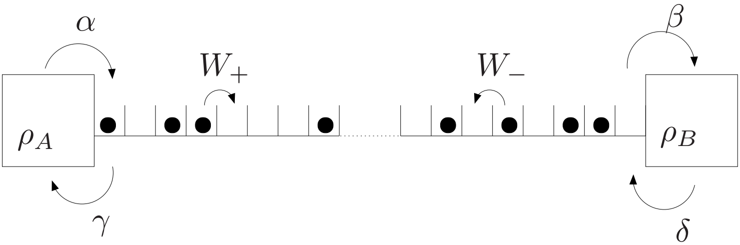
$P_l(t)$ probability that the particle is at site l at time t

$$\frac{\mathrm{d}P_l(t)}{\mathrm{d}t} = W_+P_{l-1} + W_-P_{l+1} - (W_+ + W_-)P_l = J_{l-1/2} - J_{l+1/2}$$

$$J_{l+1/2} = W_+P_l - W_-P_{l+1}; \qquad J_{l-1/2} = W_+P_{l-1} - W_-P_l$$

ASEP

the asymmetric simple exclusion process (ASEP)



$n_l = 0, 1$ occupation number, $\rho_l(t) \equiv \langle n_l(t) \rangle$

$$\frac{\mathrm{d}\rho_l(t)}{\mathrm{d}t} = \left\langle J_{l-1/2} - J_{l+1/2} \right\rangle$$

$$J_{l+1/2} = W_+ n_l(1 - n_{l+1}) - W_- n_{l+1}(1 - n_l)$$
$$J_{l-1/2} = W_+ n_{l-1}(1 - n_l) - W_- n_l(1 - n_{l-1})$$

ASEP: mean field approximation

$$\langle n_l(t) n_{l\pm1}(t)\rangle = \langle n_l(t)\rangle \langle n_{l\pm1}(t)\rangle = \rho_l(t)\rho_{l\pm1}(t)$$

$$\begin{aligned}
\frac{\mathrm{d}\rho_l(t)}{\mathrm{d}t} &= W_+\rho_{l-1}(1-\rho_l) + W_-\rho_{l+1}(1-\rho_l) - W_+\rho_l(1-\rho_{l+1}) - W_-\rho_l(1-\rho_{l-1}) \\
\frac{\mathrm{d}\rho_0(t)}{\mathrm{d}t} &= \alpha(1-\rho_0) + W_-\rho_1(1-\rho_0) - \gamma\rho_0 - W_+\rho_0(1-\rho_1) \\
\frac{\mathrm{d}\rho_N(t)}{\mathrm{d}t} &= \delta(1-\rho_N) + W_+\rho_{N-1}(1-\rho_N) - \beta\rho_N - W_-\rho_N(1-\rho_{N-1})
\end{aligned}$$

$a = L/N$, $x = la$, $\nu = a(W_+ - W_-)$, $D = a^2(W_+ + W_-)/2$

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial\rho}{\partial x} - \nu\rho(1-\rho)\right)$$

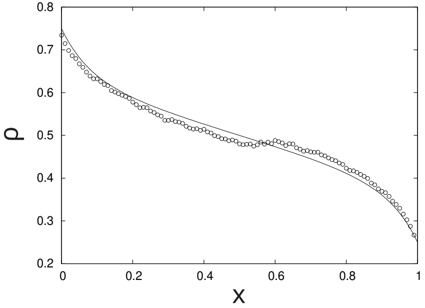
ASEP: mean field approximation II

Steady state: $\partial_t \rho = 0$
Boundary conditions

$$
\begin{aligned}
0 &= \alpha(1-\rho_0) + W_-\rho_1(1-\rho_0) - \gamma\rho_0 - W_+\rho_0(1-\rho_1) \\
0 &= \delta(1-\rho_N) + W_+\rho_{N-1}(1-\rho_N) - \beta\rho_N - W_-\rho_N(1-\rho_{N-1}) \\
\rho(0) &= \frac{W_+ - W_- + \alpha + \gamma - \sqrt{(\alpha-\gamma-W_+ + W_-)^2 + 4\alpha\gamma}}{2(W_+ - W_-)} \\
\rho(L) &= \frac{W_+ - W_- - \beta - \delta + \sqrt{(\beta-\delta-W_+ + W_-)^2 + 4\beta\delta}}{2(W_+ - W_-)}
\end{aligned}
$$

