Perfect Incompressible Fluids

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These lecture notes are designed for a 48-hour course aimed at second-year master's students, offering a comprehensive journey into the mathematical theory of the incompressible Euler equations. Our goal is to bridge the gap between the classical theory of smooth solutions and the modern, often surprising, world of weak solutions, where many of the most challenging and exciting problems in fluid dynamics reside.

The notes are divided into two parts. Part I is dedicated to the theory of *strong solutions*. We begin in Chapter 2 by establishing the foundational local-in-time existence and uniqueness result in Sobolev spaces H^s for sufficiently smooth initial data, following the classical work of Kato [9]. This provides a solid framework to investigate the central question of global regularity versus finite-time blow-up. We will then, in Chapter 3, derive the celebrated continuation criteria, including the Beale-Kato-Majda criterion, which connects the potential formation of singularities to the behavior of the fluid's vorticity. This analysis culminates in proving the global existence of strong solutions in two spatial dimensions, a result that highlights the profound difficulty of the corresponding open problem in three dimensions.

Part II ventures into the richer and more complex landscape of *weak solutions*. This framework is essential for addressing physically relevant scenarios with non-smooth initial data, such as the evolution of vortex patches, and for describing fluid motion beyond the point of a potential singularity. We will explore two distinct regimes of weak solutions. First, in Chapter 4, we delve into the "tame" setting of Yudovich's theory [15], which establishes global well-posedness in two dimensions for initial vorticity in $L^1 \cap L^\infty$. This theory is powerful enough to rigorously treat the vortex patch problem, and we will present in Chapter 5 the remarkable persistence of regularity result by Chemin [4], which shows that the boundary of a smooth patch remains smooth for all time.

Finally, in Chapter 6, we will confront the "wild" side of weak solutions by introducing the powerful geometric method of *convex integration*, as pioneered in fluid dynamics by De Lellis and Székelyhidi [5]. This technique, rooted in the work of Gromov [6], reveals an astonishing degree of non-uniqueness, allowing for the construction of infinitely many solutions with prescribed, and at times physically paradoxical, kinetic energy profiles. Chapters 4 to 6 are largely inspired by lecture notes from courses given by the author at the University of Trieste (2017 and 2019) and the Basque Center for Applied Mathematics (BCAM, 2017).

Contents

Notation

In the present lecture notes all vectors are considered to be row vectors, unless stated otherwise. Given V a smooth vector field and with V_i its components we denote with $\nabla V := \left(\partial_i V_j\right)_{ij}$ its *gradient matrix*, while with $JV := (\nabla V)^\intercal$ its *Jacobian matrix*. We denote with $\mathscr{S}(\Omega;\mathbb{K})$ the class of Schwartz functions from Ω to $\mathbb{K}$ and with $\mathscr{S}_0(\Omega;\mathbb{K})$ the class of Schwartz functions with zero moments of arbitrary order. We denote with $D := -\mathrm{i}\nabla$. We use the convention of summation on repeated indexes. Throughout, the space dimension is denoted by $\mathtt{d} \in \mathbb{N}^*$, and the ambient space is $\mathbb{R}^{\mathtt{d}}$. For vectors $a, b \in \mathbb{R}^{\mathtt{d}}$, we write $a \cdot b$ for the Euclidean scalar product and $|a|$ for its norm. For matrices $A, B \in \mathbb{R}^{\mathtt{d}\times\mathtt{d}}$, $A^\intercal$ is the transpose, Id is the identity, and the (row–column) tensor product is defined by

$$(a \otimes b)_{ij} := a_i\, b_j.$$

Chapter 1

Introduction

1.1 A glimpse into continuum mechanics: budget laws.

Fluids, as well as any object in the universe, are subjected to *motions*. The concept of motion is very intuitive and well understood by anybody, but a mathematical formulation of such natural concept may not be straightforward. We shall adopt the following definition

Definition 1.1. Let $\mathrm{d} \in \mathbb{N}^*$, Ω a $\mathcal{C}^{0,1}$ subset of $\mathbb{R}^{\mathrm{d}}$, $T > 0$ a *motion* is a $\mathcal{C}^3$ map $X : [0,T] \times \Omega \to \mathbb{R}^{\mathrm{d}}$ such that for every $t \in [0,T]$ the map $X(t, \bullet)$ is invertible . We define

- $\Omega(t) := X(t, \Omega)$ as the set which the body occupies at time t,
- $\mathcal{T} := \{(t,x) : t \in [0,T], x \in \Omega(t)\}$ is the set of *trajectories*,
- the map $X^{-t} := (X(t,\bullet))^{-1} : \Omega(t) \to \Omega$ is called the *reference map* or *back-to-label map*,
- the *velocity flow* is defined as $u(t, X(t,x)) := \frac{\mathrm{d}}{\mathrm{d}t} X(t,x) = \dot{X}(t,x)$.

We shall many times refer to a generic motion X as a *flow*.

1.1.1 The mass conservation

In the present section we assume $\mathrm{d} = 3$. We denote with V the 3D Lebesgue measure. One of the most important properties of bodies is that they possess mass. We here consider bodies whose mass is distributed continuously. No matter how severely such a body is deformed, its mass is the integral of a density field. Let us make these concept formal, a *reference density* is a function

$$\rho_0 \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}_+),$$

and the *mass* of the body Ω is defined as

$$m(\Omega) := \int_\Omega \rho_0(x) \, \mathrm{d}V(x).$$

Indeed the deformed body $\Omega(t)$ possesses as well a mass, and we assume that, no matter the deformation $X(t,\bullet)$, the mass is an invariant of the motion, i.e.

$$m(\Omega(t)) \equiv m(\Omega), \qquad \forall\ t > 0,$$

and since the mass can always be described as the integral of a density field

$$m(\Omega(t)) =: \int_{\Omega(t)} \rho(t,y) \, \mathrm{d}V(y). \tag{1.1}$$

We want to express such integral in terms of the space variable $x \in \Omega$, let us use the compact notation

$$X := X(t,x), \tag{1.2}$$

we have[1], using a standard substitution argument

$$\begin{aligned} m(\Omega(t)) &= \int_{\Omega(t)} \rho(t, X(t,x))\, \mathrm{d}V(X(t,x)), \\ &= \int_{\Omega} \rho(t, X(t,x)) \det(J_x X(t,x))\, \mathrm{d}V(x). \end{aligned} \tag{1.3}$$

Since Ω is arbitrary, comparing Eqs. (1.1) and (1.3), we can hence deduce the following relation which describes the density of the deformed body $\Omega(t)$ in terms of the reference density ρ_0 :

$$\rho_0(x) = \rho(t, X(t,x)) \det(J_x X(t,x)) \qquad \Leftrightarrow \qquad \rho(t,y) := \frac{\rho_0\left(X^{-t}(y)\right)}{\det\left(J_x X\left(t, X^{-t}(y)\right)\right)}. \tag{1.4}$$

Being the mass constant we deduce hence that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \rho(t,X)\ \mathrm{d}V(X) = 0,$$

applying Reynold's transport theorem (see [7, p. 78] for a proof)

Theorem 1.2 (Reynold's transport theorem)**.** *Let X be a motion as in Definition 1.1 (cf.* (1.2)*), Φ be a smooth (scalar or vector) field, we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \Phi(t,X)\ \mathrm{d}V(X) = \int_{\Omega(t)} \left(\dot{\Phi} + \Phi \operatorname{div}_X u\right)(t,X)\ \mathrm{d}V(X).$$

we deduce that

$$\int_{\Omega(t)} \left(\frac{\mathrm{d}}{\mathrm{d}t} \rho(t,X) + \rho(t,X)\ \operatorname{div}_X u(t,X) \right) \mathrm{d}V(X) = 0, \tag{1.5}$$

by definition of the velocity flow we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} \rho(t,X) = \frac{\mathrm{d}}{\mathrm{d}t} \rho(t, X(t,x)) = \partial_t \rho(t, X(t,x)) + u(t, X(t,x)) \cdot \nabla_X \rho(t, X(t,x)), \tag{1.6}$$

but, since Ω was arbitrary combining Eqs. (1.5) and (1.6) we finally deduce the *conservation of mass equation*

$$\partial_t \rho + \operatorname{div}\left(\rho\, u\right) = 0. \tag{1.7}$$

Definition 1.3. Let X be a motion as per Definition 1.1, let $\omega \subset \Omega$ an arbitrary $\mathcal{C}^{0,1}$ subdomain and $\omega(t) := X(t,\omega)$. Let us define $\operatorname{Vol}(\omega(t)) := \int_{\omega(t)} \mathrm{d}V(x)$. We say that a motion is *isochoric* or *volume-preserving* if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Vol}(\omega(t)) = 0\ ,$$

for all $\omega \subset \Omega$.

Lemma 1.4. *1. A motion is isochoric if and only if*

$$\operatorname{div} u = 0; \tag{1.8}$$

2. for all isochoric motion we have that

$$\det(J_x X(t,x)) = 1\ .$$

Proof. 1. It is a direct application of Theorem 1.2.

2. Indeed by a change of variables we have that for all $\omega \subset \Omega$ such that $\omega(t) = X(t,\omega)$

$$\operatorname{Vol}(\omega(t)) = \int_{\omega(t)} \mathrm{d}V(y) = \int_{\omega} \det(X(t,x))\, \mathrm{d}V(x)\ , \qquad \operatorname{Vol}(\omega) = \int_{\omega} \mathrm{d}V(x)\ .$$

But being the motion isochoric we have that $\operatorname{Vol}(\omega(t)) = \operatorname{Vol}(\omega)$, and since ω is an arbitrary subset of Ω we obtain the claim.

□

Corollary 1.5. *Let X be an isochoric motion, than the density function is given by the formula*

$$\rho(t,y) = \rho_0\left(X^{-t}(y)\right).$$

Proof. It suffice to apply Lemma 1.4 to (1.4). □

[1] We use the notation JV for the Jacobian matrix of V

1.1.2 The conservation of the linear momentum

Next we focus on the forces acting on a determinate body. From first Newton's law from each action it corresponds a second, equal with opposite direction, hence we can assume that, for a moving body $\Omega(t)$, at each $t \in [0, T]$ there is a balance of forces, i.e.

$$\mathbf{F}_{\mathrm{i}} = \mathbf{F}_{\mathrm{e}},$$

where $\mathbf{F}_{\mathrm{i}}$ are the internal forces and $\mathbf{F}_{\mathrm{e}}$ are the external forces acting on the body.

The external forces can be classified in two kinds

- forces from outside the system that act on it, which we denote with $\rho\mathbf{b}$ and;
- forces that subparcels of the of the body act upon the rest of the body itself due to deformation induces by the motion X which we denote with $\mathbf{F}_\sigma$.

In the case of a fluid we have that

$$\mathbf{F}_\sigma = \int_{\partial\Omega(t)} \sigma \mathbf{n} \, \mathrm{d}S, \qquad \sigma : \mathcal{T} \to \mathbb{R}^{3\times 3}. \tag{1.9}$$

Equation (1.9) is known as *Cauchy stress formula* and σ is the *stress tensor*, whose explicit expression depends by the physical characteristics of the material under consideration. We do not justify (1.9) and we refer the interested reader to any book of classical mechanics, such as [7]. The total external force splits into body and surface contributions:

$$\mathbf{F}_{\mathrm{e}}(t) = \int_{\Omega(t)} \rho \mathbf{b} \, \mathrm{d}V + \int_{\partial\Omega(t)} \sigma \mathbf{n} \, \mathrm{d}S, \tag{1.10}$$

where $\mathbf{b}$ is the body force per unit mass and σ is the Cauchy stress tensor.

The basic axiom connecting motion and force is the *momentum balance laws*, i.e.

$$\mathbf{F}_{\mathrm{e}}(t) = \dot{\mathbf{l}}(t), \qquad \mathbf{l}(t) = \int_{\Omega(t)} \rho\, u \, \mathrm{d}V. \tag{1.11}$$

Notice that applying Theorem 1.2 we have that

$$\dot{\mathbf{l}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho\, u \, \mathrm{d}V = \int_{\Omega(t)} \dot{(u\rho)} + \rho u \ \mathrm{div}_X\, u \, \mathrm{d}V.$$

Notice that $\dot{\Phi} = \Phi_t + u \cdot \nabla\Phi$ so that

$$\int_{\Omega(t)} \dot{(u\rho)} + \rho u \ \mathrm{div}_X\, u \, \mathrm{d}V = \int_{\Omega(t)} \left(\rho_t u + \rho u_t\right) + \left(\rho u \cdot \nabla u + u \cdot \nabla\rho \ u + \rho u \ \mathrm{div}\, u\right) \, \mathrm{d}V,$$

and

$$\rho u \cdot \nabla u + u \cdot \nabla\rho \ u + \rho u \ \mathrm{div}\, u = \rho u_j \partial_j u + u_j \partial_j \rho u + \rho \partial_j u_j u = \partial_j \left(\rho u_j u\right) = \mathrm{div}\left(\rho \ u \otimes u\right),$$

so that

$$\dot{\mathbf{l}}(t) = \int_{\Omega(t)} \partial_t \left(\rho u\right) + \mathrm{div}\left(\rho \ u \otimes u\right) \, \mathrm{d}V(X). \tag{1.12}$$

Gauss theorem allows us to turn the surface traction into a volume term:

$$\int_{\partial\Omega(t)} \sigma \mathbf{n} \, \mathrm{d}S = \int_{\Omega(t)} \mathrm{div}(\sigma) \, \mathrm{d}V. \tag{1.13}$$

Combining the results of Eqs. (1.10) to (1.13) and by arbitrarity of the integration domain we deduce the Cauchy momentum equation

$$\partial_t \left(\rho\, u\right) + \mathrm{div}\left(\rho\, u \otimes u\right) - \mathrm{div}(\sigma) = \rho \mathbf{b}. \tag{1.14}$$

1.1.3 The final assumptions

In the following we assume the fluid to be

- incompressible;
- homogeneous.

A fluid is incompressible when its motion is isochoric (cf. Definition 1.3) hence it satisfies (1.8). Equation (1.8) implies that (1.7) becomes

$$\rho_t + u \cdot \nabla \rho = 0, \tag{1.15}$$

since (1.15) is solved by (cf. Corollary 1.5)

$$\rho\left(t, y\right) = \rho_0\left(X^{-t}\left(y\right)\right), \tag{1.16}$$

we immediately obtain that if $\rho_0 \in \mathbb{R}$ is a constant the ρ defined in (1.16) is constant for all times in which the motion is defined. So, if by normalization, $\rho_0 = 1$ on top of the incompressibility condition, we obtain that the system composed of Eqs. (1.7) and (1.14) is transformed into[2]

$$\begin{cases} u_t + (u \cdot \nabla)\, u = \operatorname{div}(\sigma) + \mathbf{F} \\ \operatorname{div} u = 0 \end{cases}. \tag{1.17}$$

The stress tensor σ can assume a variety of explicit expressions that are determined by the physical characteristics of the fluid. This is an extremely active research field, for the purposes of the present course we shall consider the option

$$\sigma_{\mathrm{E}} := -p\, \mathrm{Id}. \tag{1.18}$$

The active choice of (1.18) applied to (1.17) give us the (incompressible, homogeneous) *Euler* equations (here we assume the external forces to be nil), which we endow with a suitable initial datum

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases} \tag{E}$$

Remark 1.6. Notice that if (1.8) is satisfied than

$$u \cdot \nabla u = u_i \partial_i u = \partial_i \left(u_i u\right) = \operatorname{div}\left(u \otimes u\right) \ ,$$

so that (E) can be, alternatively, written as

$$\begin{cases} u_t + \operatorname{div}\left(u \otimes u\right) + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

1.2 Functional setting

Given $f \in L^1\left(\mathbb{R}^{\mathrm{d}}; \mathbb{C}\right)$ we define, for $\xi \in \mathbb{R}^{\mathrm{d}}$

$$\hat{f}\left(\xi\right) := \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \int_{\mathbb{R}^{\mathrm{d}}} f\left(x\right) e^{-\mathrm{i} x \cdot \xi} \mathrm{d}x \ ,$$

and, if $\hat{f} \in L^1\left(\mathbb{R}^{\mathrm{d}}; \mathbb{C}\right)$, we have the classical inversion formula which is valid a.e.

$$f\left(x\right) := \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \int_{\mathbb{R}^{\mathrm{d}}} \hat{f}\left(\xi\right) e^{\mathrm{i} x \cdot \xi} \mathrm{d}\xi \ .$$

[2]Here $u \cdot \nabla := \Sigma_i u_i \partial_i$.

We use as well the notation

$$\mathscr{F} := \mathscr{F}_{x\to\xi} \qquad\qquad \mathscr{F}^{-1} := \mathscr{F}^{-1}_{\xi\to x},$$

to denote the Fourier transform a s a map between functional spaces. Notice that if a function is real-valued we have the reality condition $\overline{\hat{f}(\xi)} = \hat{f}(-\xi)$. We shall always work with real-valued functions. The Fourier transform can be extended to the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d;\mathbb{C})$: for all $f \in \mathscr{S}'(\mathbb{R}^d;\mathbb{C})$ we denote with $\hat{f}$ as the unique $g \in \mathscr{S}'(\mathbb{R}^d;\mathbb{C})$ such that

$$\langle f \mid \hat{\varphi}\rangle = \langle g \mid \varphi\rangle \qquad\qquad \forall \varphi \in \mathscr{S}(\mathbb{R}^d;\mathbb{C}) .$$

We denote, for $x \in \mathbb{R}^d$

$$\langle x\rangle := \sqrt{1+|x|^2}.$$

Definition 1.7 (Fourier multipliers)**.** Let $m \in \mathbb{R}$ we say that

i a $L^\infty_{\text{loc}}(\mathbb{R}^d;\mathbb{R})$ function $\xi \mapsto a(\xi) \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\};\mathbb{C})$ is a *non-homogeneous Fourier multiplier of order m* if for any multi-index α there exists a $C_\alpha > 0$ such that

$$\left|\partial_\xi^\alpha a(\xi)\right| \le C_\alpha \langle\xi\rangle^{m-|\alpha|}, \qquad\qquad \forall \xi \in \mathbb{R}^d . \tag{1.19}$$

We denote the space of non-homogeneous Fourier multipliers of order m as $\mathscr{M}^m$, which is endowed with the family of seminorms

$$|a|_{\mathscr{M}^m,n} := \max_{|\alpha|\le n} \sup_{\xi\in\mathbb{R}^d} \langle\xi\rangle^{|\alpha|-m} \left|\partial_\xi^\alpha a(\xi)\right|, \qquad\qquad n \in \mathbb{N}; \tag{1.20}$$

ii an m-homogeneous $L^\infty_{\text{loc}}(\mathbb{R}^d \setminus \{0\};\mathbb{R})$ function $\xi \mapsto \dot{a}(\xi) \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\};\mathbb{C})$ is a *homogeneous Fourier multiplier of order m* if for any multi-index α there exists a $C_\alpha > 0$ such that holds

$$\left|\partial_\xi^\alpha \dot{a}(\xi)\right| \le C_\alpha |\xi|^{m-|\alpha|}, \qquad\qquad \forall \xi \in \mathbb{R}^d \setminus \{0\} .$$

We denote the space of homogeneous Fourier multipliers of order m as $\dot{\mathscr{M}}^m$, which is endowed with the family of seminorms

$$|\dot{a}|_{\dot{\mathscr{M}}^m,n} := \max_{|\alpha|\le n} \sup_{\xi\in\mathbb{R}^d\setminus 0} |\xi|^{|\alpha|-m} \left|\partial_\xi^\alpha \dot{a}(\xi)\right|, \qquad\qquad n \in \mathbb{N}.$$

Definition 1.8 (Quantization of a Fourier multiplier)**.** Let $f \in \mathscr{S}(\mathbb{R}^d;\mathbb{C})$ (or $\mathscr{S}_0(\mathbb{R}^d;\mathbb{C})$), $a \in \mathscr{M}^m$ (or $\dot{\mathscr{M}}^m$) we define

$$a(D)f(x) := \mathscr{F}^{-1}_{\xi\to x}\left(a\hat{f}\right)(x)$$

Remark 1.9. By standard properties of Schwartz functions we have that $\mathscr{M}^m \subset \mathcal{L}(\mathscr{S})$.

Lemma 1.10. *Let $m \in \mathbb{R}$, $\mu \in \dot{\mathscr{M}}^m$, then $\mu(D) \in \mathcal{L}(\mathscr{S}_0)$.*

Proof. The proof is organized into three parts. First, we establish that the core of the proof is to demonstrate that for any $f \in \mathscr{S}_0$, the function $g := \mu(D)f$ also has a Fourier transform that is flat at the origin. Second, we recall and verify the necessary flatness property for any function in $\mathscr{S}_0$. Finally, we combine these facts to show that all derivatives of $\hat{g}$ vanish at the origin, confirming that all moments of g are zero.

Part 1 (Reduction to the flatness of $\widehat{\mu(D)f}$)**.** The continuity of the map $\mu(D) : \mathscr{S}_0 \to \mathscr{S}$ is a direct consequence of Remark 1.9 after applying a standard cut-off decomposition near the origin. The main challenge is to prove that the range of this operator is contained within the subspace $\mathscr{S}_0$. A function g belongs to $\mathscr{S}_0$ if and only if it is a Schwartz function and all of its moments vanish. This is equivalent to the condition that its Fourier transform, $\hat{g}$, is flat at the origin, meaning $\partial^\alpha \hat{g}(0) = 0$ for all multi-indices $\alpha \in \mathbb{N}^d$.

By definition, $\widehat{\mu(D)f}(\xi) = \mu(\xi)\hat{f}(\xi)$. Our goal is therefore to show that for any $f \in \mathscr{S}_0$, all derivatives of the product $\mu(\xi)\hat{f}(\xi)$ are zero at $\xi = 0$.

