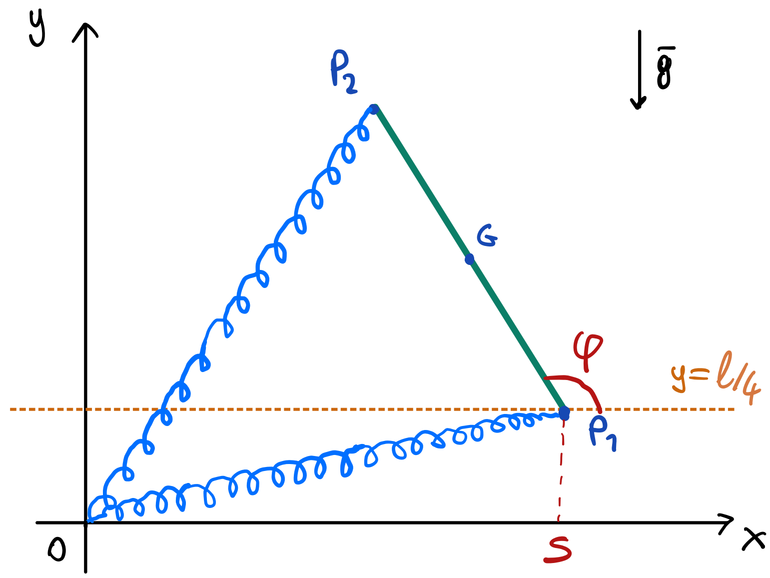


$$I_G = \int_{-\ell/2}^{\ell/2} \rho\, s^2 ds = \rho \frac{s^3}{3}\Big|_{-\ell/2}^{\ell/2} = 2\frac{\ell^3 \rho}{3 \cdot 8} = \frac{\ell^2 M}{12}$$

$$x_{P_1} = s \qquad x_{P_2} = s + \ell\cos\varphi$$
$$y_{P_1} = \frac{\ell}{4} \qquad y_{P_2} = \frac{\ell}{4} + \ell \operatorname{sen}\varphi$$
$$x_G = s + \frac{\ell}{2}\cos\varphi \qquad \dot{x}_G = \dot{s} - \dot{\varphi}\frac{\ell}{2}\operatorname{sen}\varphi$$
$$y_G = \frac{\ell}{4} + \frac{\ell}{2}\operatorname{sen}\varphi \qquad \dot{y}_G = \dot{\varphi}\frac{\ell}{2}\cos\varphi$$

$$I = \frac{m\ell^2}{12}$$

$$T = \frac{m}{2}(\dot{x}_G^2 + \dot{y}_G^2) + \frac{1}{2} I \dot{\varphi}^2 =$$
$$= \frac{m}{2}\left(\dot{s}^2 + \frac{\ell^2\dot{\varphi}^2}{4} - \ell\dot{s}\dot{\varphi}\operatorname{sen}\varphi\right) + \frac{m}{2}\left(\frac{\ell^2\dot{\varphi}^2}{12}\right)$$
$$= \frac{m}{2}\left(\dot{s}^2 + \frac{\ell^2\dot{\varphi}}{3} - \ell\dot{s}\dot{\varphi}\operatorname{sen}\varphi\right)$$

$$Q = m\begin{pmatrix} 1 & -\frac{\ell}{2}\operatorname{sen}\varphi \\ -\frac{\ell}{2}\operatorname{sen}\varphi & \ell^2/3 \end{pmatrix}$$

$$V = mg\,y_G + \frac{k}{2}\left(x_{P_1}^2 + y_{P_1}^2 + x_{P_2}^2 + y_{P_2}^2\right) =$$
$$= mg\left(\frac{\ell}{4} + \frac{\ell}{2}\operatorname{sen}\varphi\right) + \frac{k}{2}\Big(s^2 + \frac{\ell^2}{16} + s^2 + 2\ell s\cos\varphi + \frac{\ell^2}{16} + \frac{1}{2}\ell^2\operatorname{sen}\varphi + \ell^2\Big)$$
$$= \cancel{mg\frac{\ell}{4}} + \cancel{k\frac{\ell^2}{16}} + \cancel{k\ell^2} + mg\frac{\ell}{2}\operatorname{sen}\varphi + ks^2 + k\ell\left(s\cos\varphi + \frac{\ell}{4}\operatorname{sen}\varphi\right)$$

