GEOMETRIA SCRITTO 19/2/2024

SOLUZIONE

Domande

1. (c) $|-\sqrt{3} - i| = 2 \Rightarrow -\sqrt{3} - i = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$

$$= 2\left(\cos\frac{7\pi}{6} + i \sin\frac{7\pi}{6}\right)$$

2. (a) $\frac{1}{z} = \frac{\bar{z}}{|z|^2} \Rightarrow \overline{\left(\frac{1}{z}\right)} = \frac{z}{|z|^2} = \frac{1}{\bar{z}}$

3. (b) $(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & a^4-1 & 1 \\ 0 & 2 & 3 & 4 \end{array}\right) \xrightarrow{OE} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & a^4-1 & 1 \end{array}\right)$

compatibile $\Leftrightarrow a^4 \neq 1 \Leftrightarrow a \neq \pm 1, \pm i$

4. (a) $\det(A)=0 \Rightarrow rg(A) < n$

$\Rightarrow L_A : \mathbb{K}^n \to \mathbb{K}^n$ non è suriettiva

$\Rightarrow \exists\, b \in \mathbb{K}^n,\ b \notin im(L_A)$

5. (c) $\begin{pmatrix} 0 \\ -a \end{pmatrix} \in S \quad \forall a$, ma $2\begin{pmatrix} 0 \\ -a \end{pmatrix} \notin S$

se $a \neq 0 \Rightarrow S$ non è sottospazio

se $a \neq 0$.

Quando $a=0$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$ ma $\begin{pmatrix} 2 \\ 2 \end{pmatrix} \notin S$.

6. (c) I punti sono allineati

$$\iff \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix} \text{ e } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix} \text{ sono l.d.}$$

$$\iff \underbrace{\operatorname{rg} \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ -k & 1-k \end{pmatrix}}_{\shortparallel \; OE} = 1$$

$$\operatorname{rg} \begin{pmatrix} -1 & -1 \\ 0 & -1 \\ -k & 1-k \end{pmatrix} \underset{OE}{=} \operatorname{rg} \begin{pmatrix} -1 & -1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = 2 \quad \forall k$$

7. (a) $\dim\left(\mathrm{Span}\left(\begin{pmatrix}0\\k\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\right)=\dim\left(\mathrm{Span}\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\right)=2$

$\forall k$.

$$rg\begin{pmatrix}0 & 1 & 1 & 0\\ k & 0 & 1 & 1\\ 1 & 0 & 1 & 0\end{pmatrix}=3 \quad \forall k$$

$$\Rightarrow \dim\left(\mathrm{Span}\left(\begin{pmatrix}0\\k\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right)+\mathrm{Span}\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\right)=3 \quad \forall k$$

$$\Rightarrow \dim\left(\mathrm{Span}\left(\begin{pmatrix}0\\k\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\cap\mathrm{Span}\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\right)=1$$

$\forall k$

8. (b) Se f è lineare e $f(1)=1$

$$\Rightarrow f(-1)=f(-1\cdot 1)=(-1)f(1)=-1.$$

9. (b) $$P_A(t)=\det\begin{pmatrix} 1-t & i \\ i & -1-t \end{pmatrix}=t^2-1+1=t^2$$

$$m_a(0)=2 \quad , m_g(0)=2-rg(A)=1.$$

10. (a) $$f\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad f\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow M^{\mathcal{B}}_{\mathcal{B}}(f)=\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{simmetrica}} \quad \text{dove} \quad \underset{\substack{\uparrow \\ \text{ortonormale}}}{\mathcal{B}}=\left\{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\right\}$$

Esercizi

1\. (a) $\mathrm{Ker}(f) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3 \;\middle|\; \begin{matrix} y + z = 0, \\ -x + z = 0 \end{matrix} \right\}$

$$= \mathrm{Span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ è una base di $\mathrm{Ker}(f)$.

$\Rightarrow \dim(\mathrm{im}(f)) = 3 - 1 = 2$

$$\mathrm{im}(f) = \mathrm{Span}\left(f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) =$$

$= \mathrm{Span}\left(\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \Rightarrow \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ è una base di $\mathrm{im}(f)$.

Notiamo che $\mathrm{Ker}(f) \cap \mathrm{im}(f) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$$\Rightarrow \quad \mathbb{C}^3 = \mathrm{Ker}(f) \oplus \mathrm{im}(f)$$

perché $\underbrace{\mathrm{Ker}(f)}_{\text{ha dim } 1} + \underbrace{\mathrm{im}(f)}_{\text{ha dim } 2} = \mathbb{C}^3$

(b) $M^{\mathcal{C}}_{\mathcal{C}}(f) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ $\mathcal{C}$ base canonica.

$$P_f(t) = \det \begin{pmatrix} -t & 1 & 1 \\ 0 & -t & 0 \\ -1 & 0 & 1-t \end{pmatrix} = (-t)\cdot(t^2 - t + 1)$$

$$= -t \cdot \left(t - \frac{1+\sqrt{-3}}{2}\right) \cdot \left(t - \frac{1-\sqrt{-3}}{2}\right)$$

$$\Rightarrow \quad Sp(f) = \left\{ 0, \ \frac{1 \pm i\sqrt{3}}{2} \right\}$$

$\Rightarrow$ f è diagonalizzabile perché ha 3 autovalori distinti.

Autospazi:

$$V_0 = Ker(f) = Span \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$V_{\frac{1 \pm i\sqrt{3}}{2}} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3 \ \middle| \ \begin{pmatrix} -\frac{1 \pm i\sqrt{3}}{2} & 1 & 1 \\ 0 & -\frac{1 \pm i\sqrt{3}}{2} & 0 \\ -1 & 0 & 1 - \frac{1 \pm i\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= Span \begin{pmatrix} 1 \\ 0 \\ \frac{1 \pm i\sqrt{3}}{2} \end{pmatrix}$$

$$\Rightarrow \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \frac{1+i\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \frac{1-i\sqrt{3}}{2} \end{pmatrix} \right\}$$

$$M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{1-i\sqrt{3}}{2} \end{pmatrix}$$

2. (a) g è simmetrica perché $M_{\mathcal{E}}(g)$ è simmetrica.

$$g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= \begin{pmatrix} x & y & z \end{pmatrix} \cdot \begin{pmatrix} x+y \\ x+2y \\ 2z \end{pmatrix} = x^2 + 2xy + 2y^2 + 2z^2$$

$$= (x+y)^2 + y^2 + 2z^2 \geq 0 \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

ed è $=0$ se e solo se $z=0, y=0, x=0$.

$\Rightarrow$ g è definita positiva.

(b) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; (x \;\; y \;\; z) \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x + y = 0 \right\} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

(c) Per determinare $P_W^{\perp}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ dobbiamo trovare una base ortonormale di W.

Applichiamo Gram-Schmidt a $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\left\| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\| = \sqrt{g\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)} = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\tilde{v}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix} - g\left(\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} =$$

$$= \begin{pmatrix}0\\0\\1\end{pmatrix} - \underbrace{\begin{pmatrix}0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1&1&0\\1&2&0\\0&0&2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}_{\begin{pmatrix}0 & 0 & 2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} = 0} \cdot \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\|\tilde{v}_2\| = \sqrt{2} \implies v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\implies P_W^{\perp}\begin{pmatrix}0\\1\\0\end{pmatrix} = g\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} + g\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}1&1&0\\1&2&0\\0&0&2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} + \underbrace{\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}1&1&0\\1&2&0\\0&0&2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}_{0}\cdot\begin{pmatrix}0\\0\\1/2\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$