Part 2 (Flatness of $\hat{f}$ for $f \in \mathscr{S}_0$)**.** Let $f \in \mathscr{S}_0$. By definition, all its moments vanish:

$$\int_{\mathbb{R}^{\mathrm{d}}} x^{\beta} f(x)\,\mathrm{d}x = 0 \quad \text{for all multi-indices } \beta \in \mathbb{N}^{\mathrm{d}}.$$

To show that $\hat{f}$ is flat at the origin, we use the Taylor expansion of $e^{-\mathrm{i}x\cdot\xi}$ with an integral remainder. For any $N \in \mathbb{N}$:

$$e^{-\mathrm{i}x\cdot\xi} = \sum_{k=0}^{N-1} \frac{(-\mathrm{i}x\cdot\xi)^k}{k!} + \frac{(-\mathrm{i}x\cdot\xi)^N}{(N-1)!} \int_0^1 (1-t)^{N-1} e^{-\mathrm{i}tx\cdot\xi}\,\mathrm{d}t. \tag{1.21}$$

Multiplying (1.21) by $f(x)$ and integrating over $\mathbb{R}^{\mathrm{d}}$ yields $\hat{f}(\xi)$. Since every term in the finite sum corresponds to a linear combination of moments of f, the sum vanishes entirely. The expression for $\hat{f}(\xi)$ is thus equal to its remainder term, which can be bounded as follows:

$$\begin{aligned}\left|\hat{f}(\xi)\right| &= \left|\frac{(-\mathrm{i})^N}{(N-1)!}\int_{\mathbb{R}^{\mathrm{d}}} (x\cdot\xi)^N f(x) \int_0^1 (1-t)^{N-1} e^{-\mathrm{i}tx\cdot\xi}\,\mathrm{d}t\,\mathrm{d}x\right| \\ &\le \frac{|\xi|^N}{N!} \left\||\cdot|^N f\right\|_{L^1}.\end{aligned}$$

Since $f \in \mathscr{S}$, we have that $\left\||\cdot|^N f\right\|_{L^1} < \infty$. This establishes the key property that for any $N \in \mathbb{N}$, there exists a constant C_N such that:

$$\left|\hat{f}(\xi)\right| \le C_N |\xi|^N. \tag{1.22}$$

Part 3 (Conclusion)**.** We now show that all derivatives of $\widehat{\mu(D)f}(\xi) = \mu(\xi)\hat{f}(\xi)$ are zero at $\xi = 0$. By the Leibniz rule:

$$\partial^{\alpha}\left(\mu(\xi)\hat{f}(\xi)\right) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^{\beta}\mu(\xi)\,\partial^{\alpha-\beta}\hat{f}(\xi).$$

By the properties of a homogeneous multiplier of order m, we have $\left|\partial^{\beta}\mu(\xi)\right| \lesssim |\xi|^{m-|\beta|}$. From the moment conditions on f, we know that $\partial^{\alpha-\beta}\hat{f}(0) = 0$. However, to handle the singularity of μ at the origin when $m - |\beta| < 0$, we rely on the flatness established in (1.22). For any term in the sum and for any $N \in \mathbb{N}$:

$$\left|\partial^{\beta}\mu(\xi)\,\partial^{\alpha-\beta}\hat{f}(\xi)\right| \lesssim |\xi|^{m-|\beta|} \left|\partial^{\alpha-\beta}\hat{f}(\xi)\right|.$$

Applying (1.22) to the function $\bullet^{\alpha-\beta} f \in \mathscr{S}_0$, we get $\left|\partial^{\alpha-\beta}\hat{f}(\xi)\right| \lesssim_N |\xi|^N$. Thus,

$$\left|\partial^{\beta}\mu(\xi)\,\partial^{\alpha-\beta}\hat{f}(\xi)\right| \lesssim_N |\xi|^{m-|\beta|+N}.$$

For any α and β, we can choose N large enough such that $m - |\beta| + N > 0$. This ensures that every term in the Leibniz sum tends to zero as $\xi \to 0$. Therefore, $\partial^{\alpha}\left(\mu\hat{f}\right)(0) = 0$ for all α, which proves that $\mu(D)f \in \mathscr{S}_0$.

Since the map is continuous from $\mathscr{S}_0$ to $\mathscr{S}$ and its range is contained in the closed subspace $\mathscr{S}_0$, we conclude that $\mu(D) \in \mathcal{L}(\mathscr{S}_0)$.

□

Definition 1.11 (Sobolev spaces)**.** Let $f \in \mathscr{S}\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)$ and $f_0 \in \mathscr{S}_0\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)$:

i let $s \in \mathbb{R}$, $p \in (1,\infty)$, let us define

$$\|f\|_{W^{s,p}(\mathbb{R}^{\mathrm{d}};\mathbb{R})} := \left\|\langle D\rangle^s f\right\|_{L^p(\mathbb{R}^{\mathrm{d}};\mathbb{R})} \qquad \|f_0\|_{\dot{W}^{s,p}(\mathbb{R}^{\mathrm{d}};\mathbb{R})} := \left\||D|^s f_0\right\|_{L^p(\mathbb{R}^{\mathrm{d}};\mathbb{R})},$$

we define

$$W^{s,p}\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right) := \overline{\mathscr{S}\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)}^{\|\bullet\|_{W^{s,p}(\mathbb{R}^{\mathrm{d}};\mathbb{R})}} \qquad \dot{W}^{s,p}\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right) := \overline{\mathscr{S}_0\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)}^{\|\bullet\|_{\dot{W}^{s,p}(\mathbb{R}^{\mathrm{d}};\mathbb{R})}};$$

ii let $s \in \mathbb{N}$, $p = 1,\infty$ and $\alpha \in \mathbb{N}^{\mathrm{d}}$ let us define

$$\|f\|_{W^{s,p}(\mathbb{R}^{\mathrm{d}};\mathbb{R})} := \sum_{|\alpha|\le s} \left\|\partial^{\alpha} f\right\|_{L^p(\mathbb{R}^{\mathrm{d}};\mathbb{R})} \qquad \|f_0\|_{\dot{W}^{s,p}(\mathbb{R}^{\mathrm{d}};\mathbb{R})} := \sum_{|\alpha|=s} \left\|\partial^{\alpha} f\right\|_{L^p(\mathbb{R}^{\mathrm{d}};\mathbb{R})},$$

we define

$$W^{s,p}\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right) := \overline{\mathscr{S}\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)}^{\|\bullet\|_{W^{s,p}(\mathbb{R}^{\mathrm{d}};\mathbb{R})}} \qquad \dot{W}^{s,p}\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right) := \overline{\mathscr{S}_0\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)}^{\|\bullet\|_{\dot{W}^{s,p}(\mathbb{R}^{\mathrm{d}};\mathbb{R})}};$$

Definition 1.12 (Sobolev spaces when $p = 2$). 1. In the case $p = 2$ we denote $H^s := H^s\left(\mathbb{R}^d;\mathbb{R}\right) := W^{s,2}\left(\mathbb{R}^d;\mathbb{R}\right)$ and $\dot{H}^s := \dot{H}^s\left(\mathbb{R}^d;\mathbb{R}\right) := \dot{W}^{s,2}\left(\mathbb{R}^d;\mathbb{R}\right)$ and applying Plancherel theorem we have that

i $\left\|f\right\|_{H^s} := \sqrt{\int_{\mathbb{R}^d} \langle\xi\rangle^s \left|\hat{f}(\xi)\right|^2 d\xi}$;

ii $\left\|f\right\|_{\dot{H}^s} := \sqrt{\int_{\mathbb{R}^d} |\xi|^s \left|\hat{f}(\xi)\right|^2 d\xi}$.

iii A vector-valued function $F := (F_1,\dots,F_N)^\intercal \in H^s\left(\mathbb{R}^d;\mathbb{R}^N\right) \simeq (H^s)^N$ if and only if

$$\|F\|_{(H^s)^N} := \max_{i=1,\dots,N} \|F_i\|_{H^s} < \infty.$$

Similarly for vector-valued homogeneous Sobolev fields.

2. For any $s \in \mathbb{R}$ us define the subspace of *solenoidal Sobolev vector fields*

$$H^s_\sigma := \left\{u \in H^s \ : \ \operatorname{div} u = 0\right\} \hookrightarrow H^s.$$

Remark 1.13 (First properties of Sobolev spaces).

- Of course we have that $H^s \hookrightarrow \dot{H}^s$ for $s \geq 0$ and $\dot{H}^s \hookrightarrow H^s$ for $s < 0$;
- the maps $\xi \mapsto \langle\xi\rangle^s$ is a non homogeneous s-Fourier multiplier and $\xi \mapsto |\xi|^s$ is an homogeneous s-Fourier multiplier;
- thanks to the equivalence of norms in $\mathbb{R}^N$ we may use the notation $\|F\|_{H^s}$ even for vector-valued functions $F \in (H^s)^N$.

Remark 1.14.

- Definition 1.8 can be lifted by density to any Sobolev space;
- accordingly to Definition 1.8 the operator D is the quantization of the vector-valued Fourier multiplier $\xi \in \left(\mathscr{M}^1\right)^d$.

Fourier multipliers have a natural action on Sobolev spaces

Lemma 1.15. *Let $m, s \in \mathbb{R}$*

1. *let $f \in H^s$ and $a \in \mathscr{M}^m$, we have that $a(D) \in \mathcal{L}(H^s; H^{s-m})$ and*

$$\left\|a(D)f\right\|_{H^{s-m}} \leq |a|_{\mathscr{M}^m,0} \left\|f\right\|_{H^s};$$

2. *let $f \in \dot{H}^s$ and $\dot{a} \in \dot{\mathscr{M}}^m$, we have that $\dot{a}(D) \in \mathcal{L}\left(\dot{H}^s; \dot{H}^{s-m}\right)$ and*

$$\left\|\dot{a}(D)f\right\|_{\dot{H}^{s-m}} \leq |\dot{a}|_{\dot{\mathscr{M}}^m,0} \left\|f\right\|_{\dot{H}^s}.$$

Proof. The proof is immediate, in fact

$$\int \langle\xi\rangle^{2(s-m)} \left|a(\xi)\hat{f}(\xi)\right|^2 d\xi \overset{(1.19),(1.20)}{\leq} |a|_{\mathscr{M}^m,0} \int \langle\xi\rangle^{2s} \left|\hat{f}(\xi)\right|^2 d\xi,$$

the same goes for the second point. □

There is a natural classification of Sobolev spaces with integer regularity

Lemma 1.16. *Let $s \in \mathbb{N}$, then we have that*

$$\left\|f\right\|_{\dot{H}^s} \sim \sum_{|\alpha|=s} \left\|\partial^\alpha f\right\|_{L^2}.$$

Proof. Notice that, setting $|\alpha| = s$ the multi-index ∂^α is the quantization of the Fourier multiplier $\xi \mapsto (i)^s \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \in \dot{\mathscr{M}}^s$, hence we apply Lemma 1.15 and we obtain that $\|\partial^\alpha f\|_{L^2} \lesssim \|f\|_{\dot{H}^s}$. For the inverse inequality notice that by equivalence of norms in $\mathbb{R}^{\mathrm{d}}$

$$|\xi|^s \lesssim_{\mathrm{d}} \sum_{\mathrm{j}=1}^{\mathrm{d}} |\xi_{\mathrm{j}}|^s .$$

Notice that $|\xi_{\mathrm{j}}|^s = \left((\mathrm{i}\xi_{\mathrm{j}}) (-\mathrm{i}\xi_{\mathrm{j}}) \right)^{\frac{s}{2}}$ which is the quantization of $(-1)^{\frac{s}{2}} \partial_{\mathrm{j}}^s$, so that

$$\|f\|_{\dot{H}^s} \lesssim \sum_{\mathrm{j}=1}^{\mathrm{d}} \left\| \partial_{\mathrm{j}}^s f \right\|_{L^2} \lesssim \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^2} .$$

□

Theorem 1.17 (Sobolev embeddings in Lebesgue spaces)**.** *Let* $\mathrm{d} \in \mathbb{N} \setminus 0$, $s \in \left[0, \frac{\mathrm{d}}{2}\right)$, *then the space* $\dot{H}^s$ *embeds continuously in* L^{p_s} *where* $p_s := \frac{2\mathrm{d}}{\mathrm{d}-2s}$.

Theorem 1.17 is a fundamental result in mathematical analysis and it is widely used in the analysis of PDEs, its proof is based on two technical results, that we only state here. For the proof we refer the interested reader to [1]:

Lemma 1.18 (Hardy-Littlewood-Sobolev inequality)**.** *Let* $\alpha \in (0, \mathrm{d})$ *and* $(p, r) \in [1, \infty]^2$ *satisfy*

$$\frac{1}{p} + \frac{\alpha}{\mathrm{d}} = 1 + \frac{1}{r}$$

A constant C then exists such that

$$\left\| |\bullet|^{-\alpha} * f \right\|_{L^r(\mathbb{R}^{\mathrm{d}})} \leq C \|f\|_{L^p(\mathbb{R}^{\mathrm{d}})} .$$

Lemma 1.19 (Fourier transform of a power function)**.** *If* $s \in (0, \mathrm{d})$, *then* $\mathscr{F}\left(|\bullet|^{-s}\right) = c_{\mathrm{d},s} |\bullet|^{s-\mathrm{d}}$ *for some constant* $c_{\mathrm{d},s}$ *depending only on* d *and* s.

Proof of Theorem 1.17. Let $\phi \in \mathscr{S}_0$, we have that $\phi = K_s * \left(|D|^s \phi\right)$ where $K_s(x) := \mathscr{F}^{-1}_{\xi \to x}\left(|\xi|^{-s}\right)(x) = c_{\mathrm{d},s} |x|^{s-\mathrm{d}}$ thanks to Lemma 1.19. Applying Lemma 1.18 with $\alpha = \mathrm{d} - s$ and $p = 2$ we obtain the desired claim for Schwartz function. The generalization to Sobolev functions follows by density. □

Theorem 1.17 does not hold true for the endpoint case of L^∞, instead we shall use the following very simple embedding result:

Lemma 1.20. *Let* $s > \mathrm{d}/2$ *then* $H^s \hookrightarrow L^\infty$.

Proof. The proof is rather straightforward, in fact

$$f(x) \lesssim \|\hat{f}\|_{L^1} \lesssim \sqrt{\int \frac{\mathrm{d}\xi}{\langle \xi \rangle^{2s}}} \; \|\langle \bullet \rangle^s \hat{f}\|_{L^2} , \tag{1.23}$$

since the r.h.s. of Eq. (1.23) is independent of x we obtain the desired inequality. □

Definition 1.21 (Hodge decomposition)**.** Let us define

$$\mathcal{Q}(\xi) := \left(-\frac{\xi_i \xi_j}{|\xi|^2} \right)_{i,j=1,\ldots,d} \in \left(\dot{\mathscr{M}}^0 \right)^{d \times d} \qquad \mathcal{P}(\xi) := \mathrm{Id}_{\mathbb{R}^d} - \mathcal{Q}(\xi),$$

and $\mathcal{Q} := \mathcal{Q}(D)$, $\mathcal{P} := \mathcal{P}(D)$. The map $\mathcal{P} \in \mathcal{L}\left(\left(L^2\right)^{\mathrm{d}} \right)$ is known as the *Leray projector*. The decomposition

$$v = \mathcal{P}u + \mathcal{Q}v$$

is known as *Hodge decomposition*.

Lemma 1.22. *Let $s \in \mathbb{R}$ and $v \in \dot{H}^s$, we have that*

- ▷ $\operatorname{div} \mathcal{P} v = 0$;
- ▷ *there exists a $\phi \in \dot{H}^{s+1}$ such that $\mathcal{Q} v = \nabla \phi$;*
- ▷ *if $u, v \in L^2$ then $\mathcal{P}$ and $\mathcal{Q}$ are L^2–self-adjoint, i.e. $\langle \mathcal{P} v \mid \mathcal{Q} v \rangle_{L^2} = 0$.*

Remark 1.23.

1. Most of the times it is convenient to think of $\mathcal{P} = 1 - \frac{\nabla \operatorname{div}}{\Delta}$, and the same for $\mathcal{Q}$;
2. Notice that applying $\mathcal{P}$ to (E) we obtain that the system (E) is equivalent to

$$\begin{cases} u_t + \mathcal{P}(u \cdot \nabla u) = 0 \\ u|_{t=0} = u_0 \end{cases} \tag{1.24}$$

Part I

Strong solutions

Here in this first part we discuss the theory needed to construct strong solutions for (E).

After having derived the incompressible Euler equations (E) from fundamental principles of continuum mechanics in Section 1.1, we now turn to their rigorous mathematical analysis. The central question is whether these equations admit solutions that are well-defined, unique, and reflect the physical behavior they are meant to model.

This first part is dedicated to the theory of *strong solutions*. These are solutions that possess enough regularity for all terms in the equations to be interpreted in a classical, pointwise sense. The main challenge stems from the nonlinear advection term, $u \cdot \nabla u$, which can potentially lead to the formation of singularities and the breakdown of smoothness in finite time.

Our strategy will be to work within the framework of Sobolev spaces H^s, which provide a robust setting for handling the interplay between derivatives and nonlinearity. We will establish a cornerstone result by Kato on the local-in-time existence and uniqueness of solutions for sufficiently smooth initial data. Subsequently, in Chapter 3, we will investigate the crucial question of what prevents these local solutions from being extended globally in time. This will lead us to the celebrated continuation criteria, including the Beale-Kato-Majda criterion, which connects the potential blow-up of solutions to the behavior of the fluid's vorticity, Ω. Finally, we will see how these tools guarantee the global existence of strong solutions in the special case of two spatial dimensions, a result that starkly contrasts with the notoriously difficult open problem in three dimensions.

Chapter 2

The incompressible Euler equations: Well-posedness in H^s

The first natural question that we may pose is the following one: do classical solutions to (E) exist? Is so, in which space? Are they unique in such functional setting?

The main result we prove in the present section is the following one:

Theorem 2.1. *Let* $\mathrm{d} \in \mathbb{N}$, $\mathrm{d} \geq 2$, $s > \frac{\mathrm{d}}{2} + 1$ *and let* $u_0 \in H^s_\sigma$. *There exists a constant* $C_s > 0$ *such that, defining*

$$T^\star := T^\star(s, \|u_0\|_{H^s}) := \frac{1}{2C_s \|u_0\|_{H^s}}, \tag{2.1}$$

then there exists a unique solution u of (E) *in the space*

$$u \in \mathcal{C}\left(\left[0, T^\star\right]; H^s_\sigma\right) \cap \mathcal{C}^1\left(\left[0, T^\star\right]; H^{s-1}_\sigma\right),$$

that satisfies the bounds

$$\sup_{t \in [0, T^\star]} \|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s} \qquad \sup_{t \in [0, T^\star]} \|u_t(t)\|_{H^{s-1}} \leq 4C\|u_0\|^2_{H^s} T^\star. \tag{2.2}$$

The local well-posedness for the Euler equations is a classical result. The specific formulation in Sobolev spaces, as stated in Theorem 2.1, was first established by T. Kato in [9]. The proof presented here follows a classical regularization scheme, drawing its framework from modern presentations of the argument like those in [1, 2].

The regularity guaranteed by Theorem 2.1 is crucial because it ensures the velocity field u is Lipschitz continuous in space. This is precisely the condition needed to solve the ordinary differential equation governing fluid particle trajectories, which in turn defines a unique *flow map*:

Theorem 2.2 (Existence and Uniqueness of the Flow Map)**.** *Let* $u \in \mathcal{C}\left(\left[0, T^\star\right]; H^s_\sigma\right)$ *with* $s > \frac{\mathrm{d}}{2} + 1$ *be the unique solution to the Euler equations* (E) *given by Theorem 2.1. Then, there exists a unique map*

$$X : \left[0, T^\star\right] \times \mathbb{R}^\mathrm{d} \to \mathbb{R}^\mathrm{d}$$

such that for every fixed $x \in \mathbb{R}^\mathrm{d}$, *the trajectory* $t \mapsto X(t,x)$ *is of class* $\mathcal{C}^1\left(\left[0, T^\star\right]; \mathbb{R}^\mathrm{d}\right)$ *and solves the ordinary differential equation:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t,x) = u(t, X(t,x)) \\ X(0,x) = x \end{cases} \qquad \forall (t,x) \in \left[0, T^\star\right] \times \mathbb{R}^\mathrm{d}. \tag{2.3}$$

Furthermore, for each fixed $t \in \left[0, T^\star\right]$, *the map* $x \mapsto X(t,x)$ *is a homeomorphism of* $\mathbb{R}^\mathrm{d}$ *onto itself with inverse* X^{-t}.

2.1 A priori estimates

The main result we prove in the present section is the following one

Proposition 2.3. *Let* $\mathrm{d} \in \mathbb{N}$, $\mathrm{d} \geq 2$, $s > \frac{\mathrm{d}}{2} + 1$, *let* u *be a global solution of* (E) *(or, equivalently* (1.24)*) with initial datum* $u_0 \in H^s_\sigma$, *belonging to the space* $\mathcal{C}^1_{\mathrm{loc}}\left(\mathbb{R}_+; H^s_\sigma\right)$. *Then there exists a* $C, C_s > 0$ *and* $T^\star$ *defined as in* (2.1) *such that* (2.2) *holds true.*

An idea of the proof

Let us give a very brief intuitive idea of the proof of Proposition 2.3. Let us consider the case in which $\mathrm{d} = 2, 3$, $s \in \mathbb{N}$, let $\alpha \in \mathbb{N}^{\mathrm{d}}$, $|\alpha| \leq s$ be a multi-index, from (E) we have that

$$\partial^\alpha u_t + \partial^\alpha (u \cdot \nabla u) + \nabla \partial^\alpha p = 0 \tag{2.4}$$

multiplying (2.4) for $\partial^\alpha u$ and integrating give us that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \partial^\alpha u \right\|^2_{L^2} = - \left\langle \partial^\alpha (u \cdot \nabla u) + \nabla \partial^\alpha p \,\middle|\, \partial^\alpha u \right\rangle_{L^2}. \tag{2.5}$$

Notice that, integrating by parts and due to the assumption that the functions p and u decay to infinity (which is true since $u \in H^s$)

$$- \left\langle \nabla \partial^\alpha p \,\middle|\, \partial^\alpha u \right\rangle_{L^2} = - \int \sum_{i=1}^{\mathrm{d}} \partial_i \partial^\alpha p \, \partial^\alpha u_i \, \mathrm{d}x = \int \partial^\alpha p \underbrace{\sum_{i=1}^{\mathrm{d}} \partial_i \partial^\alpha u_i}_{= \partial^\alpha \operatorname{div} u = 0} \mathrm{d}x = 0,$$

so that (2.5) is, in fact

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \partial^\alpha u \right\|^2_{L^2} = - \left\langle \partial^\alpha (u \cdot \nabla u) \,\middle|\, \partial^\alpha u \right\rangle_{L^2}. \tag{2.6}$$

Notice that, in the case in which $\alpha = 0$, integrating by parts

$$\begin{aligned} - \langle u \cdot \nabla u \mid u \rangle_{L^2} = - \int u_i \partial_i u_j \; u_j \, \mathrm{d}x &\overset{(1.8)}{=\!=} - \int \partial_i \left(u_i u_j \right) \; u_j \, \mathrm{d}x \\ &= \int u_i \; \underbrace{u_j \partial_i u_j}_{= \partial_i \left(u_j^2 / 2 \right)} \, \mathrm{d}x = - \int \operatorname{div} u \, \frac{|u|^2}{2} \mathrm{d}x \overset{(1.8)}{=\!=} 0. \end{aligned} \tag{2.7}$$

so that inserting (2.7) in (2.6) we obtain that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| u \right\|^2_{L^2} = 0 \,,$$

i.e. smooth solutions of the Euler equations preserve the kinetic energy.