1)

$$L = \frac{m}{2}\left(\dot{s}^2 + \frac{\ell^2\dot{\varphi}^2}{3} - \ell\dot{s}\dot{\varphi}\operatorname{sen}\varphi\right) - mg\frac{\ell}{2}\operatorname{sen}\varphi - ks^2 - k\ell\left(s\cos\varphi + \frac{\ell}{4}\operatorname{sen}\varphi\right)$$

2)

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} = \frac{d}{dt}\left(\frac{m\ell^2\dot{\varphi}}{3} - \frac{m\ell}{2}\dot{s}\operatorname{sen}\varphi\right) = \frac{m\ell^2\ddot{\varphi}}{3} - \cancel{\frac{m\ell}{2}\dot{s}\dot{\varphi}\cos\varphi} - \frac{m\ell}{2}\ddot{s}\operatorname{sen}\varphi$$

$$\frac{\partial L}{\partial\varphi} = -\cancel{\frac{m\ell\dot{s}\dot{\varphi}}{2}}\cos\varphi - mg\frac{\ell}{2}\cos\varphi + k\ell s\operatorname{sen}\varphi - \frac{k\ell^2}{4}\cos\varphi$$

$$\frac{\ell}{3}\ddot{\varphi} - \frac{\ddot{s}}{2}\operatorname{sen}\varphi = -\frac{g}{2}\cos\varphi + \frac{k}{m}s\operatorname{sen}\varphi - \frac{k}{4m}\ell\cos\varphi$$

3) $\frac{\partial V}{\partial s} = 2ks + kl\cos\varphi \quad \longrightarrow \quad s = -\frac{l\cos\varphi}{2}$

$$\frac{\partial V}{\partial \varphi} = mg\frac{l}{2}\cos\varphi - kls\,\mathrm{sen}\,\varphi + k\frac{l^2}{4}\cos\varphi$$

$$\hookrightarrow \frac{l}{2}\left(mg + \frac{kl}{2}\right)\cos\varphi + \frac{kl^2}{2}\cos\varphi\,\mathrm{sen}\,\varphi = 0$$

$$\rightarrow \frac{kl^2}{2}\cos\varphi\left(\mathrm{sen}\,\varphi + \frac{mg}{kl} + \frac{1}{2}\right) = 0$$

$\varphi = \pm\pi/2 \quad \longrightarrow \quad s = 0$

$\varphi = \varphi^*, \pi - \varphi^*$ con φ^* t.c. $\mathrm{sen}\,\varphi^* = -\frac{mg}{kl} - \frac{1}{2}$ ← $\exists$ se $\frac{mg}{kl} + \frac{1}{2} \leq 1$ cioè $\frac{mg}{kl} \leq \frac{1}{2}$

$s = -\frac{l}{2}\cos\varphi^*, \ \frac{l}{2}\cos\varphi^*$

$(s,\varphi) =$
- $\left(0, \frac{\pi}{2}\right)$ instab
- $\left(0, -\frac{\pi}{2}\right)$ stab. se $\frac{mg}{kl} > \frac{1}{2}$
- $\left(-\frac{l}{2}\cos\varphi^*, \varphi^*\right)$, $\left(\frac{l}{2}\cos\varphi^*, \pi-\varphi^*\right)$ stab. se esistono, cioè quando $\frac{mg}{kl} < \frac{1}{2}$

$$\partial^2 V = \begin{pmatrix} 2k & -kl\,\mathrm{sen}\,\varphi \\ -kl\,\mathrm{sen}\,\varphi & -\frac{l}{4}\left(4ks\cos\varphi + (kl+2mg)\,\mathrm{sen}\,\varphi\right) \end{pmatrix}$$

$$\partial^2 V \,\Big|\, s = -\frac{l}{2}\cos\varphi = \begin{pmatrix} 2k & -kl\,\mathrm{sen}\,\varphi \\ -kl\,\mathrm{sen}\,\varphi & -\frac{l}{4}\left((kl+2mg)\,\mathrm{sen}\,\varphi\right) + \frac{l^2}{2}k\underbrace{\cos^2\varphi}_{1-\mathrm{sen}^2\varphi} \end{pmatrix}$$

