Problema FN 30/01/2018 - II

E129 bis

Altro modo. Risolviamo l'eq. differenziale $\ddot{x} - \frac{g}{L}x = 0$

$$\lambda^2 - \frac{g}{L} = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{\frac{g}{L}} \Rightarrow x = A e^{\sqrt{\frac{g}{L}}t} + B e^{-\sqrt{\frac{g}{L}}t}$$

$$v(t) = \sqrt{\frac{g}{L}} A e^{\sqrt{g/L}\,t} - \sqrt{\frac{g}{L}} B e^{-\sqrt{\frac{g}{L}}t}$$

$x(0)=a \Rightarrow A+B=a$

$v(0)=0 \Rightarrow \sqrt{\frac{g}{L}}(A-B)=0 \Rightarrow A=B$

Quindi $A=\frac{a}{2}$; $B=\frac{a}{2}$ e le eq. diventano

$$x(t) = \frac{a}{2}\left(e^{\sqrt{g/L}\,t} + e^{-\sqrt{g/L}\,t}\right) = a\cosh\left(\sqrt{\frac{g}{L}}\,t\right)$$

$$v(t) = \sqrt{g/L}\,\frac{a}{2}\left(e^{\sqrt{g/L}\,t} - e^{-\sqrt{g/L}\,t}\right) = \sqrt{g/L}\;a\sinh\left(\sqrt{g/L}\,t\right)$$

Invertiamo la funzione $x = \cosh t = \frac{e^t+e^{-t}}{2} = \frac{e^t+1/e^t}{2} = \frac{e^{2t}+1}{2e^t} \Rightarrow$

$e^{2t} - 2x e^t + 1 = 0$; ponendo $e^t = \lambda$ si ha $\lambda^2 - 2x\lambda + 1 = 0$

$\Rightarrow \lambda_{1,2} = x \pm \sqrt{x^2-1}$; $t \geq 0 \Rightarrow \lambda \geq 1 \Rightarrow \lambda_{1,2} = x + \sqrt{x^2-1} \Rightarrow t = \ln\left(x+\sqrt{x^2-1}\right)$

Quindi $\sqrt{g/L}\,t = \ln\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2}-1}\right) = \ln\frac{x+\sqrt{x^2-a}}{a} \Rightarrow$ (ponendo $x = L$)

$$\sqrt{g/L}\,t^* = \ln\frac{L+\sqrt{L^2-a^2}}{a} \Rightarrow t^* = \sqrt{\frac{L}{g}}\ln\left[\frac{L+\sqrt{L^2-a^2}}{a}\right]$$

$$v(t^*) = \sqrt{g/L}\,\frac{a}{2}\left(e^{\ln\frac{L+\sqrt{L^2-a^2}}{a}} - e^{-\ln\frac{L+\sqrt{L^2-a^2}}{a}}\right)$$

$$= \sqrt{g/L}\,\frac{a}{2}\left(\frac{L+\sqrt{L^2-a^2}}{a} - \frac{a}{L+\sqrt{L^2-a^2}}\right) = \sqrt{g/L}\,\frac{\cancel{a}}{2}\,\frac{\left(L+\sqrt{L^2-a^2}\right)^2-a^2}{\cancel{a}\left[L+\sqrt{L^2-a^2}\right]} =$$

$$= \sqrt{\frac{g}{L}}\cdot\frac{1}{2}\left(\frac{L^2+L^2-a^2+2L\sqrt{L^2-a^2}-a^2}{L+\sqrt{L^2-a^2}}\right) = \sqrt{\frac{g}{L}}\cdot\frac{1}{2}\,\frac{2L^2+2L\sqrt{L^2-a^2}-2a^2}{L+\sqrt{L^2-a^2}} =$$

$$= \sqrt{\frac{g}{L}}\cdot\frac{1}{\cancel{2}}\cdot\cancel{2}\,\frac{L^2+L\sqrt{L^2-a^2}-a^2}{L+\sqrt{L^2-a^2}} = \sqrt{\frac{g}{L}}\cdot\sqrt{L^2-a^2}\,\frac{\cancel{\left(L+\sqrt{L^2-a^2}\right)}}{\cancel{L+\sqrt{L^2-a^2}}}$$

$$v(t^*) = \sqrt{\frac{g}{L}}\sqrt{L^2-a^2}$$