Let us now consider the case fo a generic α, indeed we have the Leibniz formula

$$\partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} c_{\beta, \gamma} \partial^\beta f \partial^\beta g, \qquad\qquad c_{\beta, \gamma} \in \mathbb{N} \,,$$

and this implies that, isolating the higher order term

$$\partial^\alpha \left(u_i \partial_i u_j \right) = u_i \partial_i \partial^\alpha u_j + \sum_{\substack{\beta + \gamma = \alpha \\ \gamma < \alpha}} c_{\beta, \gamma} \, \partial^\beta u_i \partial_i \partial^\gamma u_j \,,$$

so that

$$\left\langle \partial^\alpha (u \cdot \nabla u) \,\middle|\, \partial^\alpha u \right\rangle_{L^2} = \left\langle u_i \partial_i \partial^\alpha u_j \,\middle|\, \partial^\alpha u_j \right\rangle_{L^2} + \left\langle \sum_{\beta + \gamma = \alpha, \, \gamma < \alpha} c_{\beta, \gamma} \, \partial^\beta u_i \partial_i \partial^\gamma u_j \,\middle|\, \partial^\alpha u_j \right\rangle_{L^2} =: P_1 + P_2 \,. \tag{2.8}$$

Notice thet integrating by parts and using (1.8) we have that the higher order term

$$P_1 = \int u_i \partial_i \left(\frac{\left(\partial^\alpha u_j\right)^2}{2} \right) \mathrm{d}x = -\int \frac{\left(\partial^\alpha u_j\right)^2}{2} \ \mathrm{div}\, u \ \mathrm{d}x \overset{(1.8)}{=\!=} 0 \tag{2.9}$$

Applying Hölder inequality we have that

$$P_2 \leq c_{\alpha,0} \ \left\|\partial^\alpha u_j\right\|_{L^2} \left\|\partial^\alpha u_i\right\|_{L^2} \left\|\partial_i u_j\right\|_{L^\infty} + \sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma<\alpha}} c_{\beta,\gamma} \ \left\|\partial^\alpha u_j\right\|_{L^2} \left\|\partial^\beta u_i \ \partial_i \partial^\gamma u_j\right\|_{L^2} =: P_{2,1} + P_{2,2}.$$

We indeed have that

$$P_{2,1} \lesssim_\alpha \|u\|_{\dot{W}^{1,\infty}} \|u\|_{H^s}^2 \ , \tag{2.10}$$

while for the term $P_{2,2}$ we have two options

- $\left|\beta\right| = 1$: here $\left|\gamma\right| = |\alpha| - 1$ so that using Lemma 1.16

  $$\left\|\partial^\beta u_i \ \partial_i \partial^\gamma u_j\right\|_{L^2} \leq \|u_i\|_{\dot{W}^{1,\infty}} \left\|u_j\right\|_{\dot{H}^{|\alpha|}} ,$$

  from which we easily obtain the control

  $$P_{2,2} \lesssim_\alpha \|u\|_{\dot{W}^{1,\infty}} \|u\|_{H^s}^2 \ ; \tag{2.11}$$

- $\left|\beta\right| > 1$: Here we use the embedding $H^{\mathrm{d}/4} \hookrightarrow L^4$ to deduce that

  $$\left\|\partial^\beta u_i \ \partial_i \partial^\gamma u_j\right\|_{L^2} \lesssim \left\|\partial^\beta u_i\right\|_{L^4} \left\|\partial_i \partial^\gamma u_j\right\|_{L^4} \lesssim \|u\|_{H^{|\beta|+\frac{\mathrm{d}}{4}}} \|u\|_{H^{|\alpha|-1+\frac{\mathrm{d}}{4}}} \ ,$$

  and since $\mathrm{d} = 2, 3$ we obtain the same control as in (2.11).

We can thus combine Eqs. (2.8) to (2.11) and insert them in (2.6), sum for $|\alpha| \leq s$ and obtain, after an application of Gronwall inequality, that there exists a $C_s > 0$ such that

$$\|u(t)\|_{H^s}^2 \leq \|u(0)\|_{H^s} \exp\left\{ C_s \int_0^t \left\|u\left(t'\right)\right\|_{\dot{W}^{1,\infty}} \mathrm{d}t' \right\} . \tag{2.12}$$

2.1.1 A commutator estimate

Definition 2.4 (Fourier-Lebesgue spaces)**.** Let $s \in \mathbb{R}$, $f \in \mathscr{S}\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)$ and $f_0 \in \mathscr{S}_0\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)$, let us define the norms

$$\|f\|_{FL^s} := \int \langle\xi\rangle^s \left|\hat{f}(\xi)\right| \mathrm{d}\xi \qquad\qquad \|f_0\|_{\dot{F}L^s} := \int |\xi|^s \left|\hat{f}_0(\xi)\right| \mathrm{d}\xi$$

we define the spaces

$$FL^s := \overline{\mathscr{S}\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)}^{\|\bullet\|_{FL^s}} \qquad\qquad \dot{F}L^s := \overline{\mathscr{S}_0\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)}^{\|\bullet\|_{\dot{F}L^s}} .$$

We have the following result, whose statement and proof are analogous to those of Lemma 1.15:

Lemma 2.5. *Let $m, s \in \mathbb{R}$*

1. *let $f \in FL^s$ and $a \in \mathscr{M}^m$, we have that $a \in \mathcal{L}\left(FL^s; FL^{s-m}\right)$ and*

   $$\left\|a(D) f\right\|_{FL^{s-m}} \leq |a|_{\mathscr{M}^m,0} \left\|f\right\|_{FL^s} ;$$

2. *let $f \in \dot{F}L^s$ and $\dot{a} \in \dot{\mathscr{M}}^m$, we have that $\dot{a} \in \mathcal{L}\left(\dot{F}L^s; \dot{F}L^{s-m}\right)$ and*

   $$\left\|\dot{a}(D) f\right\|_{\dot{F}L^{s-m}} \leq |\dot{a}|_{\dot{\mathscr{M}}^m,0} \left\|f\right\|_{\dot{F}L^s} .$$

The importance of the (homogeneous) Fourier-Lebesgue space is that they scale like $\dot{W}^{s,\infty}$, $s \in \mathbb{N}$, but they are a sub-algebra embedded in the space $\dot{W}^{s,\infty}$, since L^1 with the convolution has an algebra structure due to Young convolution inequality. This allows for notable simplifications in the computations.

Lemma 2.6. *1. Let $s \in \mathbb{N}$, we have that $\dot{FL}^s \hookrightarrow \dot{W}^{s,\infty}$.*

2. Let $s > -\frac{\mathrm{d}}{2}$ and $\sigma > s + \frac{\mathrm{d}}{2}$. Then we have that $H^\sigma \hookrightarrow \dot{FL}^s$.

Proof. 1. Let us restrict ourselves to the case $f \in \mathscr{S}_0$, the generic result follows by density. In such setting we have that $\hat{f} \in \mathscr{S}$ so that for any $\alpha \in \mathbb{N}^{\mathrm{d}}$, $|\alpha| = s$ we have the inversion formula

$$\partial^\alpha f(x) = \frac{1}{(2\pi)^{\mathrm{d}/2}} \int \dot{a}(\xi) \hat{f}(\xi) e^{\mathrm{i}x\cdot\xi} \mathrm{d}\xi \qquad\qquad \dot{a}(\xi) := \mathrm{i}^s \xi_1^{\alpha_1} \cdots \xi_{\mathrm{d}}^{\alpha_{\mathrm{d}}},$$

so that

$$\left\| \partial^\alpha f \right\|_{L^\infty} \lesssim \left\| \dot{a}(D) f \right\|_{FL^0},$$

we have indeed that $\dot{a} \in \dot{\mathscr{M}}^s$, so that applying Lemma 2.5 we obtain that $\left\| \partial^\alpha f \right\|_{L^\infty} \lesssim |\dot{a}|_{\dot{\mathscr{M}}^s,0} \left\| f \right\|_{FL^s}$, hence a summation over multi-index $|\alpha| = s$ give us that

$$\left\| f \right\|_{W^{\dot{s},\infty}} \leq C_s \left\| f \right\|_{\dot{FL}^s}.$$

2. Let $f \in H^\sigma\left(\mathbb{R}^{\mathrm{d}}\right)$. We split its Fourier transform at the frequency $|\xi| = 1$. Let us define

$$\underline{f} := \mathscr{F}^{-1}\left(\mathbb{1}_{|\xi|\leq 1} \hat{f}\right) \qquad \text{and} \qquad \overline{f} := \mathscr{F}^{-1}\left(\mathbb{1}_{|\xi|>1} \hat{f}\right).$$

Clearly $f = \underline{f} + \overline{f}$, and by the triangle inequality, $\left\| f \right\|_{\dot{FL}^s} \leq \left\| \underline{f} \right\|_{\dot{FL}^s} + \left\| \overline{f} \right\|_{\dot{FL}^s}$. We estimate each term separately.

For the low-frequency part $\underline{f}$, applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left\| \underline{f} \right\|_{\dot{FL}^s} &= \int_{|\xi|\leq 1} |\xi|^s \left| \hat{f}(\xi) \right| \mathrm{d}\xi \\
&\leq \left(\int_{|\xi|\leq 1} |\xi|^{2s} \mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int_{|\xi|\leq 1} \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{S}^{\mathrm{d}-1}} \int_0^1 \left(r^2\right)^{\frac{2s}{2}} r^{\mathrm{d}-1} \mathrm{d}r \mathrm{d}S \right)^{\frac{1}{2}} \left\| \hat{f} \right\|_{L^2}.
\end{aligned}$$

The radial integral $\int_0^1 r^{2s+\mathrm{d}-1} \mathrm{d}r$ converges if and only if the exponent is greater than -1, which means $2s + \mathrm{d} > 0$, or $s > -\frac{\mathrm{d}}{2}$. Under this condition, the constant is finite, and we have $\left\| \underline{f} \right\|_{\dot{FL}^s} \lesssim_{s,\mathrm{d}} \left\| f \right\|_{L^2}$.

For the high-frequency part $\overline{f}$, we use Cauchy-Schwarz again:

$$\begin{aligned}
\left\| \overline{f} \right\|_{\dot{FL}^s} &= \int_{|\xi|>1} |\xi|^{s-\sigma} |\xi|^\sigma \left| \hat{f}(\xi) \right| \mathrm{d}\xi \\
&\leq \left(\int_{|\xi|>1} |\xi|^{2(s-\sigma)} \mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int_{|\xi|>1} |\xi|^{2\sigma} \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{S}^{\mathrm{d}-1}} \int_1^\infty r^{2(s-\sigma)+\mathrm{d}-1} \mathrm{d}r \mathrm{d}S \right)^{\frac{1}{2}} \left\| f \right\|_{\dot{H}^\sigma}.
\end{aligned}$$

This radial integral converges if $2(s-\sigma) + \mathrm{d} < 0$, which is precisely the condition $\sigma > s + \frac{\mathrm{d}}{2}$. Under this assumption, we have $\left\| \overline{f} \right\|_{\dot{FL}^s} \lesssim_{\sigma,s,\mathrm{d}} \left\| f \right\|_{\dot{H}^\sigma}$.

Combining the estimates and using the fact that $\left\| f \right\|_{L^2} \leq \left\| f \right\|_{H^\sigma}$ (since $\sigma > s + \frac{\mathrm{d}}{2} > -\frac{\mathrm{d}}{2} + \frac{\mathrm{d}}{2} = 0$), we conclude that

$$\left\| f \right\|_{\dot{FL}^s} \leq \left\| \underline{f} \right\|_{\dot{FL}^s} + \left\| \overline{f} \right\|_{\dot{FL}^s} \lesssim_{s,\sigma,\mathrm{d}} \left(\left\| f \right\|_{L^2} + \left\| f \right\|_{\dot{H}^\sigma} \right) \lesssim \left\| f \right\|_{H^\sigma},$$

which proves the embedding.

□

We prove at first the following result

Lemma 2.7 (Moser tame estimates)**.** *Let $s \in \mathbb{R}$ and $u, v \in H^s \cap FL^0$ then we have that*

$$\|uv\|_{H^s} \lesssim_s \|u\|_{H^s} \|v\|_{FL^0} + \|u\|_{FL^0} \|v\|_{H^s} \ .$$

Proof. We have that

$$\|uv\|_{H^s}^2 = \int \langle \xi \rangle^{2s} \left| \int \hat{u}\left(\xi - \eta\right) \hat{v}\left(\eta\right) \, \mathrm{d}\eta \right|^2 \mathrm{d}\xi \ ,$$

notice that

$$\langle \xi \rangle^{2s} \lesssim_s \left(1 + \left|\xi - \eta\right|^2 + \left|\eta\right|^2\right)^s \lesssim \langle \max\left\{\left|\xi - \eta\right|, \left|\eta\right|\right\} \rangle^{2s} \lesssim \left\langle \xi - \eta \right\rangle^{2s} + \left\langle \eta \right\rangle^{2s} \ ,$$

and the claim follows by Young convolution inequality. □

Corollary 2.8. *If $s > \mathtt{d}/2$ the space H^s is an algebra.*

Proof. It follows from Lemmas 2.6 and 2.7. □

Definition 2.9. Let A, B be operators on a Banach space X, we define the *commutator operator*

$$[\![A, B]\!] := AB - BA.$$

Lemma 2.10 (Commutator Estimate)**.** *Let $s \in \mathbb{R}$ and let $a \in \mathscr{M}^s$ be a non-homogeneous Fourier multiplier of order s. Let $u \in H^s\left(\mathbb{R}^{\mathtt{d}};\mathbb{R}\right) \cap \dot{F}L^1\left(\mathbb{R}^{\mathtt{d}};\mathbb{R}\right)$ and $v \in H^{s-1}\left(\mathbb{R}^{\mathtt{d}};\mathbb{R}\right) \cap \dot{F}L^0\left(\mathbb{R}^{\mathtt{d}};\mathbb{R}\right)$. Then, the commutator $[\![a(D)\,, u]\!]\, v := a(D)(uv) - u(a(D)\, v)$ is well-defined and there exists a constant $C = C(s, \mathtt{d}) > 0$ such that the following estimate holds:*

$$\left\| [\![a(D)\,, u]\!]\, v \right\|_{L^2} \le C\left(\|u\|_{H^s} \|v\|_{\dot{F}L^0} + \|u\|_{\dot{F}L^1} \|v\|_{H^{s-1}}\right) \ . \tag{2.13}$$

Remark 2.11. Lemma 2.10 holds as well in Homogeneous Sobolev spaces, with homogeneous Fourier multipliers.

Proof. Let $u, v \in \mathscr{S}$, the generic result shall follow by density as usual. Let us notice that

$$\mathscr{F}\left[[\![a(D)\,, u]\!]\, v\right](\xi) = \frac{1}{(2\pi)^{\mathtt{d}/2}} \int \left(a(\xi) - a\left(\eta\right)\right) \hat{u}\left(\xi - \eta\right) \hat{v}\left(\eta\right) \mathrm{d}\eta \ ,$$

hence by inversion formula

$$[\![a(D)\,, u]\!]\, v(x) = \frac{1}{(2\pi)^{\mathtt{d}}} \iint \left(a(\xi_1 + \xi_2) - a(\xi_2)\right) \hat{u}(\xi_1)\, \hat{v}(\xi_2)\, e^{\mathrm{i}(\xi_1 + \xi_2)\cdot x} \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

Let us set a $\delta \in \left(0, \frac{1}{4}\right)$ and let

$$[\![a(D)\,, u]\!]\, v = \sum_{\mathrm{j}=1}^{3} C_{\mathrm{j}}[u]\, v, \tag{2.14}$$

where

$$\begin{aligned} C_1[u]\, v(x) &:= \frac{1}{(2\pi)^{\mathtt{d}}} \iint_{|\xi_1| \le \delta |\xi_2|} \left(a(\xi_1 + \xi_2) - a(\xi_2)\right) \hat{u}(\xi_1)\, \hat{v}(\xi_2)\, e^{\mathrm{i}(\xi_1 + \xi_2)\cdot x} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\ C_2[u]\, v(x) &:= \frac{1}{(2\pi)^{\mathtt{d}}} \iint_{|\xi_2| \le \delta |\xi_1|} \left(a(\xi_1 + \xi_2) - a(\xi_2)\right) \hat{u}(\xi_1)\, \hat{v}(\xi_2)\, e^{\mathrm{i}(\xi_1 + \xi_2)\cdot x} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\ C_3[u]\, v &:= [\![a(D)\,, u]\!]\, v - \left(C_1[u]\, v + C_2[u]\, v\right) \end{aligned}$$

We now estimate separately the action of the operators $C_{\mathrm{j}}[u]$.

For the term $C_1[u]$, notice that by Taylor $a(\xi_1 + \xi_2) = a(\xi_2) + \nabla_\xi a\left(\tilde{\xi}\right) \cdot \xi_1$ for a suitable $\tilde{\xi} \in [\xi_2, \xi_1 + \xi_2]$, so that, using (1.19) and the localization $|\xi_1| \le \delta |\xi_2|$ we obtain that

$$|a(\xi_1 + \xi_2) - a(\xi_2)| \le \left|\nabla_\xi a\left(\tilde{\xi}\right)\right| |\xi_1| \lesssim_s \left\langle \tilde{\xi} \right\rangle^{s-1} |\xi_1| \lesssim \langle \xi_2 \rangle^{s-1} |\xi_1|. \tag{2.15}$$

In proving (2.15) we have used the inequality $\langle\tilde{\xi}\rangle \sim \langle\xi_2\rangle$ for all $|\xi_1| \le \delta\,|\xi_2|$, let us see that this inequality is true. This is indeed verified if $\langle\tilde{\xi}\rangle \sim \langle\xi_2\rangle$, so let denote $\tilde{\xi} := \xi_2 + t\xi_1$, $t \in (0,1)$, indeed we have that

$$\langle\tilde{\xi}\rangle \le \sqrt{1+\xi_2^2+t^2\xi_1^2+2t\,|\xi_1|\,|\xi_2|} \le \sqrt{1+\left(1+t^2\delta^2+2t\delta\right)\xi_2^2} \le 4\,\langle\xi_2\rangle\,. \tag{2.16}$$

Similarly

$$\langle\tilde{\xi}\rangle \ge \sqrt{1+\xi_2^2-2t\,|\xi_1|\,|\xi_2|} \ge \sqrt{1+(1-2\delta)\,\xi_2^2} \ge \frac{\langle\xi_2\rangle}{2}, \tag{2.17}$$

Equations (2.16) and (2.17) prove hence that $\langle\tilde{\xi}\rangle \sim \langle\xi_2\rangle$ for all $|\xi_1| \le \delta\,|\xi_2|$. By Plancherel theorem, (2.15) and Young convolution inequality we have that

$$\|C_1\,[u]\,v\|_{L^2} \le \left\|(|\bullet|\,|\hat{u}|) * \left(\langle\bullet\rangle^{s-1}\,|\hat{v}|\right)\right\|_{L^2} \le \|u\|_{\dot{F}L^1}\,\|v\|_{H^{s-1}}\,. \tag{2.18}$$

The term $C_2\,[u]$ can be controlled in a much more straightforward way, in fact since $|\xi_2| \le \delta\,|\xi_1|$ we have that

$$|a\,(\xi_1+\xi_2) - a\,(\xi_2)| \lesssim \langle\xi_1+\xi_2\rangle^s + \langle\xi_2\rangle^s \lesssim_\delta \langle\xi_1\rangle^s\ , \tag{2.19}$$

so that by standard computations we obtain the control

$$\|C_2\,[u]\,v\|_{L^2} \lesssim \|u\|_{H^s}\,\|v\|_{\dot{F}L^0}\,. \tag{2.20}$$

The term $C_3\,[u]$ can be controlled by noticing that $|\xi_1| \sim_\delta |\xi_2|$ so that a bound like (2.19) holds true in its support as well so that

$$\|C_3\,[u]\,v\|_{L^2} \lesssim \|u\|_{H^s}\,\|v\|_{\dot{F}L^0}\,. \tag{2.21}$$

We combine Eqs. (2.18), (2.20) and (2.21) in (2.14) and obtain (5.15). □

2.2 Proof of Proposition 2.3

From (1.24) we immediately obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|_{H^s}^2 \le \left|\left\langle \langle D\rangle^s\,\mathcal{P}\,(u\cdot\nabla u)\;\middle|\;\langle D\rangle^s\,u\right\rangle\right|, \tag{2.22}$$

indeed we have that $[\langle D\rangle^s\,\mathcal{P}\,(u\cdot\nabla u)]_k = \langle D\rangle^s\,\mathcal{P}_{kj}\left(u_i\partial_i u_j\right) = u_i\partial_i\,\langle D\rangle^s\,\mathcal{P}_{kj}\,u_j + [\![\langle D\rangle^s\,\mathcal{P}_{kj}\,,\,u_i]\!]\,\partial_j u_j$. Notice that, since u satisfies (1.8), we have that $\mathcal{P}u = u$, so that

$$\langle D\rangle^s\,\mathcal{P}_{kj}\left(u_i\partial_i u_j\right) = u_i\partial_i\,\langle D\rangle^s\,u_i + [\![\langle D\rangle^s\,\mathcal{P}_{kj}\,,\,u_i]\!]\,\partial_i u_j.$$

Let us now define

$$a_{kj}^s\,(\xi) := \langle\xi\rangle^s\,\mathcal{P}_{kj}\,(\xi) = \langle\xi\rangle^s\left(\delta_{kj} + \frac{\xi_k\xi_j}{|\xi|^2}\right) \in \mathscr{M}^s\ , \tag{2.23}$$

the r.h.s. of (2.22) can be bounded as

$$\left|\left\langle \langle D\rangle^s\,\mathcal{P}\,((u\cdot\nabla)\,u)\;\middle|\;\langle D\rangle^s\,u\right\rangle\right| \le \left|\left\langle u\cdot\nabla\,\langle D\rangle^s\,u\;\middle|\;\langle D\rangle^s\,u\right\rangle\right| + \left|\left\langle \left[\!\left[a_{kj}^s\,(D)\,,\,u_i\right]\!\right]\partial_i u_j\;\middle|\;\langle D\rangle^s\,u_k\right\rangle\right|. \tag{2.24}$$

Integrating by parts and using (1.8) we obtain that

$$\left\langle u\cdot\nabla\,\langle D\rangle^s\,u\;\middle|\;\langle D\rangle^s\,u\right\rangle \equiv 0\ , \tag{2.25}$$

while for the second term, we expand the estimate. By the Cauchy-Schwarz inequality, the term is bounded by $\left\|\left[\!\left[a_{kj}^s\,(D)\,,\,u\right]\!\right]\partial_i u\right\|_{L^2}\|u\|_{H^s}$. We estimate the commutator term by applying Lemma 2.10 to $v := \partial_i u_j$, which gives

$$\left\|\left[\!\left[a_{kj}^s\,(D)\,,\,u\right]\!\right]\partial_i u\right\|_{L^2} \lesssim \|u\|_{H^s}\,\|\partial_i u\|_{\dot{F}L^0} + \|u\|_{\dot{F}L^1}\,\|\partial_i u\|_{H^{s-1}}\,.$$

Since the operator ∂_i corresponds to the symbol $i\xi_i \in \dot{\mathscr{M}}^1$, we can apply Lemmas 1.15 and 2.5

$$\|\partial_i u\|_{\dot{F}L^0} \lesssim \|u\|_{\dot{F}L^1} \qquad \text{and} \qquad \|\partial_i u\|_{H^{s-1}} \lesssim \|u\|_{H^s}\,.$$

Substituting these estimates yields $\left\| \left[\!\left[a^s_{kj}(D), u \right]\!\right] \partial_i u \right\|_{L^2} \lesssim \|u\|_{H^s} \|u\|_{\dot{F}L^1}$. Combining these steps, we obtain the final inequality:

$$\left| \left\langle \left[\!\left[a^s_{kj}(D), u_i \right]\!\right] \partial_i u_j \;\middle|\; \langle D\rangle^s u_k \right\rangle \right| \lesssim_s \|u\|_{\dot{F}L^1} \|u\|^2_{H^s}. \tag{2.26}$$

We combine Eqs. (2.24) to (2.26) in (2.22) and obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{H^s} \lesssim_s \|u\|_{\dot{F}L^1} \|u\|^2_{H^s}. \tag{2.27}$$

We use now Lemma 2.6 and transform (2.27) into

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{H^s} \leq C_s \|u\|^3_{H^s}, \tag{2.28}$$

thus setting $\|u\|^2_{H^s} := U$ we obtain by standard computations that

$$\sqrt{U} \leq \frac{\sqrt{U_0}}{1 - C_s\sqrt{U_0}\,t},$$

so we have that $\sqrt{U} \leq 2\sqrt{U_0}$ if and only if

$$t < T^\star\left(s, \|u_0\|_s\right) := \frac{1}{2C_s \|u_0\|_s}, \tag{2.29}$$

which concludes the proof of the first inequality in (2.2). For the second one let us notice that using the fact that $\mathcal{P}$ is H^s-selfadjoint and Corollary 2.8 we have that

$$\left|\langle \mathcal{P}(u\cdot\nabla u) \mid u_t\rangle_{H^{s-1}}\right| = \left|\langle u\cdot\nabla u \mid u_t\rangle_{H^{s-1}}\right| \leq C \|u\|^2_{H^s} \|u_t\|_{H^{s-1}},$$

from which the second estimate in (2.2) follows applying a Gronwall estimate and the first estimate in (2.2).

