Condensed Matter Physics I
II intermediate test a.y. 2025/26 - 18 December 2025
(*Time: 1:30 h*)

Exercise 1: *Bloch functions*

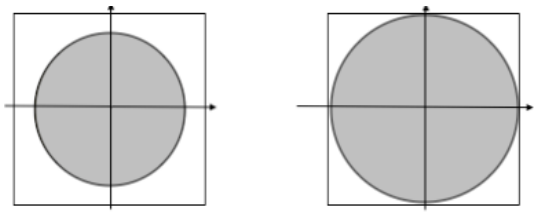
1. Does a Bloch function have the same periodicity of the Bravais lattice or not? Justify your answer. No, since, upon translation of $\mathbf{R}$, it is multiplied by a phase $\exp(i\mathbf{k}\cdot\mathbf{R})$
2. Does the spacial probability density for Bloch electrons have the periodicity of the corresponding Bravais lattice or not? Justify your answer. Yes, since the multiplying phase has modulus=1
3. What is $u_{n,\mathbf{k}}(\mathbf{r})$ in case of free electrons? A constant, the choice depending on the normalization

Exercise 2: *Density of electronic states*

A two-dimensional material has a "relativistic" dispersion relation $E(\mathbf{k}) = \alpha|\mathbf{k}|$. An example of such a system is graphene.

1. Calculate the velocity in the semiclassical model. $\mathbf{v} = \frac{\alpha}{\hbar}\,\hat{\mathbf{k}}$
2. Calculate the density of electronic states. $g(E) = \frac{E}{\pi\alpha^2}$

Exercise 3: *Nearly free electrons*



Consider a 2D square Bravais lattice of parameter a. The two figures represents the first Brillouin zone with Fermi circles ("empty" lattice model) corresponding to two different electron densities.

1. Focus on the left plot: estimate the (integer) number of electrons per unit cell. Z=1
2. Then consider the plot on the right and give the exact value of the number of electrons (not necessarily integer) per unit cell in this case. Z=$\pi/2$
3. If such number is between two integers, Z_A and Z_B, this means that it could represent the average electron number per cell of an alloy $A_{1-x}B_x$. Calculate x. x=0.57 or x=0.43, according whether one considers $Z_A \leq Z_B$ or viceversa
4. Consider a magnetic field perpendicular to the lattice. Which are the crystalline momentum velocities $\dot{\mathbf{k}}$ and the periods T of the orbit in the two cases? $\dot{\mathbf{k}} = \omega_c(-k_F|_y, k_F|_x)$, $|\dot{\mathbf{k}}| = \omega_c k_F$ with k_F=$\sqrt{2\pi}/a$ for $Z = 1$ and π/a for $Z = 2$; period $= 2\pi/\omega_c$ in any case

5. Now consider to have a small perturbing potential, tight-binding like but small:

$$V(x,y) = 2V_1(\cos(2\pi x/a) + cos(2\pi y/a)) + 4V_2\cos(2\pi x/a)cos(2\pi y/a)$$

Use the weak potential model to find the energy of the lowest electronic state at $\mathbf{k} = (\pi/a, 0)$. Writing sin and cos in terms of complex exponentials, it is evident that the Fourier components of the potential include only the following $\mathbf{K}$ reciprocal lattice vectors: $\pm\mathbf{b_1} = \pm 2\pi/a(1,0)$, $\pm\mathbf{b_2} = \pm 2\pi/a(0,1)$, $\pm(\mathbf{b_1} \pm \mathbf{b_2})$. At $\mathbf{k} = (\pi/a, 0)$ only the component for $\mathbf{b_1}$ is involved, therefore $E_-(\mathbf{k} = (\pi/a, 0)) = E^0(\mathbf{k} = (\pi/a, 0)) - V_1 = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 - V_1$