ASEP: mean field approximation III

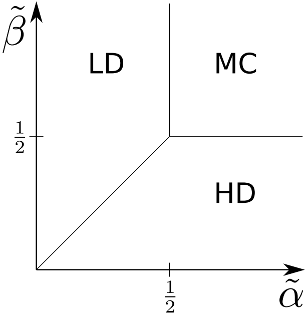
$N = 100$, $W_+ = 1$, $W_- = 0.75$, $\rho_A = 0.75$, $\rho_B = 0.25$,



ASEP: phase diagram

$\delta = \gamma = 0$

$\tilde{\alpha} = \alpha/(W_+ - W_-)$
$\tilde{\beta} = \beta/(W_+ - W_-)$



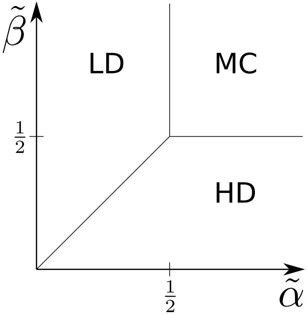
- velocity of a single-motor (in a many body system) $v = J/\rho$
- in mean-field approximation $J = (W_+ - W_-)a\rho(1 - a\rho)$, where ρ depends on the phase
- Thus

$$v = \begin{cases} v_0 - a\alpha & \text{(LD)} \\ a\beta & \text{(HD)} \\ v_0/2 & \text{(MC)} \end{cases}$$

- where $v_0 = a(W_+ - W_-)$

ASEP: phase diagram

$\delta = \gamma = 0$
$\tilde{\alpha} = \alpha/(W_+ - W_-)$
$\tilde{\beta} = \beta/(W_+ - W_-)$



ASEP: phase diagram

- Low-density phase (LD): In this regime characterized by $\tilde{\alpha} < \min(\tilde{\beta}, 1/2)$, the injection process is rate-limiting. The bulk density is $\rho = \tilde{\alpha}/a$ and the average current is $J = (W_+ - W_-)\tilde{\alpha}(1 - \tilde{\alpha})$
- High-density phase (HD): When $\tilde{\beta} < \min(\tilde{\alpha}, 1/2)$, the density $\rho = (1 - \tilde{\beta})/a$ and the current $J = (W_+ - W_-)\tilde{\beta}(1 - \tilde{\beta})$, are functions of the rate-limiting extraction step.
- Maximal current phase (MC): For $\tilde{\alpha} > 1/2$ and $\tilde{\beta} > 1/2$, the bulk behaviour is independent of the boundary conditions, and one finds $\rho = 1/(2a)$ and the maximum possible current $J = (W_+ - W_-)/4$.

ASEP: phase diagram

The low- and high-density phases are separated by the line $\tilde{\alpha} = \tilde{\beta}$, across which the bulk density is discontinuous. The density profile on this line is a mixed-state of shock profiles interpolating between the lower density $\rho = \tilde{\alpha}/a$ and the higher density $\rho = (1 - \tilde{\beta})/a$.

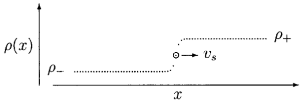


FIG. 3. Motion of a shock in an ensemble average of the lattice gas. To the left (right) of the domain wall, particles are distributed homogeneously with an average density ρ_- (ρ_+) on each lattice site. The corresponding stationary currents $j_\pm$ determine the drift velocity v_s [Eq. (7)] of the shock.

F. H. L. Essler and V. Rittenberg: J. Phys. A, 29, 3375 (1996)
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