2.3 Proof of Theorem 2.1

We now prove Theorem 2.1, the proof is divided into several standard steps:

1. **Approximate Solutions:** We first construct a sequence of approximate solutions by regularizing the initial data and the equation itself using a frequency cutoff. For each regularized problem, we obtain a local solution using the Cauchy-Lipschitz theorem in a Banach space.

2. **Uniform Estimates:** We then establish a priori estimates for these approximate solutions, proving that they are uniformly bounded in the desired function space (H^s) on a time interval that is independent of the regularization parameter.

3. **Compactness and Convergence:** The uniform bounds allow us to use a compactness argument (specifically, the Banach-Alaoglu theorem) to extract a subsequence that converges weakly to a limit object u. We then show this convergence is strong in a lower-order Sobolev space.

4. **Passage to the Limit:** We verify that this limit u is indeed a solution to the original Euler equation by showing it satisfies the integral formulation of the problem.

5. **Regularity of the Limit:** Finally, we prove that the limit solution u possesses the required H^s regularity and is continuous in time. The uniqueness of the solution is a consequence of the a priori estimates.

Before attacking the proof of Theorem 2.1 we state the following result concerning the existence of solutions for ODEs in Banach spaces

Theorem 2.12 (Cauchy-Lipschitz)**.** *Let X be a Banach space, for any $T \in [0,\infty]$ let $X(T) := \mathcal{C}([0,T];X)$. Let $F: X \to X$ such that for any $R > 0$ there exists a $C_R > 0$ such that*

$$F \in \mathcal{C}^{0,1}\left(\bar{B}_X(0,R); \bar{B}_X(0,C_R)\right), \qquad |F|_{\mathcal{C}^{0,1}\left(\bar{B}_X(0,R);\bar{B}_X(0,C_R)\right)} =: \mathsf{L}_R. \tag{2.30}$$

Then for any $u_0 \in B_X\left(0, \frac{R}{2}\right)$

i *there exists a $T_R > 0$ and a unique $u \in B_{X(T_R)}(0,R) \cap \mathcal{C}^{1,1}([0,T_R];X)$ solution of the Cauchy problem*

$$\begin{cases} \dot{u}(t) = F(u(t)) \\ u|_{t=0} = u_0 \end{cases} ; \tag{2.31}$$

ii *there exists a maximal time-span $T_R^\star \in [0,\infty]$ such that*

$$T_R^\star = \infty \qquad \text{or} \qquad \limsup_{t \nearrow T_R^\star} \|u(t)\|_X = \infty .$$

Proof. **i** The proof is rather a standard application of Banach contraction principle that we prove here for completeness. Equation (2.31) is equivalent to the integral equation

$$\begin{aligned} u &= \mathcal{T}(u) \\ \mathcal{T}(u;t) &:= u_0 + \int_0^t F\left(u\left(t'\right)\right) \mathrm{d}t' \quad \forall t \in [0,T] \end{aligned} \tag{2.32}$$

so that we have to prove that there exists a $T > 0$ (which may depend on u_0 and F) such that $\mathcal{T}$ is a contraction of $\bar{B}_{X(T)}(0,R)$ onto itself (recall that $X(T)$ is complete since X is so). Notice that for all $u, v \in \bar{B}_{X(T)}(0,R)$

$$\begin{aligned} \|\mathcal{T}(u)\|_{X(T)} &< \frac{R}{2} + \mathsf{L}_R\, T \\ \|\mathcal{T}(u) - \mathcal{T}(v)\|_{X(T)} &\leq \mathsf{L}_R T \ \|u - v\|_{X(T)} \end{aligned} \tag{2.33}$$

so that setting

$$T_R := \frac{1}{2\mathsf{L}_R} , \tag{2.34}$$

we obtain that $\mathcal{T}$ is a contraction of $\bar{B}_{X(T_R)}(0,R)$, which is complete, onto itself, so that there exists a unique $u \in \bar{B}_{X(T_R)}(0,R)$ that solves $u = \mathcal{T}(u)$. Additionally $u \in B_{X(T_R)}(0,R)$ thanks to the first estimate in (2.33) and $u \in \mathcal{C}^{1,1}([0,T_R];X)$ thanks to (2.32).

ii The existence of a maximal lifespan $T_R^\star \in [0,\infty]$ is consequence of Zorn's lemma. Let us suppose the statement in Item ii is false, i.e. $T_R^\star < \infty$ and $\limsup_{t \nearrow T_R^\star} \|u(t)\|_X =: \bar{R}_0 < \infty$. Notice that there exists a $\tilde{\varepsilon}_0 \in \left(0, T_R^\star\right)$ such that, for any $\varepsilon \in (0, \tilde{\varepsilon}_0)$, defining

$$T_{R,\varepsilon} := T_R^\star - \varepsilon, \qquad R_\varepsilon := \left\| u\left(T_{R,\varepsilon}\right) \right\|_X < \infty,$$

we have that

$$R_\varepsilon \leq 2\bar{R}_0, \qquad \forall\, \varepsilon \in (0, \tilde{\varepsilon}_0) . \tag{2.35}$$

Let

$$\tilde{u}(t) := u\left(T_{R,\varepsilon} + t\right)$$

then $\tilde{u}$ solves (2.31) with initial datum $\tilde{u}_0 := u\left(T_{R,\varepsilon}\right)$, then applying Item i we obtain that (cf, (2.34))

$$\tilde{u} \in \mathcal{C}\left(\left[0, T_{R_\varepsilon}\right]; X\right).$$

Notice now that, thanks to Eqs. (2.30) and (2.35) we have that

$$\mathsf{L}_{R_\varepsilon} \leq \mathsf{L}_{2\bar{R}_0}, \qquad \forall\, \varepsilon \in (0, \tilde{\varepsilon}_0), \tag{2.36}$$

hence $\tilde{u}$ provides an extension of u beyond $T_{R,\varepsilon}$ up to a time

$$S_{R,\varepsilon} := T_{R,\varepsilon} + T_{R_\varepsilon} = T_R^\star - \varepsilon + \frac{1}{2\mathsf{L}_{R_\varepsilon}} ,$$

and so, using (2.36), we obtain that

$$\liminf_{\varepsilon \searrow 0} S_{R,\varepsilon} \geq T_R^\star + \frac{1}{2\mathsf{L}_{2\bar{R}_0}} > T_R^\star ,$$

contradicting the fact $T_R^\star$ is a maximal timespan, which is absurd.

□

Step 1 (The regularized problem)**.** Let $n\in\mathbb{N}$ and

$$P_n := \mathscr{F}^{-1}\left(\mathbb{1}_{|\xi|\leq n}\bullet\right),$$

let $u_0\in H^s_\sigma$ and let us consider the following evolutionary equation

$$\begin{cases}\dot{u}^n = F_n(u^n)\\ u^n|_{t=0}=P_n u_0\end{cases} \qquad F_n\left(u^n\right) = P_n\mathcal{P}\operatorname{div}\left(u^n\otimes u^n\right) \ . \tag{2.37}$$

Notice that by Corollary 2.8 there exists a $C_0>0$ such that for any $R>0$, and $u,v\in\bar{B}_{H^s_\sigma}(0,R)$

$$\begin{aligned}\|F_n(u)\|_{H^s_\sigma} &\leq C_0 nR^2\\ \|F_n(u)-F_n(v)\|_{H^s_\sigma} &\leq C_0 n\|u+v\|_{H^s_\sigma}\|u-v\|_{H^s_\sigma}\leq 2C_0 nR\|u-v\|_{H^s_\sigma} \ ,\end{aligned}$$

so that we obtain that

$$F_n\in C^{0,1}\left(\bar{B}_{H^s_\sigma}(0,R);\bar{B}_{H^s_\sigma}\left(0,C_0nR^2\right)\right).$$

We can thus apply Theorem 2.12 and we obtain that for any $n\in\mathbb{N}$, $R>0$, $u_0\in B_{H^s_\sigma}\left(0,\frac{R}{2}\right)$ there exists a $T_n>0$ and a unique solution to (2.37) in the space

$$B_{X(T_n)}(0,R)\cap\mathcal{C}^{1,1}\left([0,T_n];H^s_\sigma\right) \ .$$

Step 2 (The uniform bounds)**.** Here we use estimates very similar to the ones proved in Proposition 2.3 to the regularized problem (2.37), which is regularized in such a way in order to maintain the energy structure of (1.24). Notice in fact that both P_n and $\mathcal{P}$ are self-adjoint in H^s (since they commute with $\langle D\rangle^s$ as it evident using Plancherel), moreover $u^n(t)\in H^s_\sigma$ and $\operatorname{supp}\left(\widehat{u^n}\right)\subset B(0,n)$ so that $\mathcal{P}P_nu^n=u^n$ and $\operatorname{div}(u^n\otimes u^n)=u^n\cdot\nabla u^n$ hence

$$\left\langle F_n\left(u^n\right)\mid u^n\right\rangle_{H^s} = \left\langle u^n\cdot\nabla u^n\mid u^n\right\rangle_{H^s} \ .$$

Repeating the computations of Section 2.2 we find that the estimate (2.28) holds true for u^n solution of (2.37), and in particular there exists a $T^\star>0$ defined as in (2.29) such that

$$\left\|u^n(t)\right\|_{H^s}\leq 2\|u_0\|_{H^s}, \qquad \forall\ t\in\left[0,T^\star\right],\ n\in\mathbb{N},$$

hence applying the continuation criterion stated in Theorem 2.12, Item ii we obtain that

$$u^n\in B_{X(T^\star)}\left(0,2\|u_0\|_{H^s}\right), \qquad \forall\ n\in\mathbb{N}\ . \tag{2.38}$$

Additionally we can apply the second estimate of (2.2) and obtain that

$$\left(\dot{u}^n\right)_n\subset B_{L^\infty([0,T^\star];H^{s-1})}\left(0,4C\|u_0\|_{H^s}T^\star\right) \ . \tag{2.39}$$

Step 3 (Convergence of approximate solutions)**.** Using the uniform boundsEqs. (2.38) and (2.39) and Banach-Alaoglu theorem implies that there exists an element

$$u\in L^\infty\left(\left[0,T^\star\right];H^s\right)\cap\dot{W}^{1,\infty}\left(\left[0,T^\star\right];H^{s-1}\right) \ , \tag{2.40}$$

and a subsequence $(u^{n_k})_k\subset(u^n)_n$ such that

$$u^{n_k}\xrightarrow[k\to\infty]{*}u \qquad \text{in } L^\infty\left(\left[0,T^\star\right];H^s\right)\cap\dot{W}^{1,\infty}\left(\left[0,T^\star\right];H^{s-1}\right).$$

We prove that the sequence $(u^n)_n$ of solutions of (2.37) is a Cauchy sequence in the complete space $\mathcal{C}\left(\left[0,T^\star\right];L^2\right)$ Let $n,p\in\mathbb{N}$ and $\delta u:=u^{n+p}-u^n$. Explicit computations show that δu solve the evolution equation

$$\dot{\delta u} = P_{n+p}\mathcal{P}\left(\delta u\otimes u^{n+p}\right)+P_n\mathcal{P}\left(u^n\otimes\delta u\right)+\left(P_{n+p}-P_n\right)\mathcal{P}\left(u^n\otimes u^{n+p}\right).$$

The following estimates are immediate

$$\begin{aligned}\left\langle P_{n+p}\mathcal{P}\left(\delta u\otimes u^{n+p}\right)\mid\delta u\right\rangle_{L^2} &\lesssim \left\|u^{n+p}\right\|_{L^\infty}\|\delta u\|^2_{L^2}\\ \left\langle P_n\mathcal{P}\left(u^n\otimes\delta u\right)\mid\delta u\right\rangle_{L^2} &\lesssim \left\|u^n\right\|_{L^\infty}\|\delta u\|^2_{L^2}\\ \left\langle\left(P_{n+p}-P_n\right)\mathcal{P}\left(u^n\otimes u^{n+p}\right)\mid\delta u\right\rangle_{L^2} &\lesssim \frac{1}{n^s}\left\|u^n\right\|_s\left\|u^{n+p}\right\|_s\|\delta u\|_{L^2}\end{aligned}$$

so that, since $H^s \hookrightarrow L^p$ fro any $s > \mathrm{d}/2+1$ and $p \in [1,\infty]$, $\|u(t)\|_s \lesssim \|u_0\|_s$ for any $t \in [0, T^\star]$ and applying Gronwall inequality we obtain that there exists a $C>0$ depending only on $\|u_0\|_s$ such that for any $t \in [0, T^\star]$

$$\|\delta u(t)\|_{L^2} \leq \|\delta u(0)\|_{L^2} e^{T^\star} + \frac{C T^\star e^{C T^\star}}{n^s} ,$$

moreover

$$\|\delta u(0)\|_{L^2} \leq \frac{\|u_0\|_s}{n^s},$$

and as such we obtain that

$$\|\delta u(t)\|_{L^2} = \mathcal{O}\left(n^{-s}\right) ,$$

which proves that $(u^n)_n$ is a Cauchy sequence in $\mathcal{C}\left([0,T^\star];L^2\right)$. Moreover by Sobolev interpolation

$$\|u\|_{s_\theta} \leq \|u\|_{s_0}^{\theta} \|u\|_{s_1}^{1-\theta} , \qquad s_\theta := \theta s_0 + (1-\theta) s_1, \quad \theta \in [0,1] ,$$

we obtain that $(u^n)_n$ is a Cauchy sequence in $\mathcal{C}\left([0,T^\star];H^{s'}\right)$ for all $s' \in [0,s)$, as such there exists $u \in \mathcal{C}\left([0,T^\star];H^{s'}\right)$ such that

$$u^n \xrightarrow{n\to\infty} u \qquad \text{in } \mathcal{C}\left([0,T^\star];H^{s'}\right), \quad \forall s' \in [0,s). \tag{2.41}$$

Step 4 (The limit point are classical solutions of (1.24))**.** Let us define

$$\Delta(t,x) := u(t,x) - u_0(x) - \int_0^t \mathcal{P}\left(\left(u\left(t',x\right)\cdot\nabla u\left(t',x\right)\right)\right) \mathrm{d}t',$$

and we show that

$$\sup_{(t,x)\in[0,T^\star]\times\mathbb{R}^{\mathrm{d}}} |\Delta(t,x)| = 0.$$

Notice that for any $n \in \mathbb{N}$ thanks to (2.37)

$$\Delta = \left(u-u^n\right) + \left(u_0 - P_n u_0\right) + \Delta_n \quad \Delta_n(t,x) := \int_0^t \left(\mathcal{P}\left(\left(u\left(t',x\right)\cdot\nabla u\left(t',x\right)\right)\right) - P_n\mathcal{P}\left(\left(u^n\left(t',x\right)\cdot\nabla u^n\left(t',x\right)\right)\right)\right)\mathrm{d}t' . \tag{2.42}$$

Indeed we have that

$$\begin{aligned} \left|\left(u-u^n\right)(t,x)\right| &\lesssim \sup_{t\in[0,T^\star]} \left\|\left(u-u^n\right)(t,\bullet)\right\|_{H^2} \xrightarrow[(2.41)]{n\to\infty} 0,\\ \left|\left(u_0-P_n u_0\right)(t,x)\right| &\lesssim \frac{1}{n^{s-1}}\|u_0\|_{H^s} \xrightarrow{n\to\infty} 0. \end{aligned} \tag{2.43}$$

Let now

$$\Delta_n = \Delta_n^1 + \Delta_n^2 + \Delta_n^3 \tag{2.44}$$

where

$$\begin{aligned} \delta u(t) &:= u(t) - u^n(t)\\ \Delta_n^1(t) &:= \int_0^t \left(1-P_n\right)\mathcal{P}\,\mathrm{div}\left(u\left(t'\right)\otimes u\left(t'\right)\right)\mathrm{d}t'\\ \Delta_n^2(t) &:= \int_0^t P_n\mathcal{P}\,\mathrm{div}\left(\delta u\left(t'\right)\otimes u\left(t'\right)\right)\mathrm{d}t'\\ \Delta_n^3(t) &:= \int_0^t P_n\mathcal{P}\,\mathrm{div}\left(u^n\left(t'\right)\otimes \delta u\left(t'\right)\right)\mathrm{d}t' \end{aligned}$$

Notice that using $[\![1-P_n, \mathcal{P}\,\mathrm{div}]\!]=0$, the embedding $H^{\frac{s}{2}} \hookrightarrow L^\infty$, $\mathcal{P}\,\mathrm{div} \in \mathscr{M}^1$, Lemma 1.15 and Corollary 2.8 we obtain that

$$\left|\left(1-P_n\right)\mathcal{P}\,\mathrm{div}\left(u\left(t',x\right)\otimes u\left(t',x\right)\right)\right| \lesssim \left\|\left(1-P_n\right)\left(u\left(t'\right)\otimes u\left(t'\right)\right)\right\|_{H^{\frac{s}{2}+1}} \lesssim \frac{1}{n^{\frac{s}{2}-1}}\left\|u\left(t'\right)\right\|_{H^s}^2$$

so that using (2.2)

$$\left|\Delta_n^1(t,x)\right| \lesssim \frac{4T^\star}{n^{\frac{s}{2}-1}} \|u_0\|_{H^s}^2 \xrightarrow{n\to\infty} 0\,. \tag{2.45}$$

Similar computations show that

$$\left|\Delta_n^2(t,x)\right| \lesssim \sup_t \|\delta u(t)\|_{H^{s'}} \sup_t \|u(t)\|_{H^{s'}} \qquad \left|\Delta_n^3(t,x)\right| \lesssim \sup_t \|\delta u(t)\|_{H^{s'}} \sup_t \left\|u^n(t)\right\|_{H^s}$$

which complemented with Eqs. (2.38) and (2.41) prove that

$$\left|\Delta_n^{\mathrm{j}}(t,x)\right| \xrightarrow{n\to\infty} 0 \qquad \mathrm{j}=2,3. \tag{2.46}$$

Equations (2.44) to (2.46) prove hence that

$$|\Delta_n(t,x)| \xrightarrow{n\to\infty} 0\,. \tag{2.47}$$

We combine Eqs. (2.42), (2.43) and (2.47) and obtain that

$$\Delta(t,x)=0 \qquad \forall\ t\in\left[0,T^\star\right]\times\mathbb{R}^{\mathrm{d}}\,,$$

thus proving that u is a classical solution of (1.24).

Step 5 (Regularity of the limit point)**.** Recall that a consequence of the Banach-Alaoglu theorem is the fact that for every reflexive Banach space X, every bounded sequence in $(x^n)_n \subset X$ is weakly convergent to an element $x\in X$ and the *Fatou property*

$$\|x\| \le \liminf_n \left\|x^n\right\|\,,$$

holds true. So we get that

$$\|u(t)\|_{H^s} \le \liminf_n \left\|u^n(t)\right\|_{H^s}\,, \qquad \forall t\in\left[0,T^\star\right]\,.$$

and, as a consequence

$$\sup_{t\in[0,T^\star]} \|u(t)\|_{H^s} < \infty\,. \tag{2.48}$$

Let us now prove that $u\in\mathcal{C}\left(\left[0,T^\star\right];H^s\right)$. From (2.48) we obtain that $u_t\in L^1\left(\left[0,T^\star\right];H^{-M}\right)$ for $M\gg 1$, and this implies that $u\in\mathcal{C}\left(\left[0,T^\star\right];H^{-M}\right)$. Let now $R>0$ and let us define $u^{[R]} := \mathcal{F}^{-1}_{\xi\to x}\left(\mathbb{1}_{|\xi|\le R}(\xi)\,\hat{u}(\xi)\right)(x)$, so we have that

$$\left\|u^{[R]}(t_1)-u^{[R]}(t_2)\right\|_{H^s}^2 \sim \int_{|\xi|\le R} \langle\xi\rangle^{2s} |\hat{u}(t_1,\xi)-\hat{u}(t_2,\xi)|^2\,\mathrm{d}\xi \lesssim R^{2(s+M)}\|u(t_1)-u(t_2)\|_{H^{-M}}\,,$$

so that $\left(u^{[R]}\right)_{R>0}\subset\mathcal{C}\left(\left[0,T^\star\right];H^s\right)$. Let us now prove that

$$\sup_{t\in[0,T^\star]} \left\|u-u^{[R]}\right\|_{H^s} \xrightarrow{R\to\infty} 0\,, \tag{2.49}$$

thus proving that u is the uniform limit of a sequence of continuous functions, which implies the time-continuity for u. Let

$$g_R(t,\xi) := \mathbb{1}_{|\xi|\le R}(\xi)\,\langle\xi\rangle^{2s}|\hat{u}(t,\xi)|^2\,, \qquad g(\xi) := \sup_{t\in[0,T^\star]} \langle\xi\rangle^{2s}|\hat{u}(t,\xi)|^2\,.$$

Indeed (2.48) implies that $g\in L^1$ so that, by dominated convergence

$$\forall\ t\in\left[0,T^\star\right] \quad g_R(t,\bullet) \xrightarrow[R\to\infty]{L^1} 0\,,$$

which implies (2.49) and concludes the proof. □

2.4 Proof of Theorem 2.2

The proof of Theorem 2.2 is very close to the proof of Theorem 2.12 but must be restated and reproved since the ODE for the flow map is non-autonomous, contrarily to (2.31).

Let us restate (2.3) as the integral equation

$$X(t,x) = x + \int_0^t u\left(t', X\left(t',x\right)\right) \mathrm{d}t'. \tag{2.50}$$

Denoting with

$$\mathcal{T}(X;t,x) := x + \int_0^t u\left(t', X\left(t',x\right)\right) \mathrm{d}t',$$

solving (2.50) is equivalent to find a fixed point for $\mathcal{T}$, i.e. for all $x \in \mathbb{R}^{\mathrm{d}}$ we want to prove that there exists $T > 0$ and $X \in \mathcal{C}^0\left([0,T];\mathbb{R}^{\mathrm{d}}\right)$ such that $X = \mathcal{T}(X)$. Notice that, being u continuous in t, then $t \mapsto \mathcal{T}(X;t,x) \in \mathcal{C}^1\left([0,T];\mathbb{R}^{\mathrm{d}}\right)$, and

$$\begin{aligned}
|\mathcal{T}(X;t,x) - \mathcal{T}(Y;t,x)| \leq \int_0^t \left|u\left(t', X\left(t'\right)\right) - u\left(t', Y\left(t'\right)\right)\right| \mathrm{d}t' &\leq \int_0^t \left|u\left(t',\bullet\right)\right|_{\mathcal{C}^{0,1}\left(\mathbb{R}^{\mathrm{d}}\right)} \left|X\left(t'\right) - Y\left(t'\right)\right| \mathrm{d}t' \\
&\leq t \sup_{t'\in[0,t]} \left|u\left(t',\bullet\right)\right|_{\mathcal{C}^{0,1}\left(\mathbb{R}^{\mathrm{d}}\right)} \|X - Y\|_{\mathcal{C}([0,t])}.
\end{aligned}$$

The map $\mathcal{T}$ is indeed a contraction onto $\mathcal{C}([0,t])$ if

$$t < \frac{1}{\|u\|_{L^\infty\left([0,t];\mathcal{C}^{0,1}\left(\mathbb{R}^{\mathrm{d}}\right)\right)}}. \tag{2.51}$$

notice that the r.h.s. of (2.51) (cf. (2.2)) is t-dependent, fortunately a consequence of Theorem 2.1 is the fact that is $t \leq T^\star$ defined as in (2.1) then

$$\|u\|_{L^\infty\left([0,t];\mathcal{C}^{0,1}\right)} \leq \|u\|_{L^\infty([0,T^\star];H^s)} \leq 2\|u_0\|_s,$$

so that if $t \leq T^\star$ then (2.51) is true and there exists a unique

$$X(\bullet,x) \in \mathcal{C}^1\left(\left[0,T^\star\right]\right), \tag{2.52}$$

that solves (2.50). Notice that $T^\star$, defined in (2.1) is *independent on x*, so that the flow solution in (2.52) has a lifespan that *does not depend on the starting point x* and is uniform in $\mathbb{R}^{\mathrm{d}}$.

