SCRITTO 9/1/2024
SOLUZIONE

Domande a risposta multipla

1. b $\quad \dfrac{1+i}{3-2i} = \dfrac{(1+i)\cdot(3+2i)}{(3-2i)\cdot(3+2i)} = \dfrac{1+5i}{13}$

2. a $\quad \left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)^{13} = \left(\cos\left(\dfrac{\pi}{6}\right) + i\sin\left(\dfrac{\pi}{6}\right)\right)^{13} =$

$$= \cos\left(\frac{13}{6}\pi\right) + i\sin\left(\frac{13}{6}\pi\right) =$$

$$= \cos\left(2\pi + \frac{\pi}{6}\right) + i\sin\left(2\pi + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

3. c

4. c $\quad (A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & a^3-1 & 1 \end{array}\right)$

$\Rightarrow$ SL compatibile se e solo se $a^3-1 \neq 0$,

cioè $\forall\, a \neq 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

5 a $\quad U \cap W = \{0\}$ perché $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}\right)$

6. C

$$A^{-1} = \frac{1}{\det A} {}^t\mathrm{cof}(A) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

7. B Dal teorema della dimensione abbiamo che $2 = \dim(\mathbb{R}^2) = \dim(\mathrm{Ker}(f)) + rg(f)$

$$\Rightarrow \quad rg(f) \leq 2 < 3.$$

8. B $M^{\mathcal{E}}_{\mathcal{E}}(f) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ non è simmetrica.

$\mathcal{E}$ canonica

9. B A simmetrica $\Rightarrow$ $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$$P_A(t) = \det\begin{pmatrix} a-t & b \\ b & c-t \end{pmatrix} = t^2 - t(a+c) + ac - b^2$$

Se $a = 1$, $c = -1$, $b = i \Rightarrow Sp(A) = \{0\}$, $m_a(0) = 2$, $m_g(0) = 1$

$\Rightarrow$ tale A non è diagonalizzabile!

10. a

Giacitura di $\mathbb{r} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{l} 2x - y + z = 0 \\ x + y + z = 0 \end{array} \right\} =$

$= \mathrm{Span} \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

$$\Updownarrow$$

$$\begin{cases} -3y - z = \\ x + y + z = 0 \end{cases}$$

Giacitura di $\mathbb{P} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; 3x + 2z = 0 \right\}$

Si vede che $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ soddisfa l'equazione

$3x + 2z = 0 \Rightarrow \mathbb{r}$ e $\mathbb{P}$ sono paralleli

Esercizi

1. a) Risolviamo il SLO per trovare una base di U :

$$\begin{cases} x_1 + x_2 + x_4 = 0 \\ 2x_1 - x_2 - x_3 = 0 \\ x_2 + x_4 = 0 \end{cases} \xrightarrow{OE1} \begin{cases} x_1 + x_2 + x_4 = 0 \\ x_2 + x_4 = 0 \\ -3x_2 - x_3 - 2x_4 = 0 \end{cases}$$

$$\xrightarrow{OE3} \begin{cases} x_1 + x_2 + x_4 = 0 \\ -3x_2 - x_3 - 2x_4 = 0 \\ x_2 + x_4 = 0 \end{cases} \xrightarrow{OE3} \begin{cases} x_1 + x_2 + x_4 = 0 \\ x_2 + x_4 = 0 \\ -x_3 + x_4 = 0 \end{cases}$$

$$\Rightarrow U = \mathrm{Span}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \dim(U) = 1 .$$

$$V = \mathrm{Span}\left(\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}\right).$$

OSS: v_1, v_2 l.i. $\Rightarrow \dim(V) = 2$.

$$U + V = \mathrm{Span}\left(\underbrace{\begin{pmatrix}0\\-1\\1\\1\end{pmatrix}}_{\in V}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}\right) =$$

$$= \mathrm{Span}\left(\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}\right) = V$$

$\Rightarrow U \cap V = U$, $U + V = V$

$\dim(U \cap V) = 1$, base di $U \cap V$: $\left\{\begin{pmatrix}0\\-1\\1\\1\end{pmatrix}\right\}$,

$\dim(U+V)=2$, base di $U+V$: $\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}\right\}$.

e) Aggiungiamo alla base trovata di $U+V$ i primi due vettori della base canonica

$$rg\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & -1 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}=4$$

$\Rightarrow \left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\}$ è base di $\mathbb{R}^4$.

(c) $U \cup V = V$ in questo caso

$\Rightarrow$ è un sottospazio vettoriale.

2. (a)

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\Rightarrow f(e_1) = \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}4\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}3\\2\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad f(e_2) = \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}4\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}3\\2\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad f(e_3) = -\frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}4\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}3\\2\\0\end{pmatrix} = \begin{pmatrix}3\\1\\0\end{pmatrix} \Rightarrow M_{\mathcal{E}}^{\mathcal{E}}(f) = \begin{pmatrix}1&0&3\\0&1&1\\0&0&0\end{pmatrix}$$

(b) $M^{\mathcal{E}}_{\mathcal{E}}(f)$ è a scala, $rg\left(M^{\mathcal{E}}_{\mathcal{E}}(f)\right) = 2$

$\Rightarrow rg(f) = 2$

$im(f) = Span\left(f(e_1), f(e_2)\right) = \langle e_1, e_2 \rangle$

$\Rightarrow \{e_1, e_2\}$ è base di $im(f)$

$\dim Ker(f) = 3 - 2$ (teorema della dimensione)

$$Ker(f) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = Span \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \right\}$ è base di $Ker(f)$.

(c) $P_f(t) = \det \begin{pmatrix} 1-t & 0 & 3 \\ 0 & 1-t & 1 \\ 0 & 0 & -t \end{pmatrix} = (1-t)^2 \cdot (-t)$

$\Rightarrow Sp(f) = \{1, 0\}$ $m_a(1) = 2$, $m_a(0) = 1$.

$\Rightarrow m_g(0) = 1$.

$m_g(1) = 3 - rg \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} = 2 = m_a(1)$

$\Rightarrow$ f è diagonalizzabile.

$V_1 = Span(e_1, e_2)$, $V_0 = Span \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

$\Rightarrow \left\{ e_1, e_2, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \right\}$ è una base di $\mathbb{R}^3$ che diagonalizza f.