GEO SCRITTO 29/7/2024

Domande

1. (c) $$P_A(t) = \det\begin{pmatrix} -t & 0 & 1 \\ 0 & 1-t & 0 \\ -1 & 0 & -t \end{pmatrix} = -t(1-t)(-t) +$$
$$+(1-t) = (1-t)(t^2+1) = (1-t)(t-i)(t+i)$$
$$\Rightarrow \mathbb{K} = \mathbb{C}$$

2. (a) $$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = i \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$$

3. (b) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ sono l. i.

4. (b) Una base di V è

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

5. (c) $f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

6. (a)

7. (b) $|z| = \sqrt{81 + 3 \cdot 81} = 18$

$$\frac{z}{|z|} = -\frac{1}{2} - i\,\frac{3^{5/2}}{2 \cdot 3^2} = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2}$$

$$= \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}$$

8. (a)

9. (b)

10. (a)

<u>Esercizi</u>

1(a) $\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} \Rightarrow f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1/2\\1\end{pmatrix} + \begin{pmatrix}1/2\\-1/2\\0\end{pmatrix} + \begin{pmatrix}1/2\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} \Rightarrow f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1/2\\-1\end{pmatrix} + \begin{pmatrix}1/2\\-1/2\\0\end{pmatrix} - \begin{pmatrix}1/2\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\-2\end{pmatrix}$

$\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} \Rightarrow f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1/2\\1\end{pmatrix} + \begin{pmatrix}-1/2\\1/2\\0\end{pmatrix} - \begin{pmatrix}1/2\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

$$\Rightarrow M^{\mathcal{E}}_{\mathcal{E}}(f) = \begin{pmatrix}1 & 0 & -1\\ 0 & -1 & 1\\ 2 & -2 & 0\end{pmatrix}$$

(b) $$\mathrm{Ker}(f) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3 \;\middle|\; \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x = z,\ y = z,\ x = y \right\} = \mathrm{Span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow rg(f) = 2$, una base di $im(f)$ è $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \right\}$.

(c) $$P_f(t) = \det \begin{pmatrix} 1-t & 0 & -1 \\ 0 & -1-t & 1 \\ 2 & -2 & -t \end{pmatrix} = (1-t)(t^2+t+2) +$$

$$+2(-1-t) = t^2+t+2 - t(t^2+t+2+2) - 2$$

$$= \cancel{t^2} + t + \cancel{2} - t^3 - \cancel{t^2} - 4t - \cancel{2}$$

$$= -t(t^2+3) = -t(t+i\sqrt{3})(t-i\sqrt{3})$$

$\Rightarrow f$ è diagonalizzabile perché ha 3 autovalori distinti

2. (a) $M_e(g)$ è simmetrica $\Rightarrow$ g è simmetrica

$$g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x + y \\ x + 2y + z \\ y + 2z \end{pmatrix} = x^2 + xy + yx + 2y^2 + yz + zy + 2z^2$$

$$= (x+y)^2 + (y+z)^2 + z^2 \geq 0 \quad \forall x, y, z \text{ ed è } = 0$$

$$\Leftrightarrow x + y = y + z = z = 0 \Leftrightarrow x = y = z = 0$$

Quindi g è definita positiva.

(e) $$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0, \right.$$

$$\left. \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \right\} =$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \underbrace{x + y = 0, \quad x + 2y + 2z = 0}_{\Updownarrow \atop x+y=0, \quad y+2z=0} \right\}$$

$$= \operatorname{Span} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

(c) Una base ortonormale di W è

$$\left\{ \frac{1}{\left\| \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\|} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$$

norma di g

$$\Rightarrow P^{\perp}_{W} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = g\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right) \cdot \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \underbrace{\left(-\sqrt{2} + \frac{1}{\sqrt{2}} \right)}_{-1/\sqrt{2}} \cdot \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1/2 \end{pmatrix}.$$