Let us now prove that for every $t \in \left[0,T^\star\right]$ there exists an inverse back-to-label map. Let $y \in \mathbb{R}^{\mathrm{d}}$ and us consider the ODE

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau} Y\left(t;\tau,y\right) = -u\left(t-\tau, Y\left(t;\tau,y\right)\right) \\ Y\left(t;0,y\right) = y \end{cases}, \qquad \forall \tau \in [0,t],\ y \in \mathbb{R}^{\mathrm{d}}, \tag{2.53}$$

using again Banach contraction principle it is possible to prove that there exists a unique $\tau \mapsto Y\left(\tau,y\right) \in \mathcal{C}^1\left([0,t];\mathbb{R}^{\mathrm{d}}\right)$ of (2.53).

We define the candidate inverse map, the *back-to-label map*, using the solution of (2.53) evaluated at time $\tau = t$:

$$X^{-t}\left(y\right) := Y\left(t;t,y\right).$$

To prove that X^{-t} is the inverse of the map $\Phi_t(x) := X(t,x)$, we verify that their composition yields the identity. Let us show that $X\left(t, X^{-t}\left(y\right)\right) = y$.

Let $x_0 := X^{-t}\left(y\right)$. By definition, x_0 is the point that, evolved backward in time from y according to (2.53), is the starting position at time 0. Now, consider the trajectory $\gamma(s)$ obtained by re-parameterizing the backward solution in forward time $s \in [0,t]$:

$$\gamma(s) := Y\left(t;t-s,y\right).$$

By construction, this path starts at $\gamma(0) = Y\left(t; t, y\right) = x_0$ and ends at $\gamma(t) = Y\left(t; 0, y\right) = y$. Let us find the ODE it satisfies. Using the chain rule on (2.53) with $\tau = t - s$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) = \left.\frac{\mathrm{d}Y}{\mathrm{d}\tau}\right|_{\tau=t-s} \cdot \frac{\mathrm{d}\tau}{\mathrm{d}s} = \left.\left(-u\left(t-\tau, Y\right)\right)\right|_{\tau=t-s} \cdot (-1) = u\left(s, \gamma(s)\right).$$

Thus, $\gamma(s)$ is the solution to the *forward* initial value problem:

$$\dot{\gamma}(s) = u\left(s, \gamma(s)\right) \quad \text{with} \quad \gamma(0) = x_0.$$

However, the unique solution to this problem is, by definition, the forward flow map $X(s, x_0)$. By the uniqueness guaranteed by the Cauchy-Lipschitz theorem, we must have $\gamma(s) = X(s, x_0)$ for all $s \in [0, t]$. Evaluating at the final time $s = t$, we get:

$$X(t, x_0) = \gamma(t) = y.$$

Substituting back the definition $x_0 = X^{-t}\left(y\right)$, we have proven that $X\left(t, X^{-t}\left(y\right)\right) = y$. The proof of the other composition, $X^{-t}\left(X(t, x)\right) = x$, follows symmetrically. Finally, the continuity of the map $x \mapsto X(t, x)$ and its inverse $y \mapsto X^{-t}\left(y\right)$ is a standard result of the continuous dependence on initial data for ODEs with a Lipschitz vector field. Therefore, for each $t \in \left[0, T^\star\right]$, the map $X(t, \bullet)$ is a homeomorphism. □

Chapter 3

Continuation criteria

Now that we know that, thanks to Theorem 2.1, local classical solutions of Eq. (E) exists for sufficiently regular initial data, we can perform, rigorously, the computations of Section 2.2 to the solutions constructed in Theorem 2.1. In particular (2.27) holds true, hence an application og Gronwall inequality give us the following control

$$\|u(t)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 \exp\left\{\int_0^t \left\|u(t')\right\|_{\dot{F}L^1} \mathrm{d}t'\right\} \qquad \forall t \in \left[0, T^\star\right], \ s > \frac{\mathrm{d}}{2} + 1 \,. \tag{3.1}$$

The Equation (3.1) implies the following: *as long as* $u \in L^1\left([0,t]; \dot{F}L^1\right)$ *then* u *belongs to* $L^\infty([0,t]; H^s)$ *for arbitrarily large* s. In particular the quantity $L^1\left([0,t]; \dot{F}L^1\right)$ is *critical* for the Euler equations in the sense that singularities, is they appear, shall be traceable at this level of regularity, and conversely a global control of $L^1\left(\mathbb{R}_+; \dot{F}L^1\right)$ implies global propagation of smoothness of initial data.

3.1 An heuristic discussion on the scaling invariance for (E)

Here we shall discuss some heuristic considerations that despite not being mathematically rigorous are very useful as a roadmap when we study (E). In the paragraph at the beginning of Chapter 3 we understood, using the computations performed in Section 2.2, that there are specific regularities that, if controlled, allow us to control any regularity for the Euler equations.

Let $\lambda > 0$, and consider the transformation

$$\left(u(t,x), p(t,x)\right) \mapsto \left(u_\lambda(t,x), p_\lambda(t,x)\right) := \left(\lambda^\beta u\left(\lambda^\alpha t, \lambda x\right), \lambda^\gamma p\left(\lambda^\alpha t, \lambda x\right)\right), \qquad \alpha, \beta, \gamma \in \mathbb{R}\,. \tag{3.2}$$

The *scaling transformation* (3.2) highlights, when $\lambda \to \infty$ the effects of the unknowns at small scale (x small becomes $\lambda x \sim 1$). In particular we ask ourselves the following question: if $\left(u, p\right)$ solves (E) then for which parameters α, β, γ the couple $\left(u_\lambda, p_\lambda\right)$ solves (E)? Indeed this happens if the identity

$$\lambda^{\alpha+\beta} u_t\left(\lambda^\alpha t, \lambda x\right) + \lambda^{1+2\beta} u\left(\lambda^\alpha t, \lambda x\right) \cdot \nabla u\left(\lambda^\alpha t, \lambda x\right) = -\lambda^{1+\gamma} \nabla p\left(\lambda^\alpha t, \lambda x\right) \tag{3.3}$$

is verified. Equation (3.3) is indeed true if

$$\beta := \alpha - 1 \qquad\qquad \gamma := \frac{\alpha - 1}{2}\,,$$

hence (E) are *scaling invariant* w.r.t. the family of transformations

$$\left(u(t,x), p(t,x)\right) \mapsto \left(u_\lambda(t,x), p_\lambda(t,x)\right) := \left(\lambda^{\alpha-1} u\left(\lambda^\alpha t, \lambda x\right), \lambda^{\frac{\alpha-1}{2}} p\left(\lambda^\alpha t, \lambda x\right)\right), \qquad \forall \alpha \in \mathbb{R}, \lambda > 0. \tag{3.4}$$

So we have proved that if $\left(u, p\right)$ solve (E) then for any $\lambda > 0$ the couple $\left(u_\lambda, p_\lambda\right)$ solves (E). Let us now suppose that $\left(u, p\right)$ belong to a certain functional space $X\left([0,T]; Y\left(\mathbb{R}^{\mathrm{d}}\right)\right)$. Equation (3.4) implies that

$$\left(u, p\right) \in X\left([0,T]; Y\left(\mathbb{R}^{\mathrm{d}}\right)\right) \quad \stackrel{(3.4)}{\Longrightarrow} \quad \left(u_\lambda, p_\lambda\right) \in X\left(\left[0, \lambda^{-\alpha} T\right]; Y\left(\mathbb{R}^{\mathrm{d}}\right)\right)\,. \tag{3.5}$$

As usual, suppressing the pressure as in (1.24), u is the only unknown that matters so that we can focus on the family of transformations

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\alpha-1} u\left(\lambda^\alpha t, \lambda x\right) \qquad \forall\, \lambda > 0, \alpha \in \mathbb{R}\,. \tag{3.6}$$

Notice that there is an entire set of spaces whose norms are invariant w.r.t. the transformation (3.6) an example (but not only) is provided by

$$L^q\left([0,T]\,;\dot{W}^{s,p}\left(\mathbb{R}^{\mathrm{d}}\right)\right) \qquad s = s\left(p,q,\alpha\right) := \left(\frac{\mathrm{d}}{p} + \frac{\alpha}{q}\right) + 1 - \alpha\,. \tag{3.7}$$

From the relation (3.7) it appears clearly that the case in which $q = 1$ occupies a position of particular importance, since in fact the map $\alpha \mapsto s\left(p,1,\alpha\right)$ is constant, and as such the space $L^1\left([0,T]\,;\dot{W}^{s(p,1),p}\right)$ is invariant for any transformation (indexed by λ and α) of (3.6). Notice that the space $\dot{W}^{s(p,1),p}$ is invariant (in the sense that its norm is such) w.r.t. the transformation

$$v(x) \mapsto \frac{1}{\lambda} v(\lambda x)\,. \tag{3.8}$$

Direct computations show that the space $\dot{FL}^1$ is also invariant w.r.t. (3.8), and so in particular, as argued above, $L^1\left([0,T]\,;\dot{FL}^1\right)$ is invariant w.r.t. (3.6). This makes sense, in fact since the space $\dot{H}^s$ is invariant with respect to the transformation

$$u(x) \mapsto \lambda^{\frac{\mathrm{d}}{2}-s} u(\lambda x)$$

and the solutions of (1.24) are invariant w.r.t. the scaling transformation (3.6), then substituting u with u_λ in (3.1) we obtain that (here we set $\alpha = 0$ so that (3.6) is the scaling invariance transformation for $L^\infty([0,T])$)

$$\lambda^{s-\left(\frac{\mathrm{d}}{2}+1\right)} \left\| u_\lambda \right\|_{L^\infty([0,\lambda^{-\alpha}T];\dot{H}^s)} \leq \lambda^{s-\left(\frac{\mathrm{d}}{2}+1\right)} \left\| u_{0,\lambda} \right\|_{\dot{H}^s} \exp\left\{ \left\| u_\lambda \right\|_{L^1([0,\lambda^{-\alpha}T];\dot{FL}^1)} \right\}\,.$$

Now, the scaling invariance of $L^1\left([0,T]\,;\dot{FL}^1\right)$ w.r.t. (3.6) with $\alpha = 0$ ensures that

$$\left\| u_\lambda \right\|_{L^1([0,T];\dot{FL}^1)} = \left\| u \right\|_{L^1([0,T];\dot{FL}^1)}\,,$$

which combined with the scaling properties of $\dot{H}^s$ norms yields

$$\left\| u_\lambda \right\|_{L^\infty([0,T];\dot{H}^s)} \leq \left\| u_{0,\lambda} \right\|_{\dot{H}^s} \exp\left\{ \left\| u \right\|_{L^1([0,T];\dot{FL}^1)} \right\}\,.$$

Therefore, the estimate (3.1) is preserved under scaling transformations if and only if we work in scaling-invariant spaces. Any deviation from this scaling invariance would introduce spurious λ-dependence in the constants of our estimates, which would contradict the fundamental scaling symmetry of (1.24). This mathematical necessity, rather than mere convenience, establishes $L^1\left([0,T]\,;\dot{FL}^1\right)$ as the natural functional framework for the critical regularity theory of the Euler equations.

3.2 Continuation criteria

As suggested by the scaling analysis, the potential breakdown of a smooth solution in finite time must be linked to the blow-up of a norm that is invariant under the Euler scaling. In this section, we formalize this principle by establishing rigorous continuation criteria. These theorems identify specific quantities which, if they remain controlled in time, are sufficient to prevent the formation of singularities and extend the lifespan of the local solution.

Theorem 3.1 (Lipschitz continuation criterion)**.** *Let $s > \frac{\mathrm{d}}{2} + 1$ and $u_0 \in H^s_\sigma$ there is a $C_s > 0$ such that the maximal solution of Theorem 2.1 satisfies for all $t \in \left[0, T^\star\right)$*

$$\left\| u(t) \right\|_{H^s}^2 \leq \left\| u_0 \right\|_{H^s}^2 \exp\left\{ \int_0^t \left\| \nabla u\left(t'\right) \right\|_{L^\infty} \mathrm{d}t' \right\}.$$

Remark 3.2. As a result of Theorem 3.1 either $T^\star = \infty$ or

$$\int_0^{T^\star} \left\| \nabla u\left(t'\right) \right\|_{L^\infty} \mathrm{d}t' = \infty\,.$$

If $s \in \mathbb{N}$ the computations leading to (2.12) prove the result. In order to extend the result to an arbitrary $s \in \mathbb{R}$ we need some additional tool of harmonic analysis.

3.2.1 Harmonic analysis toolbox: Dyadic analysis

Littlewood-Paley dyadic decomposition Let $\chi \in \mathcal{C}^\infty\left(\mathbb{R}^{\mathrm{d}};[0,1]\right)$ a non-increasing smooth function such that

$$\chi(\xi) := \begin{cases} 1 & \text{if } |\xi| \in [0,1] \\ \exp\left\{\frac{1}{2-|\xi|}-1\right\} & \text{if } |\xi| \in (1,2) \\ 0 & \text{if } |\xi| \geq 2 \end{cases} . \tag{3.9}$$

For any $j \in \mathbb{Z}$ let us define

$$\varphi_j(\xi) := \chi\left(\frac{\xi}{2^j}\right) - \chi\left(\frac{\xi}{2^{j-1}}\right). \tag{3.10}$$

Notice that the family $\left(\varphi_j(\xi)\right)_{j\in\mathbb{Z}}$ is a partition of the unity for any $|\xi|>0$, in fact, fixed $\xi \neq 0$, there exists only two indexed $j_-(\xi)$ and $j_+(\xi)$ for which $\varphi_j(\xi) \neq 0$. For any $\xi \neq 0$ there exists a unique $j \in \mathbb{N}$ such that $|\xi| \in \left[2^{j-1}, 2^j\right)$ so that

$$j_-(\xi) := \left\lfloor \log_2|\xi| + 1 \right\rfloor \qquad\qquad j_+(\xi) := \left\lceil \log_2|\xi| \right\rceil .$$

Let us define the operator

$$\triangle_j u(x) := \begin{cases} \mathscr{F}^{-1}_{\xi\to x}\left(\chi(\xi)\,\hat{u}(\xi)\right)(x) & \text{if } j=0 \\ \mathscr{F}^{-1}_{\xi\to x}\left(\varphi_j(\xi)\,\hat{u}(\xi)\right)(x) & \text{if } j \geq 1 \end{cases} ,$$

so that

$$u = \sum_{n\in\mathbb{N}} \triangle_n u . \tag{3.11}$$

An important property of the dyadic decomposition in (3.11) is the following one: for any $\xi \in \mathbb{R}$ there exists only two $n_\pm(\xi) \in \mathbb{N}$ such that $\triangle_n \hat{u}(\xi) \neq 0$ and they are defined as

$$n_-(\xi) := \begin{cases} 0 & \text{if } |\xi| \in [0,1) \\ \left\lfloor \log_2|\xi| + 1 \right\rfloor & \text{if } |\xi| \geq 1 \end{cases} \qquad n_+(\xi) := \begin{cases} 1 & \text{if } |\xi| \in [0,1) \\ \left\lceil \log_2|\xi| \right\rceil & \text{if } |\xi| \geq 1 \end{cases} ,$$

and there is the obvious relation

$$n_-(\xi) = n_+(\xi) - 1 .$$

Lemma 3.3 (Bernstein inequality)**.** *Let $\lambda > 1$, $\alpha \geq 0$, $p, r \in [1,\infty]$ with $p \leq r$, and $m_\alpha \in \mathscr{M}^\alpha$. Then for any $f \in L^p$ we have that*

$$\left\|\chi\left(\lambda^{-1}D\right) m_\alpha(D) f\right\|_{L^r} \lesssim \lambda^{\alpha+\mathrm{d}\left(\frac{1}{p}-\frac{1}{r}\right)} \left\|f\right\|_{L^p} . \tag{3.12}$$

Remark 3.4. Lemma 3.3 holds true also for homogeneous Fourier multipliers.

Proof of Lemma 3.3 .

Step 1 (The case in which $m_\alpha(\xi) \equiv 1$)**.** Indeed $\chi\left(\lambda^{-1}D\right) f$ can be written as a convolution operator as

$$\chi\left(\lambda^{-1}D\right) f(x) = \lambda^{\mathrm{d}} \int \check{\chi}\left(\lambda(x-y)\right) f(y)\,\mathrm{d}y$$

so that by Young inequality

$$\left\|\chi\left(\lambda^{-1}D\right) f\right\|_{L^r} \lesssim \lambda^{\mathrm{d}} \left\|\check{\chi}(\lambda\bullet)\right\|_{L^q} \left\|f\right\|_{L^p} , \tag{3.13}$$

and

$$\left\|\chi(\lambda\bullet)\right\|_{L^q}^q = \int \left|\check{\chi}(\lambda y)\right|^q \mathrm{d}y = \lambda^{-\mathrm{d}} \left\|\check{\chi}\right\|_{L^q}^q \tag{3.14}$$

thus combining Eqs. (3.13) and (3.14) and the fact that $1-\frac{1}{q}=\frac{1}{p}-\frac{1}{r}$ we obtain the desired result.

Step 2 (The case for a generic Fourier multiplier)**.** Let now $g_\alpha(\lambda;x) := \chi\left((2\lambda)^{-1}D\right)m_\alpha(D)f$. Applying the first step we obtain that

$$\left\|\chi\left(\lambda^{-1}D\right)g_\alpha(\lambda;\bullet)\right\|_{L^r} \lesssim \lambda^{\mathrm{d}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|g_\alpha(\lambda;\bullet)\right\|_{L^p} \tag{3.15}$$

notice that we have that

$$\chi\left(\lambda^{-1}D\right)g_\alpha(\lambda;\bullet) = \chi\left(\lambda^{-1}D\right)\chi\left((2\lambda)^{-1}D\right)m_\alpha(D)f = \chi\left(\lambda^{-1}D\right)m_\alpha(D)f$$

and, by Young

$$\left\|g_\alpha(\lambda;\bullet)\right\|_{L^p} = \left\|\left(\chi\left((2\lambda)^{-1}\bullet\right)m_\alpha(\bullet)\right)^\vee * f\right\|_{L^p} \lesssim \left\|\left(\chi\left((2\lambda)^{-1}\bullet\right)m_\alpha(\bullet)\right)^\vee\right\|_{L^1}\left\|f\right\|_{L^p} \tag{3.16}$$

Let us now fix $R>0$ and let us define $K := \left(\chi\left((2\lambda)^{-1}\bullet\right)m_\alpha(\bullet)\right)^\vee$

$$\underline{K} := \chi\left(R^{-1}\bullet\right)K \qquad\qquad \overline{K} := \left(1-\chi\left(R^{-1}\bullet\right)\right)K\,,$$

certainly we have that

$$\left\|K\right\|_{L^1} \leq R^{\mathrm{d}}\left\|K\right\|_{L^\infty} + \left\|\overline{K}\right\|_{L^1}. \tag{3.17}$$

A direct computation shows that

$$\left\|K\right\|_{L^\infty} \leq \left\|\hat{K}\right\|_{L^1} \lesssim \lambda^{\alpha+\mathrm{d}}\,, \tag{3.18}$$

while

$$\begin{aligned}\left\|\underline{K}\right\|_{L^1} = \int\left(1-\chi\left(R^{-1}x\right)\right)\left|K(x)\right|\mathrm{d}x \lesssim \int_{|x|\geq R}\frac{\mathrm{d}x}{|x|^{2\mathrm{d}}}\sup_{x\in\mathbb{R}^{\mathrm{d}}}\left\{\left||x|^{2\mathrm{d}}K(x)\right|\right\} \lesssim R^{-\mathrm{d}}\int\left|\Delta^{\mathrm{d}}\left(\chi\left(2\lambda^{-1}\xi\right)m_\alpha(\xi)\right)\right| \\ \lesssim R^{-\mathrm{d}}\sum_{d_1+d_2=2\mathrm{d}}\lambda^{\alpha-d_2}\int_{|\xi|\lesssim\lambda}\langle\xi\rangle^{-d_1}\mathrm{d}\xi \overset{(3.20)}{\lesssim} R^{-\mathrm{d}}\lambda^{\alpha-\mathrm{d}}\end{aligned} \tag{3.19}$$

Notice that in (3.19) we used the inequality

$$\int_{|\xi|\lesssim\lambda}\langle\xi\rangle^{-d_1}\mathrm{d}\xi \lesssim \int_0^\lambda\frac{r^{\mathrm{d}-1}}{\left(1+r^2\right)^{\frac{d_1}{2}}}\mathrm{d}r \lesssim 1+\int_1^\lambda r^{\mathrm{d}-1-d_1}\mathrm{d}r \lesssim \lambda^{\mathrm{d}-d_1}\,. \tag{3.20}$$

We put together Eqs. (3.17) to (3.19) and set $R := \lambda^{-1}$ and obtain that

$$\left\|\left(\chi\left((2\lambda)^{-1}\bullet\right)m_\alpha(\bullet)\right)^\vee\right\|_{L^1} \lesssim \lambda^\alpha\,. \tag{3.21}$$

We put together now Eqs. (3.15), (3.16) and (3.21) and we obtain (3.12).

□

We have the following result

Lemma 3.5. *For all $s\in\mathbb{R}$ and $p\in(1,\infty)$ we have that*

$$\left\|u\right\|_{W^{s,p}} \sim \left(\sum_{n\in\mathbb{N}}2^{nps}\left\|\triangle_n u\right\|_{L^p}^p\right)^{\frac{1}{p}}.$$

Proof. We prove the theorem for $p=2$, which is the main setting for these notes. The proof consists in showing two opposite inequalities.

Part 1 (Proof of the inequality $\sum_n 2^{2ns}\left\|\triangle_n u\right\|_{L^2}^2 \lesssim \left\|u\right\|_s^2$)**.** This is the easier direction. By Plancherel's theorem and the definition of $\triangle_n$, we have

$$\sum_n 2^{2ns}\left\|\triangle_n u\right\|_{L^2}^2 = \sum_n 2^{2ns}\int\varphi_n(\xi)^2\left|\hat{u}(\xi)\right|^2\mathrm{d}\xi. \tag{3.22}$$

On the support of φ_n, we have the localization $\langle\xi\rangle \sim 2^n$. Therefore, we can bound the term 2^{2ns} from above by $\langle\xi\rangle^{2s}$ and swap the sum and integral (by Fubini-Tonelli theorem):

$$\sum_n 2^{2ns}\|\triangle_n u\|_{L^2}^2 \lesssim \sum_n \int \langle\xi\rangle^{2s}\varphi_n(\xi)^2|\hat{u}(\xi)|^2\,\mathrm{d}\xi = \int \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2\left(\sum_n \varphi_n(\xi)^2\right)\mathrm{d}\xi. \tag{3.23}$$

Indeed $\sum_n \varphi_n(\xi)^2 \le c_2$, this gives

$$\sum_n 2^{2ns}\|\triangle_n u\|_{L^2}^2 \lesssim \int \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2\,\mathrm{d}\xi \sim \|u\|_s^2,$$

which concludes this part.