$$\partial^2 V(0, \pi/2) = \begin{pmatrix} 2k & -kl \\ -kl & \underbrace{-\frac{l}{4}(kl+2mg)}_{<0} \end{pmatrix} \rightarrow \text{instab.}$$

$$\partial^2 V\left(0, -\frac{\pi}{2}\right) = \begin{pmatrix} 2k & kl \\ kl & \frac{l}{4}(kl+2mg) \end{pmatrix}$$

tr > 0 $\quad$ det $= \frac{kl}{2}(kl + 2mg) - k^2l^2$

$= \frac{kl}{2}(2mg - kl)$ $\quad$ stab. se $\frac{mg}{kl} > \frac{1}{2}$

$\partial^2 V\left(\frac{\ell}{2}\cos\varphi^*, \varphi^*\right) = \longrightarrow \begin{pmatrix} 2k & \frac{k\ell}{2}+mg \\ \frac{k\ell}{2}+mg & \frac{k\ell^2}{2} \end{pmatrix}$ $Tr > 0$

$\cos^2\varphi_* = 1 - \operatorname{sen}^2\varphi_* = 1 - \left(\frac{mg}{k\ell} + \frac{1}{2}\right)^2 =$

$= 1 - \frac{1}{4} - \frac{mg}{k\ell} + \left(\frac{mg}{k\ell}\right)^2$

$\det = k^2\ell^2 - \left(\frac{k\ell}{2}+mg\right)^2 =$

$= \left(\frac{k\ell}{2} - mg\right)\left(\frac{3k\ell}{2} + mg\right)$

stab $\Leftrightarrow$ $\frac{mg}{k\ell} < \frac{1}{2}$ cioè quando $\exists$.

4) $k = \frac{2mg}{3\ell}$ $\quad \frac{mg}{k\ell} = \frac{3}{2} > \frac{1}{2}$ $\quad$ pto stab. è $\left(0, -\frac{\pi}{2}\right)$

$$B = \begin{pmatrix} 2k & k\ell \\ k\ell & \frac{\ell}{4}(k\ell + 2mg) \end{pmatrix} = \begin{pmatrix} 2k & k\ell \\ k\ell & \frac{k\ell^2}{4}\left(1 + 2\frac{mg}{k\ell}\right) \end{pmatrix} = \begin{pmatrix} 2k & k\ell \\ k\ell & k\ell^2 \end{pmatrix}$$

$$A = m\begin{pmatrix} 1 & -\frac{\ell}{2}\operatorname{sen}\varphi \\ -\frac{\ell}{2}\operatorname{sen}\varphi & \ell^2/3 \end{pmatrix}\Bigg|_{\varphi = -\frac{\pi}{2}} = \begin{pmatrix} m & \frac{m\ell}{2} \\ \frac{m\ell}{2} & m\ell^2/3 \end{pmatrix}$$

$$0 = \det(B - \lambda A) = \det m^2\begin{pmatrix} \frac{4g}{3\ell} - \lambda & \frac{2g}{3} - \frac{\lambda}{2}\ell \\ \frac{2g}{3} - \frac{\lambda}{2}\ell & \frac{2g}{3}\ell - \lambda\ell^2/3 \end{pmatrix}$$

$$\cancel{m^2\ell^2}\left\{\left(\frac{4}{3}\frac{g}{\ell} - \lambda\right)\left(\frac{2}{3}\frac{g}{\ell} - \frac{\lambda}{3}\right) - \frac{1}{2}\left(\frac{2}{3}\frac{g}{\ell} - \frac{\lambda}{2}\right)\left(\frac{4}{3}\frac{g}{\ell} - \lambda\right)\right\} = 0$$

$$\left(\frac{4}{3}\frac{g}{\ell} - \lambda\right)\Big[\underbrace{\frac{2}{3}\frac{g}{\ell} - \frac{\lambda}{3} - \frac{1}{3}\frac{g}{\ell} + \frac{\lambda}{4}}_{\frac{1}{3}\frac{g}{\ell} - \frac{\lambda}{12}}\Big] = 0$$

