GEO SCRITTO 16/7/2024

Domande

1. a)

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{13}=\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)^{13}$$

$$=\cos\left(\frac{13\pi}{3}+i\sin\frac{13\pi}{3}\right)=$$

$$=\cos\left(\frac{12+1}{3}\pi\right)+i\sin\left(\frac{12+1}{3}\pi\right)$$

$$=\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)$$

2. c)

3. c)

4. a)

5. a)

6. b)

7. a)

8. b)

9. b) Ad es $A=\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ ha $P_A(t) = \det\begin{pmatrix} 1-t & i \\ i & -1-t \end{pmatrix}$

$= t^2 \Rightarrow Sp(A) = \{0\}, \; m_a(0) = 2, \; m_g(0) = 1.$

10. a) $\left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 0 & 2 & 0 \end{array}\right) \xrightarrow{OE} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & 3/2 & 1/2 & -3/2 \\ 0 & 3/2 & 1/2 & -3/2 \end{array}\right)$

Esercizi

1. (a) $Sp(f) = \{ 0 , 1 \}$

Ker(f) è autospazio relativo all'autovalore 0

$m_g(0) = 2, \quad m_g(1) = 1 \Rightarrow 3 = \dim Ker(f) + \dim V_1$

$V_1 \cap Ker(f) = \{0\} \Rightarrow \mathbb{R}^3 = Ker(f) \oplus V_1$

b) Sì

Base di Ker(f): $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

Base di V_1: $\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$

$$\Rightarrow \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\} \text{ diagonalizza } f$$

$$M^{\mathcal{B}}_{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) f non è autoaggiunto perché $\ker(f)$ e V_1 non sono ortogonali tra di loro.

2. (a)

g è simmetrica perché $M_e(g)$ è simmetrica.

$$g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = (x \quad y \quad z)\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= (x \quad y \quad z)\begin{pmatrix} x+y \\ x+2y \\ 2z \end{pmatrix} = x^2 + xy + yx + 2y^2 + 2z^2$$

$$= (x+y)^2 + y^2 + 2z^2 \geq 0 \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ ed è } = 0$$

se e solo se $\begin{cases} x+y=0 \\ y=0 \\ z=0 \end{cases} \Longleftrightarrow x=y=z=0.$

$\Rightarrow$ g è definita positiva $\Rightarrow$ g è un prodotto scalare.

b) $$W = \left(\mathrm{Span}\begin{pmatrix}0\\0\\1\end{pmatrix}\right)^{\perp} = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, (x\ y\ z)\begin{pmatrix}1&1&0\\1&2&0\\0&0&2\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = 0\right\}$$

$$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, 2z = 0\right\} = \mathrm{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$$

c) Troviamo una base ortonormale di W:

$$\left\|\begin{pmatrix}1\\0\\0\end{pmatrix}\right\| = \sqrt{g\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right)} = 1 \Rightarrow v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\tilde{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} - g\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - (0\ 1\ 0)\begin{pmatrix}1&1&0\\1&2&0\\0&0&2\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\| \tilde{v}_2 \| = \sqrt{1} = 1 \Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$P^1_W \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = g\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + g\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$