Part 2 (Proof of the inequality $\|u\|_s^2 \lesssim \sum_n 2^{2ns}\|\triangle_n u\|_{L^2}^2$)**.** For the reverse inequality, we start from the Sobolev norm and use the fact that $\sum_n \varphi_n(\xi)^2 \ge c_1 > 0$, which is the other half of the almost-orthogonality property.

$$\|u\|_s^2 \sim \int \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2\,\mathrm{d}\xi \lesssim \int \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2\left(\sum_n \varphi_n(\xi)^2\right)\mathrm{d}\xi.$$

We can now perform the same steps as before: swap sum and integral, and then use the frequency localization $\langle\xi\rangle \sim 2^n$ on the support of each φ_n:

$$\begin{aligned}\int \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2\left(\sum_n \varphi_n(\xi)^2\right)\mathrm{d}\xi &= \sum_n \int \langle\xi\rangle^{2s}\varphi_n(\xi)^2|\hat{u}(\xi)|^2\,\mathrm{d}\xi\\ &\sim \sum_n 2^{2ns}\int \varphi_n(\xi)^2|\hat{u}(\xi)|^2\,\mathrm{d}\xi = \sum_n 2^{2ns}\|\triangle_n u\|_{L^2}^2.\end{aligned}$$

This proves the second inequality.

Combining the two inequalities gives the desired norm equivalence. □

Lemma 3.6. *1. Let $s \in \mathbb{R}$, $p \in (1,\infty)$, let $u \in W^{s,p}$, there exists a sequence $(a_n)_n \in \ell^p(\mathbb{N};\mathbb{R})$ such that $\|\triangle_n u\|_{L^p} \lesssim a_n 2^{-ns}\|u\|_{W^{s,p}}$;*

2. if $s \in \mathbb{N}$ and $u \in W^{s,\infty}$ then $\|\triangle_n u\|_{L^\infty} \lesssim 2^{-ns}\|u\|_{L^\infty}$.

Proof. The first point follows directly from Lemma 3.5, while the second one is a consequence of the fact that $\triangle_n u = \check{\varphi}_n * u$, then the estimate follows applying Young inequality and the fact that $\check{\varphi}_n \in L^1$ since it is the inverse FT of a Schwartz function. □

Bony paraproduct decomposition Let u, v be tempered distributions sufficiently regular so that the product uv is well defined (alternatively, let them be Schwartz functions and argue by density). We have that

$$uv = \sum_{n,m\in\mathbb{N}} \triangle_n u\,\triangle_m v\,.$$

Let $N_0 \ge 5$, we have that

$$\begin{aligned}\sum_{n,m\in\mathbb{N}} \triangle_n u\,\triangle_m v &= \sum_{|n-m|\ge N_0} \triangle_n u\,\triangle_m v + \sum_{|n-m|\le N_0} \triangle_n u\,\triangle_m v\\ &= \sum_{n\in\mathbb{N}}\sum_{m\le n-N_0} \triangle_n u\,\triangle_m v + \sum_{m\in\mathbb{N}}\sum_{n\le m-N_0} \triangle_n u\,\triangle_m v + \sum_{|n-m|\le N_0} \triangle_n u\,\triangle_m v\end{aligned}$$

Definition 3.7 (Bony paraproduct decomposition)**.** Let us define

$$\begin{aligned} T_v u &:= \sum_{n\in\mathbb{N}} S_n v \triangle_n u && := \sum_{n\in\mathbb{N}} \triangle_n u \sum_{m\le n-N_0} \triangle_m v\\ T_u v &:= \sum_{m\in\mathbb{N}} S_m u \triangle_m v && := \sum_{m\in\mathbb{N}} \triangle_m v \sum_{n\le m-N_0} \triangle_n u\,,\\ R[u,v] &:= \sum_{|n-m|\le N_0} \triangle_n u\,\triangle_m v \end{aligned}$$

the terms $T_v u$ and $T_u v$ are the *(Bony) paraproducts*, while the term $R[u,v]$ is the *(Bony) remainder*.

Indeed Definition 3.7 and the computations above provide us with the *(Bony) paraproduct decomposition*

$$uv = T_v u + T_u v + R[u,v] \ .$$

Notice that choosing N_0 sufficiently large, and since the support of a convolution is the sum of the supports, we have that the terms $S_n v \triangle_n u$ and $S_m u \triangle_m v$ are supported, in the Fourier side, on annuli whose both interior and exterior radius is comparable to 2^n and 2^m respectively.

We have the following result

Lemma 3.8. *The following estimates are true*

$$\begin{aligned}\|T_v u\|_s &\lesssim \|v\|_{L^\infty} \|u\|_s\\ \|R[u,v]\|_s &\lesssim \min\left\{\|u\|_{L^\infty}\|v\|_s \ , \ \|v\|_{L^\infty}\|u\|_s\right\}\end{aligned}$$

Proof. Notice, as mentioned above, that the term $S_n v \triangle_n u$ is localized on an annulus of size comparable to 2^n, hence there exists a $\tilde{N}_0$ such that for any $q \in \mathbb{N}$

$$2^{qs}\left\|\triangle_q T_v u\right\|_{L^2} \lesssim 2^{qs} \sum_{n:|n-q|\leq \tilde{N}_0} \|S_n v \triangle_n u\|_{L^2} \overset{\text{Lemmas 3.3 and 3.6}}{\lesssim} \left(\sum_{n:|n-q|\leq \tilde{N}_0} 2^{(q-n)s} a_n(u)\right) \|v\|_{L^\infty} \|u\|_s \ .$$

We thus conclude the proof of the first estimate applying Young convolution inequality for series and Lemma 3.5. Next, we turn our attention to the second inequality, notice that again by localization we have that there exists a $\bar{N}_0$ such that

$$\triangle_q(\triangle_m v \triangle_n u) \neq 0 \qquad \Leftrightarrow \qquad q \leq n + \bar{N}_0 \ ,$$

so that

$$2^{qs}\|R[u,v]\|_{L^2} \lesssim 2^{qs} \sum_{\substack{|n-m|\leq N_0\\ n\geq q-\bar{N}_0}} \|\triangle_m v\|_{L^\infty}\|\triangle_n u\|_{L^2} \overset{\text{Lemmas 3.3 and 3.6}}{\lesssim} \left(\sum_{\substack{|n-m|\leq N_0\\ n\geq q-\bar{N}_0}} 2^{(q-n)s} a_n(u)\right) \|v\|_{L^\infty}\|u\|_s ,$$

and the result thus follows applying Young convolution inequality for series and Lemma 3.5. □

3.2.2 Dyadic commutator estimates

Recall that the commutator of two operators is defined in Definition 2.9.

Lemma 3.9 (L^p commutator estimate)**.** *Let $\theta \in \mathcal{C}^1(\mathbb{R}^{\mathrm{d}};\mathbb{R})$ be a function such that $(1+|\cdot|)\hat{\theta} \in L^1$. There exists a constant $C>0$ such that for any Lipschitz function a with gradient $\nabla a \in L^p(\mathbb{R}^{\mathrm{d}})$ and for any function $b \in L^q(\mathbb{R}^{\mathrm{d}};\mathbb{R})$, the following estimate holds for any $\lambda > 0$:*

$$\left\|\left[\!\left[\theta\left(\lambda^{-1}D\right), a\right]\!\right] b\right\|_{L^r} \leq C\lambda^{-1}\|\nabla a\|_{L^p}\|b\|_{L^q}, \quad \textit{with} \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}. \tag{3.24}$$

Proof. To prove the lemma, it is sufficient to rewrite the action of the commutator using the convolution operator associated with θ. Let $k := \mathscr{F}^{-1}\theta$. The action of the commutator is given by

$$\begin{aligned}\left[\!\left[\theta\left(\lambda^{-1}D\right), a\right]\!\right] b(x) &= \theta\left(\lambda^{-1}D\right)(ab)(x) - a(x)\theta\left(\lambda^{-1}D\right)b(x)\\ &= \lambda^{\mathrm{d}}\int_{\mathbb{R}^{\mathrm{d}}} k\left(\lambda(x-y)\right)\left(a(y)-a(x)\right)b(y)\,\mathrm{d}y.\end{aligned}$$

Using the first-order Taylor formula, we have $a(y)-a(x) = \int_0^1 \nabla a\left(x+\tau(y-x)\right)\cdot(y-x)\,\mathrm{d}\tau$. Setting $z := x-y$, we obtain:

$$|a(x-z)-a(x)| \leq |z|\int_0^1 |\nabla a(x-\tau z)|\,\mathrm{d}\tau.$$

Let $k_1(z) := |z||k(z)|$. The hypothesis on θ implies that $k_1 \in L^1$, since $\mathscr{F}\left(z_j k(z)\right)(\xi) = \mathrm{i}\partial_j \theta(\xi)$. Consequently, the norm of the commutator can be bounded as follows:

$$\left|\left[\!\left[\theta\left(\lambda^{-1}D\right), a\right]\!\right] b(x)\right| \leq \int_{\mathbb{R}^{\mathrm{d}}} \lambda^{\mathrm{d}} |k(\lambda z)||a(x-z)-a(x)||b(x-z)|\,\mathrm{d}z.$$

Using the Taylor estimate and the definition of k_1,

$$\left|\left[\!\left[\theta\left(\lambda^{-1}D\right), a\right]\!\right] b(x)\right| \leq \lambda^{-1}\int_0^1\int_{\mathbb{R}^{\mathrm{d}}} \lambda^{\mathrm{d}} k_1(\lambda z)|\nabla a(x-\tau z)||b(x-z)|\,\mathrm{d}z\mathrm{d}\tau.$$

Applying the L^r norm, the translation invariance of the Lebesgue measure, and Young's inequality for convolution, we obtain

$$\left\|\left[\!\left[\theta\left(\lambda^{-1}D\right), a\right]\!\right] b\right\|_{L^r} \leq \lambda^{-1}\left\|k_1\right\|_{L^1}\left\|\nabla a\right\|_{L^p}\left\|b\right\|_{L^q},$$

which is the desired result. □

Corollary 3.10. *Let a be a Lipschitz function with $\nabla a \in L^p\left(\mathbb{R}^{\mathrm{d}}\right)$ and let $b \in L^q\left(\mathbb{R}^{\mathrm{d}};\mathbb{R}\right)$, with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. There exists a constant $C>0$, independent of j, such that for any $j \in \mathbb{N}$ the following estimate holds:*

$$\left\|\left[\!\left[\triangle_j, a\right]\!\right] b\right\|_{L^r} \leq C 2^{-j}\left\|\nabla a\right\|_{L^p}\left\|b\right\|_{L^q}. \tag{3.25}$$

Proof. The operator $\triangle_j$ is defined as the quantization of the symbol $\varphi_j(|\xi|)$, where $\varphi_j(r)$ is defined in (3.10). Let us set $\theta_j(\xi) := \varphi_j(|\xi|)$. For $j \geq 1$, θ_j is a $\mathcal{C}^\infty$ function with compact support in the annulus $\left\{\xi \in \mathbb{R}^{\mathrm{d}} : 2^{j-1} \leq |\xi| \leq 2^{j+1}\right\}$. For $j=0$, θ_0 is supported in the ball $\{|\xi| \leq 2\}$. In all cases, $\mathscr{F}^{-1}\theta_j$ is a Schwartz function, which implies that the condition $(1+|\cdot|)\mathscr{F}^{-1}\theta_j \in L^1$ is satisfied.
We can rewrite the operator as $\triangle_j = \varphi_0\left(2^{-j}|D|\right)$, which allows us to apply Lemma 3.9 with the symbol $\theta(\xi) := \varphi_0(|\xi|)$ and $\lambda = 2^j$. □

3.2.3 Proof of Theorem 3.1

Using Eq. (E) and Definition 2.9 we obtain that $u_q := \triangle_q u$ solves the evolution equation

$$\partial_t u_q + u\cdot\nabla u_q + \left[\!\left[\triangle_q, u\cdot\nabla\right]\!\right] u = \nabla p_q. \tag{3.26}$$

An L^2 energy estimate on (3.26) combined with the high order cancellation due to the incompressibility of the flow

$$\left\langle u\cdot\nabla u_q \mid u_q\right\rangle_0 = 0$$

gives us that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_q\right\|_{L^2}^2 = -\left\langle\left[\!\left[\triangle_q, u\cdot\nabla\right]\!\right] u \mid u_q\right\rangle_0 \leq \left\|\left[\!\left[\triangle_q, u\cdot\nabla\right]\!\right] u\right\|_{L^2}\left\|u_q\right\|_{L^2}. \tag{3.27}$$

Let us now use Definition 3.7 to get

$$\left[\!\left[\triangle_q, u\cdot\nabla\right]\!\right] u = \left[\!\left[\triangle_q, T_{u_j}\right]\!\right]\partial_j u + \left(\triangle_q T_{\partial_j u} u_j - T_{\triangle_q\partial_j u} u_j\right) + \left(\triangle_q R\left[u_j, \partial_j u\right] - R\left[u_j, \triangle_q\partial_j u\right]\right) \tag{3.28}$$

Notice now that

$$\triangle_q T_{\partial_j u} u_j - T_{\triangle_q\partial_j u} u_j = \sum_n \triangle_q\left(S_n\partial_j u\triangle_n u_j\right) - S_n\triangle_q\partial_j u\triangle_n u_j = \sum_n \left[\!\left[\triangle_q, S_n\right]\!\right]\partial_j u\,\triangle_n u_j\,,$$

but $\triangle_q$ and S_n commute so that $\left[\!\left[\triangle_q, S_n\right]\!\right] = 0$ and we obtain that

$$\triangle_q T_{\partial_j u} u_j - T_{\triangle_q\partial_j u} u_j = 0. \tag{3.29}$$

additionally, from the explicit expression of the Bony remainder and the localization properties of the dyadic blocks we have that there exists a $\bar{N}_0 \in \mathbb{N}$ such that

$$\triangle_q R\left[u_j, \partial_j u\right] - R\left[u_j, \triangle_q\partial_j u\right] = \sum_{\substack{n \geq q-\bar{N}_0 \\ |n-m| \leq N_0}} \left[\!\left[\triangle_q, \triangle_n u_j\right]\!\right]\triangle_m\partial_j u =: \tilde{R}_q[u,u]\,. \tag{3.30}$$

We insert Eqs. (3.29) and (3.30) in (3.28) and obtain that

$$\llbracket \triangle_q , u\cdot\nabla \rrbracket u = \llbracket \triangle_q , T_{u_j} \rrbracket \partial_j u + \tilde{R}_q [u,u] \ . \tag{3.31}$$

Notice that, thanks to the localization there exists a $\tilde{N}_0 \in \mathbb{N}$ such that

$$\llbracket \triangle_q , T_{u_j} \rrbracket \partial_j u = \sum_{|n-q|\leq \tilde{N}_0} \llbracket \triangle_q , S_n u_j \rrbracket \triangle_n \partial_j u$$

hence the commutator is composed by a *finite* sum of dyadic contributions. Applying Corollary 3.10

$$\left\| \llbracket \triangle_q , T_{u_j} \rrbracket \partial_j u \right\|_{L^2} \leq \sum_{|n-q|\leq \tilde{N}_0} \left\| \llbracket \triangle_q , S_n u_j \rrbracket \triangle_n \partial_j u \right\|_{L^2} \lesssim 2^{-q} \sum_{|n-q|\leq \tilde{N}_0} \|\nabla u\|_{L^\infty} \|\triangle_n \nabla u\|_{L^2} .$$

By localization and Lemma 3.6 we have that $\|\triangle_n \nabla u\|_{L^2} \sim 2^n \|\triangle_n u\|_{L^2} \sim 2^{(1-s)n} a_n(u) \|u\|_{H^s}$, moreover since the summation if defined on $|n-q| \leq \tilde{N}_0$ we have that $2^n \sim_{\tilde{N}_0} 2^q$ so that

$$\left\| \llbracket \triangle_q , T_{u_j} \rrbracket \partial_j u \right\|_{L^2} \lesssim \left(\sum_{|n-q|\leq \tilde{N}_0} 2^{-ns} a_n(u) \right) \|\nabla u\|_{L^\infty} \|u\|_s \sim 2^{-qs} \|\nabla u\|_{L^\infty} \|u\|_s \ . \tag{3.32}$$

Let us now focus on the term $\tilde{R}_q[u,u]$, we apply again Corollary 3.10 and the localization of the Bony remainder to obtain that

$$\left\| \tilde{R}_q [u,u] \right\|_{L^2} \lesssim 2^{-q} \sum_{\substack{n \geq q-\tilde{N}_0 \\ |n-m|\leq N_0}} \|\triangle_n \nabla u\|_{L^\infty} \|\triangle_m \nabla u\|_{L^2} \ ,$$

here again we use Lemma 3.6 and obtain that $\|\triangle_n \nabla u\|_{L^\infty} \|\triangle_m \nabla u\|_{L^2} \lesssim 2^{-(s-1)n} \|\nabla u\|_{L^\infty} \|u\|_s$, hence thanks to the localization $n \geq q - \tilde{N}_0$ we have that

$$\left\| \tilde{R}_q [u,u] \right\|_{L^2} \lesssim 2^{-qs} \|\nabla u\|_{L^\infty} \|u\|_s \ . \tag{3.33}$$

Plugging Eqs. (3.31) to (3.33) in (3.27) and summing in q we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_s^2 \lesssim \|\nabla u\|_{L^\infty} \|u\|_s^2 \ , \tag{3.34}$$

from which th claim follows applying Gronwall inequality. □

3.3 Beyond Lipschitz: The Beale-Kato-Majda blowup criterion

We want now to further refine the result in Theorem 3.1

Definition 3.11 (Vorticity)**.** Let $\mathtt{d} \in \mathbb{N}$, $\mathtt{d} \geq 2$, $u \in \mathscr{S}'(\mathbb{R}^{\mathtt{d}})$, we define the *vorticity* as

$$\Omega := Ju - \nabla u \ .$$

Remark 3.12. Notice that we have the identification $\Omega \rightsquigarrow \omega$ where

- $\omega := -\partial_2 u_1 + \partial_1 u_2$ if $\mathtt{d} = 2$;
- $\omega := \nabla \times u$ if $\mathtt{d} = 3$.

The Lipschitz continuation criterion in Theorem 3.1 confirms the principle from our scaling analysis: regularity persists as long as the time integral of a critical, scale-invariant quantity like the Lipschitz norm, $\|\nabla u\|_{L^\infty}$, remains bounded. This raises a natural question: are there other, more physical quantities that share this crucial scaling property? The *vorticity* is the primary candidate, as its L^∞ norm is also invariant under the same scaling. This observation is the key insight leading to the celebrated Beale-Kato-Majda blow-up criterion.

Theorem 3.13 (Beale-Kato-Majda continuation criterion)**.** *Let* $\mathtt{d} \in \mathbb{N}$, $\mathtt{d} \geq 2$, $s > \frac{\mathtt{d}}{2} + 1$, $u_0 \in H^s_\sigma$ *and* $T^\star > 0$ *the maximal lifespan identified in Theorem 2.1, then*

$$\limsup_{t \nearrow T^\star} \|u(t)\|_s < \infty \tag{3.35}$$

if and only if

$$\int_0^t \left\| \Omega(t') \right\|_{L^\infty} \mathrm{d}t' < \infty \ . \tag{3.36}$$

A closer look to the vorticity: the Biot-Savart law Here it is convenient to interpret the vorticity as the anti-symmetric part of the Jacobian of u (cf. Definition 3.11), i.e.

$$\Omega = Ju - \nabla u \qquad \Leftrightarrow \qquad \Omega_{ij} = \partial_j u_i - \partial_i u_j \ . \tag{3.37}$$

From (3.37) and $\operatorname{div} u = 0$ we have that

$$\operatorname{div}\Omega := \left(\partial_i \Omega_{ij}\right)_j = \left(\partial_{ij} u_i - \partial_i^2 u_j\right)_j = \left(-\Delta u_j\right)_j = -\Delta u \ ,$$

so that, inverting

$$u = -\Delta^{-1} \operatorname{div}\Omega \qquad \Leftrightarrow \qquad u_j = -\Delta^{-1}\partial_i \Omega_{ij} \ . \tag{3.38}$$

The relation (3.38) in known as *Biot-Savart law*, and allows us to recover u from Ω. Notice in particular that

$$\nabla u = -\Delta^{-1}\nabla \operatorname{div}\Omega =: M_0(D)\,\Omega \ , \tag{3.39}$$

with $M_0 \in \left(\mathscr{M}^0\right)^{\mathrm{d}\times\mathrm{d}}$ with entries

$$M_{0;ik}(\xi) = -\frac{\xi_i \xi_k}{|\xi|^2}$$

The L^∞ action of 0-order Fourier multipliers here we prove the following technical lemma that is the crux of the proof of BKM blowup criterion

Lemma 3.14 (Logarithmic Sobolev inequality)**.** *Let* $\mathrm{d} \in \mathbb{N}^*$*,* $s > \frac{\mathrm{d}}{2}$*, and let* $f \in H^s\left(\mathbb{R}^{\mathrm{d}};\mathbb{C}\right)$*. Let* $m_0 \in \mathscr{M}^0$*. There exists a constant* $C > 0$*, depending only on* s*,* d*, and the seminorms of* m_0*, such that the following estimate holds:*

$$\left\|m_0(D) f\right\|_{L^\infty} \leq C\left(1 + \log\left(\frac{\|f\|_s}{\|f\|_{L^\infty}}\right)\right)\|f\|_{L^\infty} \ .$$

Proof. Using the dyadic decomposition we have that

$$\left\|m_0(D) f\right\|_{L^\infty} \leq \sum_{n\in\mathbb{N}} \left\|\triangle_n m_0(D) f\right\|_{L^\infty} \ .$$

Step 1 (Bound for the low frequencies)**.** Let $n_0 \in \mathbb{N}\setminus 0$, we apply again Lemma 3.3 and obtain that

$$\left\|\triangle_n m_0(D) f\right\|_{L^\infty} \lesssim \|f\|_{L^\infty} \qquad \forall n \in \mathbb{N} \ ,$$

hence

$$\sum_{n=0}^{n_0} \left\|\triangle_n m_0(D) f\right\|_{L^\infty} \lesssim n_0 \|f\|_{L^\infty} \tag{3.40}$$

Step 2 (Bound for the high frequencies)**.** If we let $s > \mathrm{d}/2$ applying Lemmas 3.3 and 3.6

$$\left\|\triangle_n m_0(D) f\right\|_{L^\infty} \lesssim 2^{-\left(s-\frac{\mathrm{d}}{2}\right)n} a_n(f)\, \|f\|_s$$

so that

$$\sum_{n>n_0} \left\|\triangle_n m_0(D) f\right\|_{L^\infty} \lesssim 2^{-\left(s-\frac{\mathrm{d}}{2}\right)n_0} \|f\|_s \tag{3.41}$$

Step 3 (Optimization)**.** We combine Eqs. (3.40) and (3.41) and obtain that

$$\left\|m_0(D) f\right\|_{L^\infty} \lesssim n_0 \|f\|_{L^\infty} + 2^{-\left(s-\frac{\mathrm{d}}{2}\right)n_0} \|f\|_s \ . \tag{3.42}$$

Let $A, B, \alpha > 0$, $r \geq 0$ the function $F(A,B,\alpha;r) := Ar + B2^{-\alpha r}$ admits a minimum at $-\frac{1}{\alpha}\log_2\left(\frac{A}{B\alpha\log 2}\right)$ so that with the substitution $n_0 \rightsquigarrow -\left(s - \frac{\mathrm{d}}{2}\right)^{-1}\log_2\left(\frac{\|f\|_{L^\infty}}{\|f\|_s}\right)$ (3.42) becomes

$$\left\|m_0(D) f\right\|_{L^\infty} \lesssim \left(1 + \frac{1}{s - \mathrm{d}/2}\log_2\left(\frac{\|f\|_s}{\|f\|_{L^\infty}}\right)\right)\|f\|_{L^\infty} \ ,$$

the proof is then concluded by a change of basis in the logarithm.