$\lambda_1 = \frac{4}{3}\frac{g}{\ell}$ $\qquad \lambda_2 = 4\frac{g}{\ell}$

5)

$$L = \frac{m}{2}\left(\dot s^2 + \frac{\ell^2\dot\varphi^2}{3} - \ell\dot s\dot\varphi\,\mathrm{sen}\,\varphi\right) - mg\frac{\ell}{2}\mathrm{sen}\,\varphi$$

s coord. ciclica $\quad P_s = m\left(\dot s - \frac{\ell}{2}\dot\varphi\,\mathrm{sen}\varphi\right)$

$$\Rightarrow \dot s = \frac{P_s}{m} + \frac{\ell}{2}\dot\varphi\,\mathrm{sen}\varphi$$

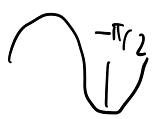
$$L_{eff} = \frac{m}{2}\left(\left(\frac{P_s}{m} + \frac{\ell}{2}\dot\varphi\,\mathrm{sen}\varphi\right)^2 + m\frac{\ell^2}{6}\dot\varphi^2 - \ell\dot\varphi\,\mathrm{sen}\varphi\left(\frac{P_s}{m} + \frac{\ell}{2}\dot\varphi\,\mathrm{sen}\varphi\right)\right) - mgl\,\mathrm{sen}$$

$$- P_s\left(\frac{P_s}{m} + \frac{\ell}{2}\dot\varphi\,\mathrm{sen}\varphi\right)$$

$$= \frac{m\ell^2\dot\varphi^2}{2\cdot 3} - \frac{m}{2}\left(\left(\frac{P_s}{m}\right)^2 + \frac{P_s\ell}{m}\dot\varphi\,\mathrm{sen}\varphi + \frac{\ell^2}{4}\dot\varphi^2\mathrm{sen}^2\varphi\right) - mgl\,\mathrm{sen}\,\varphi$$

$$= \frac{m\ell^2}{2}\left(\frac{1}{3} - \frac{1}{4}\mathrm{sen}^2\varphi\right)\dot\varphi^2 - mgl\,\mathrm{sen}\,\varphi \underbrace{- \frac{P_s\ell}{2}\dot\varphi\,\mathrm{sen}\,\varphi}_{= \frac{d}{dt}\left(\frac{P_s\ell}{2}\cos\varphi\right)}$$

$V_{eff} = mgl\,\mathrm{sen}\,\varphi$



$\varphi_{min} \sim \frac{\pi}{2}$ $\quad \mathrm{sen}\left(-\frac{\pi}{2} + \delta\varphi\right) = -\cos\delta\varphi \simeq -1 + \frac{\delta\varphi^2}{2} + \ldots$

$$L_{lin} = \frac{m\ell^2}{2}\left(\frac{1}{3} - \frac{1}{4}\right)\delta\dot\varphi^2 - \frac{mg\ell}{2}\delta\varphi^2$$

$$\frac{m\ell^2}{2}\frac{1}{12}\delta\dot\varphi^2 - \frac{mg\ell}{2}\delta\varphi^2 \qquad A = \frac{m\ell^2}{12} \quad B = mg\ell$$

$$\lambda = B/A = 12\frac{g}{\ell}$$

ES1 6)
$$q = \sqrt{\tilde{p}}\, e^{\alpha \tilde{q}}$$
$$p = -2\tilde{p}^{\beta} e^{\tilde{q}/2}$$

$$1 = \{q, p\} = \{\sqrt{\tilde{p}}\, e^{\alpha\tilde{q}}, -2\tilde{p}^{\beta} e^{\tilde{q}/2}\} =$$
$$= \left(\alpha\sqrt{\tilde{p}}\, e^{\alpha\tilde{q}}\right)\left(-2\beta\,\tilde{p}^{\beta-1} e^{\tilde{q}/2}\right) - \left(\frac{1}{2\sqrt{\tilde{p}}} e^{\alpha\tilde{q}}\right)\left(-\tilde{p}^{\beta} e^{\tilde{q}/2}\right)$$