□

Proof of Theorem 3.13

Part 1 ((3.35) implies (3.36))**.** By Lemma 1.20 and the fact that $s > \frac{\mathrm{d}}{2} + 1$ we have that

$$\|\Omega\|_{L^\infty} \lesssim \|\nabla u\|_{L^\infty} \lesssim \|u\|_s$$

hence the embedding $L^\infty([0,T]) \hookrightarrow L^1([0,T])$ for $T \in (0,\infty)$ concludes the proof of the first implication.

Part 2 ((3.36) implies (3.35))**.** Using Eq. (3.39) and Lemma 3.14 and the fact that $s > \frac{\mathrm{d}}{2} + 1$ we obtain that

$$\|\nabla u\|_{L^\infty} \lesssim \left(1 + \log\left(\frac{\|\Omega\|_{s-1}}{\|\Omega\|_{L^\infty}}\right)\right) \|\Omega\|_{L^\infty} \ , \tag{3.43}$$

moreover $\|\Omega\|_{s-1} \sim \|u\|_s$ for all $s \geq 1$ thanks to Eqs. (3.37) and (3.39) and Plancherel theorem. Hence defining

$$\log_+ r := \log(e + r)$$

then (3.43) becomes

$$\|\nabla u\|_{L^\infty} \lesssim \left(1 + \log_+ \|\Omega\|_{L^\infty} + \log_+ \|u\|_s\right) \|\Omega\|_{L^\infty} \ . \tag{3.44}$$

We plug (3.44) in (3.34) and obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_s^2 \lesssim \left(1 + \log_+ \|\Omega\|_{L^\infty} + \log_+ \|u\|_s\right) \|\Omega\|_{L^\infty} \|u\|_s^2 \ . \tag{3.45}$$

Divide (3.45) for $\left(e + \|u\|_s^2\right) \log_+ \|u\|_s^2$ and we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} \log_+ \log_+ \|u\|_s \lesssim \left(1 + \frac{\log_+ \|\Omega\|_{L^\infty}}{\log_+ \|u\|_s}\right) \|\Omega\|_{L^\infty} \ . \tag{3.46}$$

Next notice that from Lemma 1.20 and Eq. (3.38) and the fact that $s > \frac{\mathrm{d}}{2} + 1$ we have that

$$\|\Omega\|_{L^\infty} \lesssim \|\Omega_{s-1}\| \lesssim \|u\|_{H^s}$$

so that $\frac{\log_+ \|\Omega\|_{L^\infty}}{\log_+ \|u\|_s} \lesssim_s 1$ and (3.46) becomes, after integration in time

$$\log_+^2 \|u(t)\|_s \leq \log_+^2 \|u_0\|_s + C_s \int_0^t \left\|\Omega\left(t'\right)\right\|_{L^\infty} \mathrm{d}t', \tag{3.47}$$

which concludes the proof.

□

3.4 Global existence in dimension two

In the present section we address the specific problem of the existence of solutions when $\mathrm{d} = 2$. The result we prove is the following one

Theorem 3.15. *Let $s > 2$, $u \in \mathcal{C}\left([0,T]; H^s_\sigma\left(\mathbb{R}^2\right)\right)$ be the unique maximal solution of* (E) *stemming from $u_0 \in H^s_\sigma\left(\mathbb{R}^2\right)$ then $T = \infty$ and in particular there exists a $C_s > 0$ such that*

$$\|u(t)\|_s \leq \|u_0\|_s \, e^{C_s e^{C_s t}} \qquad \forall t > 0 \, .$$

The main observation is that in dimension two the structure of the equation for the vorticity is fundamentally simpler, in fact explicit computations show that

$$\Omega_t + u^\intercal \nabla \Omega + Ju\, \Omega - \Omega^\intercal \, \nabla u = 0 \, , \tag{3.48}$$

but, when $\mathrm{d} = 2$ and ω is as in Remark 3.12 we obtain from Eqs. (3.38) and (E) that

$$\omega_t + u \cdot \nabla \omega = 0, \qquad u = \Delta^{-1} \nabla \omega \, . \tag{3.49}$$

Notice that (3.49) is a *transport* equation, so that we have the following result

Lemma 3.16. *Let* $\mathrm{d}=2$*,* $s>2$*,* $u_0 \in H^s_\sigma$ *and let* $u \in \mathcal{C}\left([0,T];H^s_\sigma\right)$ *be the unique solution of* (E) *stemming from* u_0 *constructed in Theorem 2.1. Let* ω *be as in* (3.49)*, then we have that*

$$\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p} \ , \qquad\qquad \forall\, t \in [0,T]\,,\ p \in [1,\infty] \ .$$

Proof. Let us recall that by construction in Theorem 2.1 $T \sim \|u_0\|_s^{-1}$. The case $p \in [1,\infty)$ follows by a simple L^p energy estimate on (3.49), in fact since $\operatorname{div} u = 0$ we have that

$$\int u\cdot\nabla\omega \ |\omega|^{p-1}\omega \,\mathrm{d}x = \frac{1}{p}\int u\cdot\nabla|\omega|^p = 0\ .$$

For $p=\infty$ we argue differently, let X be the flow map constructed in Theorem 2.2, it is a matter of straighforward computations to check that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega\left(t,X(t,x)\right)=0 \qquad\qquad \Leftrightarrow \qquad\qquad \omega\left(t,X(t,x)\right)\equiv\omega_0(x)\,,$$

so that, setting $y := X(t,x)$ we obtain that

$$\omega\left(t,y\right)=\omega_0\left(X^{-t}\left(y\right)\right)\ , \tag{3.50}$$

hence

$$\sup_y\left|\omega\left(t,y\right)\right| = \sup_y\left|\omega_0\left(X^{-t}\left(y\right)\right)\right| = \sup_z|\omega_0(z)|\ ,$$

where the last identity is justified by the fact that X^{-t} is an homeomorphism of $\mathbb{R}^{\mathrm{d}}$ and, as such, it is surjective. □

Proof of Theorem 3.15. Equation (3.47) implies that there exists a $C_s>0$ such that

$$\|u(t)\|_s \le \|u_0\|_s \exp\left\{C_s \exp\left\{C_s\int_0^t \left\|\omega\left(t'\right)\right\|_{L^\infty}\mathrm{d}t'\right\}\right\}$$

which we combine with Lemma 3.16, concluding. □

Part II

Weak solutions

The theory of strong solutions developed in the first part provides a solid foundation for understanding the behavior of the Euler equations under the assumption of sufficient smoothness. However, this framework has two significant limitations. First, it requires highly regular initial data (e.g., $u_0 \in H^s$ with $s > \frac{d}{2} + 1$), a condition that excludes many physically relevant scenarios, such as the evolution of a *vortex patch*, where the initial vorticity $\omega_0 = \mathbb{1}_{D_0}$ is merely a characteristic function of a domain D_0. Second, the global existence of strong solutions in three dimensions remains one ofthe greatest open problems in mathematical physics.

To address these limitations, we turn to the concept of *weak solutions*. This broader class of solutions allows for rougher initial data and provides a framework for describing fluid motion even after the potential formation of singularities. However, this generality comes at a cost: uniqueness is often lost, and the space of weak solutions can harbor physically paradoxical behaviors, such as solutions that arise from a state of rest without any external force, thus violating the conservation of energy.

This second part navigates the rich and complex landscape of weak solutions. We will begin by studying a foundational result by Yudovich [15], which establishes the global existence and uniqueness of weak solutions in two dimensions for initial vorticity in the space $L^1 \cap L^\infty$. This theory is powerful enough to handle the vortex patch problem, and we will explore the remarkable result by Chemin [4], which shows that the boundary of the patch maintains its initial regularity for all time.

Finally, we will venture into the more "wild" aspects of weak solutions by introducing the powerful method of *convex integration*, as pioneered by De Lellis and Székelyhidi in [5]. This geometric technique reveals a surprising flexibility in the Euler equations, allowing for the construction of infinitely many weak solutions that can be prescribed to have specific, and at times physically startling, kinetic energy profiles. This exploration will underscore the profound challenges that remain in selecting the "physically correct" solution among a universe of mathematical possibilities.

Chapter 4

Yudovich Theory for Bounded Vorticity

As we saw in Theorem 3.15, the two-dimensional case for the Euler equations is special due to the structure of the vorticity equation. When $\mathtt{d} = 2$, the *vorticity*, defined as the scalar quantity $\omega = \partial_1 u_2 - \partial_2 u_1$, evolves according the a pure transport equation (3.49). This absence of the *vortex stretching term* $Ju\,\Omega - (Ju\,\Omega)^\intercal$, appearing in (3.48), prevents the local amplification of vorticity and is the key to global regularity for strong solutions.

This structure also allows us to construct global solutions for initial data that are far less regular than the H^s spaces required by Theorem 2.1. The seminal work of V. I. Yudovich [15] established a global well-posedness theory for initial vorticity ω_0 merely in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. This framework is robust enough to describe singular structures like vortex patches, where the vorticity is discontinuous across an interface. For a comprehensive treatment, we refer the reader to the monographs [11, 12].

4.1 The Vorticity-Stream Formulation

To handle weak solutions, it is convenient to reformulate the Euler equations entirely in terms of the vorticity, as it is done in (3.49). The velocity field u can be recovered from the vorticity ω. Since the flow is incompressible ($\operatorname{div} u = 0$) in a simply connected domain like $\mathbb{R}^2$, there exists a *stream function* ψ such that:

$$u = \nabla^\perp \psi := \begin{bmatrix} -\partial_2 \psi \\ \partial_1 \psi \end{bmatrix}.$$

Taking the curl of this expression, we find the relationship between the vorticity and the stream function:

$$\omega = \partial_1 u_2 - \partial_2 u_1 = \partial_1^2 \psi + \partial_2^2 \psi = \Delta \psi.$$

Formally inverting the Laplacian gives $\psi = \Delta^{-1}\omega$. The fundamental solution for the Laplacian in $\mathbb{R}^2$ is $\frac{1}{2\pi}\log|x|$, which allows us to express the velocity field u directly in terms of ω:

$$\begin{aligned} u(x) &= \nabla^\perp \left(\Delta^{-1}\omega\right)(x) \\ &= \frac{1}{2\pi}\nabla^\perp \int_{\mathbb{R}^2} \log|x-y|\,\omega(y)\,\mathrm{d}y \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,\mathrm{d}y. \end{aligned} \tag{4.1}$$

Equation (4.1) is the 2D analogous of (3.38), which we can write concisely as a convolution:

$$u = K \star \omega, \quad \text{where} \quad K(x) := \frac{1}{2\pi}\frac{x^\perp}{|x|^2}. \tag{4.2}$$

This formulation allows us to define a notion of a weak solution that does not rely on the differentiability of u.

Definition 4.1 (Yudovich Weak Solution)**.** Let $\omega_0 \in L^1(\mathbb{R}^2;\mathbb{R}) \cap L^\infty(\mathbb{R}^2;\mathbb{R})$. A pair (ω, u) is a *weak solution* of (3.49) if:

- For any $T > 0$, the vorticity ω belongs to the space $L^\infty\left([0,T];L^1\left(\mathbb{R}^2;\mathbb{R}\right)\cap L^\infty\left(\mathbb{R}^2;\mathbb{R}\right)\right)$.
- The velocity u is determined by the Biot-Savart law $u = K\star\omega$, and conversely $\omega = \nabla\times u$ in the sense of distributions.
- For any test function $\phi\in\mathcal{C}^1\left([0,T];\mathcal{C}_c^1\left(\mathbb{R}^2;\mathbb{R}\right)\right)$, the following integral identity holds:

$$\int_{\mathbb{R}^2}\omega(T,x)\phi(T,x)\mathrm{d}x-\int_{\mathbb{R}^2}\omega_0(x)\phi(0,x)\mathrm{d}x=\int_0^T\int_{\mathbb{R}^2}\omega(t,x)\left(\partial_t+u\cdot\nabla\right)\phi(t,x)\mathrm{d}x\mathrm{d}t. \tag{4.3}$$

The main result of this section is the celebrated theorem by Yudovich, cf. [15], which we state here. We will focus on proving existence first and address uniqueness later.

Theorem 4.2 (Yudovich)**.** *Let $\omega_0\in L^1\left(\mathbb{R}^2;\mathbb{R}\right)\cap L^\infty\left(\mathbb{R}^2;\mathbb{R}\right)$. Then there exists a unique weak solution (ω,u) to the 2D Euler equations* (3.49) *in the sense of Definition 4.1.*

Remark 4.3. The proof of uniqueness requires a separate line of argument based on the logarithmic regularity of the velocity field, which we will postpone. Our immediate goal is to construct a solution. For detailed proofs of both existence and uniqueness, see [11,12].

4.2 Existence of Yudovich Weak Solutions

The proof of existence relies on a *regularization scheme*. The idea is to approximate the rough initial data $\omega_0\in L^1\cap L^\infty$ with a sequence of smooth initial data ω_0^ε, solve the Euler equations for each smooth datum, and then pass to the limit as the regularization parameter $\varepsilon\to 0$. Let $\eta\in\mathcal{C}_c^\infty\left(\mathbb{R}^2;\mathbb{R}\right)$ be a standard *mollifier*, i.e., a non-negative function supported in the unit ball $B(0,1)$ with $\int_{\mathbb{R}^2}\eta(x)\mathrm{d}x=1$. For any $\varepsilon>0$, we define the rescaled mollifier $\eta_\varepsilon(x):=\varepsilon^{-2}\eta(x/\varepsilon)$ and the mollified initial vorticity:

$$\omega_0^\varepsilon(x):=\left(\eta_\varepsilon\star\omega_0\right)(x)=\int_{\mathbb{R}^2}\eta_\varepsilon\left(x-y\right)\omega_0\left(y\right)\mathrm{d}y.$$

By standard properties of convolutions, the family $\left(\omega_0^\varepsilon\right)_{\varepsilon>0}$ satisfies:

Lemma 4.4 (Properties of the mollified initial data)**.** *We have*

1. *for any $p\in[1,\infty]$, $\left\|\omega_0^\varepsilon\right\|_{L^p}\le\left\|\eta_\varepsilon\right\|_{L^1}\|\omega_0\|_{L^p}=\|\omega_0\|_{L^p}$;*
2. *$\omega_0^\varepsilon\to\omega_0$ strongly in $L^1\left(\mathbb{R}^2\right)$ as $\varepsilon\to 0$;*
3. *For each $\varepsilon>0$, ω_0^ε is a smooth function in $\mathcal{C}^\infty\left(\mathbb{R}^2;\mathbb{R}\right)$ and belongs to $H^s\left(\mathbb{R}^2;\mathbb{R}\right)$ for all s.*

Since ω_0^ε is smooth, the corresponding initial velocity $u_0^\varepsilon:=K\star\omega_0^\varepsilon$ is also smooth. Therefore, for each $\varepsilon>0$, the conditions of Theorem 3.15 are met. This guarantees the existence of a unique global strong solution $(\omega^\varepsilon,u^\varepsilon)$ to the 2D Euler equations, which also satisfies the weak formulation (4.3).

The core of the proof is to establish *uniform bounds* on the sequence of approximate solutions $(\omega^\varepsilon,u^\varepsilon)$ that are independent of ε. These bounds will allow us to use a compactness argument to extract a convergent subsequence whose limit is the desired weak solution. These key steps are summarized in the following proposition.

Proposition 4.5. *Let $(\omega^\varepsilon,u^\varepsilon)$ be the sequence of global strong solutions stemming from the mollified initial data ω_0^ε constucted in Theorem 3.15. The following hold:*

1. ***Uniform Bounds:*** *For any $t>0$, the solutions are uniformly bounded:*

$$\left\|u^\varepsilon(t,\cdot)\right\|_{L^\infty}\lesssim\left\|\omega^\varepsilon(t,\cdot)\right\|_{L^1\cap L^\infty}\le\|\omega_0\|_{L^1\cap L^\infty}. \tag{4.4}$$

2. ***Convergence:*** *There exists a pair (ω,u) with $\omega\in L^\infty\left([0,T];L^1\cap L^\infty\right)$ and $u=K\star\omega$, and a subsequence (not relabeled) such that for any $T>0$:*

$$\omega^\varepsilon\to\omega\quad\text{strongly in }\mathcal{C}\left([0,T];L^1_{\mathrm{loc}}\left(\mathbb{R}^2\right)\right), \tag{4.5a}$$
$$u^\varepsilon\to u\quad\text{strongly in }\mathcal{C}\left([0,T];L^\infty_{\mathrm{loc}}\left(\mathbb{R}^2\right)\right). \tag{4.5b}$$

Once this proposition is established, the existence part of Theorem 4.2 follows by passing to the limit in the integral formulation (4.3) for $(\omega^\varepsilon, u^\varepsilon)$. The uniform bounds on ω^ε and the strong local convergence of u^ε are sufficient to justify passing to the limit in the nonlinear term using the dominated convergence theorem.

Proof of Theorem 4.2: existence. The proof of uniqueness is postponed; we focus here on constructing a solution. The strategy is to show that the limit pair (ω, u), whose existence is guaranteed by Proposition 4.5, satisfies the weak formulation given in Definition 4.1.

By construction, each approximate solution $(\omega^\varepsilon, u^\varepsilon)$ is a global strong solution and therefore satisfies the weak formulation (4.3) for any test function $\phi \in \mathcal{C}^1\left([0,T];\mathcal{C}^1_c\left(\mathbb{R}^2;\mathbb{R}\right)\right)$:

$$\int_{\mathbb{R}^2} \omega^\varepsilon(T,x)\phi(T,x)\mathrm{d}x - \int_{\mathbb{R}^2} \omega_0^\varepsilon(x)\phi(0,x)\mathrm{d}x = \int_0^T \int_{\mathbb{R}^2} \omega^\varepsilon(t,x)\left(\partial_t + u^\varepsilon \cdot \nabla\right)\phi(t,x)\mathrm{d}x\mathrm{d}t. \tag{4.6}$$

Our goal is to pass to the limit as $\varepsilon \to 0$ in each term. Let $\mathcal{K}$ be a compact set containing the support of $\phi(t,\cdot)$ for all $t \in [0,T]$.

Convergence of the Terms

1. **Initial and Final Time Terms**: From Lemma 4.4, $\omega_0^\varepsilon \to \omega_0$ strongly in $L^1(\mathbb{R}^2)$. From Proposition 4.5, $\omega^\varepsilon \to \omega$ strongly in $\mathcal{C}\left([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^2)\right)$. Since ϕ is compactly supported and smooth, the boundary terms converge as desired:

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} \omega_0^\varepsilon(x)\phi(0,x)\mathrm{d}x = \int_{\mathbb{R}^2} \omega_0(x)\phi(0,x)\mathrm{d}x,$$
$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} \omega^\varepsilon(T,x)\phi(T,x)\mathrm{d}x = \int_{\mathbb{R}^2} \omega(T,x)\phi(T,x)\mathrm{d}x.$$

2. **The Integral Term**: The linear part involving $\partial_t\phi$ converges for the same reason. For the nonlinear term, we must show that:

$$\lim_{\varepsilon\to 0}\int_0^T \int_{\mathcal{K}} \omega^\varepsilon u^\varepsilon \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathcal{K}} \omega u \cdot \nabla\phi\,\mathrm{d}x\mathrm{d}t.$$

   We analyze the difference by splitting it into two parts:

$$\int_0^T \int_{\mathcal{K}} \left(\omega^\varepsilon u^\varepsilon - \omega u\right)\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathcal{K}} \left(\omega^\varepsilon - \omega\right) u^\varepsilon \cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathcal{K}} \omega\left(u^\varepsilon - u\right)\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t.$$

   For the first integral on the right-hand side, we use the uniform L^∞ bound on u^ε from Proposition 4.5:

$$\left|\int_0^T \int_{\mathcal{K}} \left(\omega^\varepsilon - \omega\right) u^\varepsilon \cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t\right| \le C \sup_{t\in[0,T]} \left\|u^\varepsilon(t,\cdot)\right\|_{L^\infty} \int_0^T \left\|\omega^\varepsilon(t,\cdot) - \omega(t,\cdot)\right\|_{L^1(\mathcal{K})}\mathrm{d}t \xrightarrow{\varepsilon\to 0} 0.$$

   For the second integral, we use the strong local convergence of u^ε:

$$\left|\int_0^T \int_{\mathcal{K}} \omega\left(u^\varepsilon - u\right)\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t\right| \le C\|\omega\|_{L^\infty([0,T];L^1(\mathcal{K}))} \int_0^T \left\|u^\varepsilon(t,\cdot) - u(t,\cdot)\right\|_{L^\infty(\mathcal{K})}\mathrm{d}t \xrightarrow{\varepsilon\to 0} 0.$$

   Thus, the nonlinear term converges as required.

Conclusion Since every term in (4.6) converges to its counterpart for the limit pair (ω, u), we conclude that (ω, u) satisfies the weak formulation (4.3). The pair also satisfies the other conditions of Definition 4.1 by construction. This completes the proof of existence for Theorem 4.2. ☐

4.3 Proof of Proposition 4.5

The proof is divided into two main parts: establishing the uniform bounds and then using them to prove convergence via a compactness argument.

Part 1: Uniform Bounds This part corresponds to the first statement of the proposition. The conservation of L^p norms for the vorticity equation (3.49) is a classical result for smooth solutions, as seen in the proof of Lemma 3.16. Since each $(\omega^\varepsilon, u^\varepsilon)$ is a strong solution, this property holds. For any $p \in [1,\infty]$ and for all $t > 0$:

$$\left\|\omega^\varepsilon(t,\cdot)\right\|_{L^p} = \left\|\omega_0^\varepsilon\right\|_{L^p}.$$

Using the properties of mollification, we have $\left\|\omega_0^\varepsilon\right\|_{L^p} \le \|\omega_0\|_{L^p}$. Combining these gives the uniform bound for the vorticity:

$$\left\|\omega^\varepsilon(t,\cdot)\right\|_{L^1\cap L^\infty} \le \|\omega_0\|_{L^1\cap L^\infty}. \tag{4.7}$$

To bound the velocity $u^\varepsilon = K \star \omega^\varepsilon$, we split the Biot-Savart kernel $K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$ into its local and far-field parts. Let χ be as in (3.9). We write $u^\varepsilon = u_1^\varepsilon + u_2^\varepsilon$, where

$$u_1^\varepsilon := (\chi K)\star\omega^\varepsilon, \qquad u_2^\varepsilon := ((1-\chi)K)\star\omega^\varepsilon.$$

Applying Young's convolution inequality, we estimate each part:

$$\begin{aligned}\left\|u_1^\varepsilon\right\|_{L^\infty} &\le \left\|\chi K\right\|_{L^1}\left\|\omega^\varepsilon\right\|_{L^\infty},\\ \left\|u_2^\varepsilon\right\|_{L^\infty} &\le \left\|(1-\chi)K\right\|_{L^\infty}\left\|\omega^\varepsilon\right\|_{L^1}.\end{aligned}$$

Since $K \in L^1_{\mathrm{loc}}$ and decays like $|x|^{-1}$ at infinity (cf. (4.2)), both $\left\|\chi K\right\|_{L^1}$ and $\left\|(1-\chi)K\right\|_{L^\infty}$ are finite constants, this gives

$$\left\|u^\varepsilon(t,\cdot)\right\|_{L^\infty} \lesssim \left\|\omega^\varepsilon(t,\cdot)\right\|_{L^1\cap L^\infty}. \tag{4.8}$$

Equations (4.7) and (4.8) prove (4.4).