$\alpha = -1/2 \qquad \beta = 1/2$
$$= \left(-\frac{1}{2}\right)\left(-2\cdot\frac{1}{2}\right) - \left(\frac{1}{2}\right)(-1) = 1$$

$$q = \sqrt{\tilde{p}}\, e^{-\tilde{q}/2} \qquad\qquad H = \frac{\omega}{2} p q$$
$$p = -2\sqrt{\tilde{p}}\, e^{\tilde{q}/2}$$
$$K = \omega\tilde{p}$$
$$\downarrow$$
$$e^{\tilde{q}/2} = \sqrt{\tilde{p}}/q \rightarrow p = -2\tilde{p}/q \qquad \tilde{q} = 2\log\sqrt{\tilde{p}}/q$$

8)
Funzione generatrice di tipo $F_2(\tilde{p}, q)$
$$p = \frac{\partial F_2}{\partial q} = -2\tilde{p}/q \rightarrow F_2 = -2\tilde{p}\log q + f(\tilde{p})$$
$$\tilde{q} = \frac{\partial F_2}{\partial \tilde{p}} = 2\log\sqrt{\tilde{p}}/q = \log\tilde{p} - 2\log q$$
$$= -2\log q + f'(\tilde{p}) \Rightarrow f(\tilde{p}) = -\tilde{p} + \tilde{p}\log\tilde{p}$$

$$\Rightarrow F_2(\tilde{p}, q) = -2\tilde{p}\log q + \tilde{p}\log\tilde{p} - \tilde{p} \qquad \cancel{x}$$
$$= 2\tilde{p}\log\sqrt{\tilde{p}}/q - \tilde{p}$$

7) Risolvere eq. di $H = \frac{\omega}{2} p q$ $\quad \dot{p} = \frac{\omega}{2} p \quad \dot{q} = -\frac{\omega}{2} q$

$K = \tilde{p}\omega \quad \dot{\tilde{p}} = 0 \quad \dot{\tilde{q}} = \omega \qquad \tilde{p}(t) = \tilde{p}_0 \quad \tilde{q}(t) = \omega t + \tilde{q}_0$

$$p(t) = -2\sqrt{\tilde{p}_0}\, e^{\frac{\omega t}{2} + \tilde{q}_0/2} = p_0 e^{\omega t/2}$$
$$q(t) = \sqrt{\tilde{p}_0}\, e^{-\omega t/2 - \tilde{q}_0/2} = q_0 e^{-\omega t/2} \qquad (*)$$

9) $G(p,q) = -\frac{\omega}{2} p q$

$\delta p = \varepsilon \{ p, -\frac{\omega}{2} p q \} = \frac{\varepsilon \omega}{2} p$ $(\circ)$ oppure espandendo trasf. finite

$\delta q = \varepsilon \{ q, -\frac{\omega}{2} p q \} = -\varepsilon \frac{\omega}{2} q$

Trasf. finite $\quad p = \tilde{p}\, e^{-\omega\varepsilon/2}$ $\quad$ (da $(*)$ o integrando $(\circ)$)

$\qquad q = \tilde{q}\, e^{+\omega\varepsilon/2}$

10) M^2, M_3, H sono <u>cost. del moto</u> in <u>involuzione</u>

5) $H = \frac{\bar{p}^2}{2m} - V(r) \qquad M_i = \sum_{j,k} \epsilon_{ijk} q_j p_k$

$$\{M_i, \bar{p}^2\} = \sum_{j,k,m} \epsilon_{ijk} p_k \{q_j, p_m^2\} =$$
$$= \sum_{i,j,m} 2\epsilon_{ijk} p_k p_m \underbrace{\{q_j, p_m\}}_{\delta_{jm}} = \sum_{i,j} 2\underbrace{\epsilon_{ijk}}_{\text{antisi}} \underbrace{p_k p_j}_{\text{sim}} = 0$$

$$\{M_i, V(r)\} = \sum_h \left(\cancel{\frac{\partial M_i}{\partial q_h} \frac{\partial V(r)}{\partial p_h}} - \frac{\partial M_i}{\partial p_h} \frac{\partial V(r)}{\partial q_h} \right)$$
$$= -\sum_h \left(\sum_j \epsilon_{ijh} q_j \right) V'(r) \frac{\partial \sqrt{q_1^2+q_2^2+q_3^2}}{\partial q_h} =$$
$$= \sum_{h,j} \underbrace{\epsilon_{ijh}}_{\text{antisim}} \overbrace{q_j \frac{V'(r)}{2r} 2 q_h}^{\text{sim.}} = 0$$