Part 2: Convergence The proof of convergence relies on the properties of the particle flow maps. For each $\varepsilon > 0$, let $X_\varepsilon(t,\alpha)$ be the unique flow map generated by u^ε, whose existence is guaranteed by Theorem 2.2. Since ω^ε is transported by the flow and solves (3.49), we can write it in terms of the initial data and the *backward flow map* $X_\varepsilon^{-t}(\cdot) := X_\varepsilon(t,\cdot)^{-1}$ as (cf. (3.50)):

$$\omega^\varepsilon(t,x) = \omega_0^\varepsilon\left(X_\varepsilon^{-t}(x)\right). \tag{4.9}$$

The strategy is to show that the family of maps $\left(X_\varepsilon^{-t}\right)_{\varepsilon>0}$ is compact, allowing us to extract a limit X^{-t}. The key ingredients are the following estimates on the velocity and the resulting flow, whose proofs we postpone for clarity.

Lemma 4.6 (Logarithmic Lipschitz Estimate for Velocity)**.** *Let $\omega^\varepsilon(t,\cdot) \in L^1 \cap L^\infty$ for any $t \in [0,T]$ and $u^\varepsilon = K \star \omega^\varepsilon$, then u^ε is uniformly quasi-Lipschitz, i.e.*

$$\left|u^\varepsilon(t,x_1) - u^\varepsilon(x_2,t)\right| \leqslant C\|\omega_0\|_{L^1\cap L^\infty}|x_1-x_2|\left(1-\min\{0,\ \log|x_1-x_2|\}\right).$$

Remark 4.7. It is crucial to pause and appreciate the significance of Lemma 4.6. Recall that in the proof of Theorem 2.2 for strong solutions, the existence and uniqueness of the flow map was a direct consequence of the velocity field being *Lipschitz continuous.* This property fails for Yudovich solutions; the estimate in the lemma shows that the Lipschitz norm of the velocity is formally infinite due to the logarithmic divergence as $|x_1 - x_2| \to 0$.

However, this failure is exceptionally mild. A logarithmic divergence is, in a sense, the weakest possible singularity that breaks the Lipschitz condition. This *near-Lipschitz* regularity is precisely the critical threshold needed to retain many of the desirable properties typically associated with strong solutions. Most importantly, the log-Lipschitz condition is sufficient to guarantee the existence and uniqueness of particle trajectories via *Osgood's criterion* for ODEs.

Therefore, while the initial vorticity can be merely bounded and discontinuous, the resulting flow is remarkably well-behaved, ensuring that the fluid particles follow unique, non-intersecting paths. This is why Yudovich solutions, despite being weak solutions, are considered to be on the "tame" side of the weak solution spectrum, bridging the gap between the classical theory and more singular behaviors.

Lemma 4.8 (Equicontinuity of Flow Maps)**.** *Let the family of velocity fields* $(u^\varepsilon)_{\varepsilon>0}$ *be uniformly bounded in* L^∞ *and uniformly log-Lipschitz. Then the corresponding backward flow maps* $\left(X_\varepsilon^{-t}\right)$ *are uniformly bounded and equicontinuous on any compact subset of* $[0,T]\times\mathbb{R}^2$.

With these lemmas, we can apply the Arzelà-Ascoli theorem. Since the maps $X_\varepsilon^{-t}(x)$ are uniformly bounded and equicontinuous, there exists a subsequence (which we do not relabel) and a limit map $X^{-t}\in \mathcal{C}\left([0,T]\times\mathbb{R}^2;\mathbb{R}^2\right)$ such that for any compact set $\mathcal{K}\Subset\mathbb{R}^2$:

$$X_\varepsilon^{-t}\to X^{-t}\quad \text{uniformly on } [0,T]\times\mathcal{K}. \tag{4.10}$$

Notice, in particular, that since $\operatorname{div} u^\varepsilon=0$ then by Lemma 1.4 we have that $\det\left(\nabla X_\varepsilon(t,x)\right)\equiv 1$ for all $t\in[0,T]$. We now define the limit vorticity and velocity as:

$$\omega(t,x):=\omega_0\left(X^{-t}(x)\right),\qquad\qquad u(t,x):=(K\star\omega)(t,x). \tag{4.11}$$

We shall need later the following result on the limit back-to-label map:

Lemma 4.9. *For all* $f\in L^1\left(\mathbb{R}^2;\mathbb{R}\right)$ *and* $t\in[0,t]$ *we have that*

$$\int f\left(X^{-t}(y)\right)\mathrm{d}y=\int f(x)\,\mathrm{d}x \tag{4.12}$$

Proof. W.l.o.g. we can assume $f\geq 0$, let f^δ a δ-approximating sequence of $\mathcal{C}_c^\infty$ functions of f, indeed since $\operatorname{div} u^\varepsilon=0$ we have, thanks to Lemma 1.4, that $\det\nabla X_\varepsilon\equiv 1$ so that

$$\int f(x)\,\mathrm{d}x=\int f\left(X_\varepsilon^{-t}(y)\right)\mathrm{d}y,\qquad\qquad \forall\varepsilon>0. \tag{4.13}$$

Moreover we have that

$$\int f\left(X_\varepsilon^{-t}(y)\right)\mathrm{d}y=\int\left(f\left(X_\varepsilon^{-t}(y)\right)-f^\delta\left(X_\varepsilon^{-t}(y)\right)\right)\mathrm{d}y+\int f^\delta\left(X_\varepsilon^{-t}(y)\right)\mathrm{d}y. \tag{4.14}$$

certainly we have that

$$\int\left(f\left(X_\varepsilon^{-t}(y)\right)-f^\delta\left(X_\varepsilon^{-t}(y)\right)\right)\mathrm{d}y\xrightarrow{\delta\to 0}0 \tag{4.15}$$

thanks to integration by substitution and Lemma 4.4. Next, being f^δ compactly supported and smooth we obtain that $f^\delta\circ X_\varepsilon^{-t}$ is compactly supported and continuous and we have that, pointwise, thanks to (4.10)

$$f^\delta\left(X_\varepsilon^{-t}(y)\right)\xrightarrow{\varepsilon\to 0}f^\delta\left(X^{-t}(y)\right),\qquad\qquad \forall y\in\mathbb{R}^2,\ t\in[0,T].$$

By dominated convergence

$$\int f^\delta\left(X^{-t}(y)\right)\mathrm{d}y=\lim_{\varepsilon\to 0}\int f^\delta\left(X_\varepsilon^{-t}(y)\right)\mathrm{d}y\overset{(4.13)}{=\!=\!=}\int f^\delta(x)\,\mathrm{d}x \tag{4.16}$$

We combine now Eqs. (4.14) to (4.16) and obtain, after a passage for $\varepsilon\to 0$ that

$$\int f(x)\,\mathrm{d}x=o(\delta)+\int f^\delta\left(X^{-t}(y)\right)\mathrm{d}y. \tag{4.17}$$

In order to obtain (4.12) it suffice to pass to the limit for $\delta\to 0$ in (4.17) and use the fact that $f^\delta\xrightarrow[\delta\to 0]{L^1}f$. □

The final step is to show that $\omega^\varepsilon\to\omega$ and $u^\varepsilon\to u$ in the topologies specified in Eqs. (4.5a) and (4.5b). Set $\mathcal{K}\Subset\mathbb{R}^2$, for the vorticity, we estimate the $L^1(\mathcal{K})$ norm of the difference for any fixed $t\in[0,T]$:

$$\begin{aligned}\left\|\omega^\varepsilon(t,\cdot)-\omega(t,\cdot)\right\|_{L^1(\mathcal{K})}&=\left\|\omega_0^\varepsilon\left(X_\varepsilon^{-t}\right)-\omega_0\left(X^{-t}\right)\right\|_{L^1(\mathcal{K})}\\&\leq\left\|\omega_0^\varepsilon\left(X_\varepsilon^{-t}\right)-\omega_0^\varepsilon\left(X^{-t}\right)\right\|_{L^1(\mathcal{K})}+\left\|\omega_0^\varepsilon\left(X^{-t}\right)-\omega_0\left(X^{-t}\right)\right\|_{L^1(\mathcal{K})}\quad=:J_1^\varepsilon(t)+J_2^\varepsilon(t).\end{aligned} \tag{4.18}$$

The convergence in (4.10) and the fact that ω_0^ε is smooth implies that

$$\omega_0^\varepsilon \circ X_\varepsilon^{-t} - \omega_0^\varepsilon \circ X^{-t} \xrightarrow{\varepsilon\to 0} 0 \qquad \text{in } L^\infty_{\text{loc}}\left([0,T]\times\mathbb{R}^2\right),$$

which implies, combined with the embedding $L^\infty(\mathcal{K}) \hookrightarrow L^1(\mathcal{K})$ that

$$\sup_{t\in[0,T]} J_1^\varepsilon(t) \xrightarrow{\varepsilon\to 0} 0\,. \tag{4.19}$$

Next, we analyze the term $J_2^\varepsilon(t)$, we apply Lemmas 4.4 and 4.9 and obtain that

$$\sup_{t\in[0,T]} J_2^\varepsilon(t) \xrightarrow{\varepsilon\to 0} 0\,. \tag{4.20}$$

We combine Eqs. (4.19) and (4.20) in (4.18) and, by arbitrarity of $\mathcal{K}$, we prove (4.5a).

It remains now to prove (4.5b), let χ be as in (3.9) and with $\chi_\delta(x) = \chi\left(x/\delta\right)$, so that we have the following

$$\left|u^\varepsilon(t,x) - u(t,x)\right| \leqslant \left|\left(\chi_\delta K\right)\star\left(\omega^\varepsilon-\omega\right)(t,x)\right| + \left|\left(\left(1-\chi_\delta\right)K\right)\star\left(\omega^\varepsilon-\omega\right)(t,x)\right|,$$

and for all $(t,x)\in[0,T]\times\mathcal{K}$ we have

$$\begin{aligned}
\left|\left(\chi_\delta K\right)\star\left(\omega^\varepsilon-\omega\right)(t,x)\right| &\leqslant \left\|\chi_\delta K\right\|_{L^1}\left\|\omega^\varepsilon-\omega\right\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leqslant C\left\|\chi_\delta K\right\|_{L^1} \xrightarrow{\delta\to 0} 0,\\
\left|\left(\left(1-\chi_\delta\right)K\right)\star\left(\omega^\varepsilon-\omega\right)(t,x)\right| &\leqslant \left\|\left(1-\chi_\delta\right)K\right\|_{L^\infty}\left\|\omega^\varepsilon-\omega\right\|_{L^\infty([0,T];L^1(\mathcal{K}))} \xrightarrow{\varepsilon\to 0} 0,
\end{aligned}$$

which proves (4.5b). □

4.4 Proof of Lemma 4.6

Recall that the velocity field u^ε is recovered from the vorticity ω^ε via the Biot-Savart law (4.2):

$$u^\varepsilon(t,x) = \left(K\star\omega^\varepsilon\right)(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-z)^\perp}{|x-z|^2}\omega^\varepsilon(t,z)\,\mathrm{d}z.$$

Let us select two distinct points $x_1, x_2\in\mathbb{R}^2$ and assume, without loss of generality, that the distance $d := |x_1-x_2| < 1$. The difference can be written as:

$$u^\varepsilon(t,x_1) - u^\varepsilon(t,x_2) = \int_{\mathbb{R}^2}\left[K(x_1-z) - K(x_2-z)\right]\omega^\varepsilon(t,z)\,\mathrm{d}z.$$

We split the integral into three regions based on the distance from x_1:

$$\begin{aligned}
\left|u^\varepsilon(t,x_1) - u^\varepsilon(t,x_2)\right| \le \left(\int_{\mathbb{R}^2\setminus B(x_1,2)} + \int_{B(x_1,2)\setminus B(x_1,2d)} + \int_{B(x_1,2d)}\right)\left|K(x_1-z)-K(x_2-z)\right|\left|\omega^\varepsilon(t,z)\right|\mathrm{d}z\\
=: \left(J_1+J_2+J_3\right)(t;x_1,x_2)\,.
\end{aligned}$$

We now estimate each term, using the uniform bounds on the vorticity from Proposition 4.5.

Part 1 (Estimate for the far field (J_1))**.** For $z\in\mathbb{R}^2\setminus B(x_1,2)$, we have $|x_1-z|\ge 2$ and $|x_2-z|\ge|x_1-z|-|x_1-x_2|\ge 2-d>1$. Using the identity $|K(a)-K(b)|\sim\frac{|a-b|}{|a||b|}$ and Lemma 3.16, we get:

$$J_1(t;x_1,x_2) \lesssim |x_1-x_2|\int_{\mathbb{R}^2\setminus B(x_1,2)}\frac{|\omega^\varepsilon(t,z)|}{|x_1-z|\,|x_2-z|}\mathrm{d}z \lesssim d\,\|\omega_0\|_{L^1}\,.$$

Part 2 (Estimate for the intermediate field (J_2))**.** In the region $B(x_1,2)\setminus B(x_1,2d)$, we apply the mean value theorem to the kernel K:

$$|K(x_1-z)-K(x_2-z)| \le \sup_{\xi\in(0,1)}\left|\nabla K\left(x_1-z+\xi(x_2-x_1)\right)\right||x_1-x_2| \lesssim \frac{d}{|x_1-z|^2}.$$

Integrating in polar coordinates around x_1, we find:

$$J_2(t;x_1,x_2) \lesssim \|\omega_0\|_{L^\infty}\,d\int_{B(x_1,2)\setminus B(x_1,2d)}\frac{\mathrm{d}z}{|x_1-z|^2} \sim \|\omega_0\|_{L^\infty}\,d\int_{2d}^2\frac{r}{r^2}\mathrm{d}r = \|\omega_0\|_{L^\infty}\,d\log\left(\frac{1}{d}\right).$$

Part 3 (Estimate for the near field (J_3)). For the innermost region, we use the triangle inequality:

$$J_3(t;x_1,x_2) \le \int_{B(x_1,2d)} |K(x_1-z)|\left|\omega^\varepsilon(t,z)\right| \mathrm{d}z + \int_{B(x_1,2d)} |K(x_2-z)|\left|\omega^\varepsilon(t,z)\right| \mathrm{d}z$$
$$\lesssim \|\omega_0\|_{L^\infty}\left(\int_0^{2d}\frac{r}{r}\mathrm{d}r + \int_0^{3d}\frac{r}{r}\mathrm{d}r\right) \lesssim \|\omega_0\|_{L^\infty} d.$$

Note that the second integral's domain is $B(x_2,3d)$, which covers $B(x_1,2d)$.

Combining the three estimates yields:

$$\left|u^\varepsilon(t,x_1)-u^\varepsilon(t,x_2)\right| \lesssim \|\omega_0\|_{L^1} d + \|\omega_0\|_{L^\infty} d\left(1+\log(1/d)\right).$$

Since $d=|x_1-x_2|$, this is precisely the logarithmic Lipschitz estimate. □

4.5 Proof of Lemma 4.10

We prove a stronger result than the one stated in Lemma 4.10, which is the following one:

Lemma 4.10 (Equicontinuity and Hölder Regularity of Flow Maps). *Let the family of velocity fields $(u^\varepsilon)_{\varepsilon>0}$ be uniformly bounded in L^∞ and uniformly log-Lipschitz as per Lemma 4.6. Then the corresponding backward flow maps $\left(X_\varepsilon^{-t}\right)$ are uniformly bounded and uniformly Hölder continuous on any compact subset of $[0,T]\times\mathbb{R}^2$. Specifically, there exists a constant $C>0$ and a time-dependent exponent $\beta(t) := \exp\left\{-C\|\omega_0\|_{L^1\cap L^\infty} t\right\}$ such that for any $\varepsilon>0$, $t,t_1,t_2\in[0,T]$ with $t_1\le t_2$ and any $x,x_1,x_2\in\mathbb{R}^2$:*

$$\left|X_\varepsilon^{-t}(x_1)-X_\varepsilon^{-t}(x_2)\right| \leqslant C|x_1-x_2|^{\beta(t)}, \tag{4.21a}$$
$$\left|X_\varepsilon^{-t_1}(x)-X_\varepsilon^{-t_2}(x)\right| \leqslant C(t_2-t_1)^{\beta(t_2)}. \tag{4.21b}$$

Proof. The proof relies on applying Osgood's criterion to the ODEs defining the particle trajectories, using the logarithmic Lipschitz estimate on the velocity field.

Part 1 (Hölder Continuity in Space). Let us analyze the distance between two backward trajectories starting from points x_1 and x_2. Let $y_1(s) := X_\varepsilon^{-s}(x_1)$ and $y_2(s) := X_\varepsilon^{-s}(x_2)$. These solve the ODE $\frac{\mathrm{d}}{\mathrm{d}s}y_i(s) = -u^\varepsilon\left(s,y_i(s)\right)$. Let $\rho(s) := \left|y_1(s)-y_2(s)\right|$. Its time evolution is governed by (cf. Eq. (2.53)):

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(s) = \frac{\left(y_1(s)-y_2(s)\right)}{\left|y_1(s)-y_2(s)\right|}\cdot\left(-\frac{\mathrm{d}}{\mathrm{d}t}y_1(s)+\frac{\mathrm{d}}{\mathrm{d}t}y_2(s)\right) \le \left|u^\varepsilon\left(s,y_1(s)\right)-u^\varepsilon\left(s,y_2(s)\right)\right|.$$

Applying the logarithmic Lipschitz estimate from Lemma 4.6, we obtain the differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(s) \leqslant C_0\|\omega_0\|_{L^1\cap L^\infty}\rho(s)\left(1-\min\left\{0,\log\rho(s)\right\}\right).$$

Assuming, without loss of generality, that $\rho(s)<1$, the inequality simplifies to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(s) \leqslant C_0\|\omega_0\|_{L^\infty}\rho(s)\left(1-\log\rho(s)\right).$$

Let $z(s) := \log\rho(s)$. The inequality becomes $\frac{\mathrm{d}}{\mathrm{d}t}z(s) \le C_0\|\omega_0\|_{L^\infty}(1-z(s))$. By Grönwall's inequality, this implies $1-z(s) \ge (1-z(0))\,e^{-C_0\|\omega_0\|_{L^\infty}s}$, which translates back to:

$$\log\left(\frac{1}{\rho(s)}\right) \ge \log\left(\frac{1}{\rho(0)}\right)e^{-C_0\|\omega_0\|_{L^\infty}s}.$$

Exponentiating both sides gives $\rho(s) \le \rho(0)^{\exp\{-C_0\|\omega_0\|_{L^\infty}s\}}$. Setting $s=t$ and noting that $\rho(0)=|x_1-x_2|$ proves (4.21a).

Part 2 (Hölder Continuity in Time)**.** To prove the temporal regularity, we use a trick involving the spatial estimate. Let $0 \le t_1 \le t_2 \le T$. We want to estimate $\left|X_\varepsilon^{-t_1}(x) - X_\varepsilon^{-t_2}(x)\right|$. Let $\alpha := X_\varepsilon^{-t_2}(x)$, which means $x = X_\varepsilon(t_2, \alpha)$. By the semigroup property of the flow, we have $X_\varepsilon^{-t_1}(x) = X_\varepsilon^{-t_1}(X_\varepsilon(t_2,\alpha)) = X_\varepsilon(t_2 - t_1, \alpha)$. Therefore, the difference is:

$$|X_\varepsilon(t_2 - t_1, \alpha) - \alpha| = \left|\int_0^{t_2-t_1} u^\varepsilon(s, X_\varepsilon(s,\alpha))\,\mathrm{d}s\right| \le \left\|u^\varepsilon\right\|_{L^\infty}(t_2 - t_1).$$

While correct, this only gives Lipschitz continuity. To get the Hölder estimate, we relate the distance to a spatial separation. Let $\alpha^\star := X_\varepsilon^{-t_1}(x)$. Then

$$\left|X_\varepsilon^{-t_1}(x) - X_\varepsilon^{-t_2}(x)\right| = \left|\alpha^\star - \alpha\right| = \left|X_\varepsilon^{-t_2}\left(X_\varepsilon\left(t_2, \alpha^\star\right)\right) - X_\varepsilon^{-t_2}(x)\right|.$$

Using the spatial Hölder continuity (4.21a) at time t_2:

$$\begin{aligned}\left|\alpha^\star - \alpha\right| &\le C\left|X_\varepsilon\left(t_2, \alpha^\star\right) - x\right|^{\beta(t_2)} \\ &= C\left|X_\varepsilon\left(t_2, X_\varepsilon^{-t_1}(x)\right) - X_\varepsilon\left(t_1, X_\varepsilon^{-t_1}(x)\right)\right|^{\beta(t_2)} \\ &= C\left|X_\varepsilon(t_2 - t_1, x) - x\right|^{\beta(t_2)}.\end{aligned}$$

The last term is the distance a particle starting at x travels in time $t_2 - t_1$.

$$|X_\varepsilon(t_2 - t_1, x) - x| = \left|\int_0^{t_2-t_1} u^\varepsilon(s, X_\varepsilon(s,x))\,\mathrm{d}s\right| \le \left\|u^\varepsilon\right\|_{L^\infty}(t_2 - t_1).$$

Combining these estimates proves (4.21b).

The uniform boundedness of the flow map follows from the uniform L^∞ bound on the velocity:

$$\left|X_\varepsilon^{-t}(x) - x\right| = \left|-\int_0^t u^\varepsilon\left(s, X_\varepsilon^{-s}(x)\right)\mathrm{d}s\right| \le t\left\|u^\varepsilon\right\|_{L^\infty} \le TC\left\|\omega_0\right\|_{L^1\cap L^\infty}.$$

Thus, the family of maps $\left(X_\varepsilon^{-t}\right)$ is uniformly bounded and equicontinuous on any compact set, as required for the Arzelà-Ascoli theorem. □

4.6 Uniqueness of weak solutions

Before starting to prove uniqueness of weak solutions let us remark that that since $u(t,\cdot) \in L^\infty$ for each $t \in [0,T]$ there exists a $\mathsf{L} = \mathsf{L}(T)$ s.t.

$$\bigcup_{t\in[0,T]} \operatorname{supp}\omega(t,\cdot) \subset B(0,\mathsf{L}).$$

We will use the following technical lemma whose proof is omitted:

Lemma 4.11. *Let u be weak solution of* (E) *constructed in Theorem 4.2, then for each* $p \in (1,\infty)$

$$\|\nabla u\|_{L^p} \leqslant C_0\left(\|\omega_0\|_{L^\infty}\right)\, p.$$

Let u_1, u_2 be the solutions of (E) stemming from the same initial data u_0 and let us define

$$E(t) = \|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2}^2,$$

since the evolution equation for $w = u_1 - u_2$ is

$$\partial_t w + w\cdot\nabla w + u_2\cdot\nabla w + w\cdot\nabla u_2 + \nabla\left(p_1 - p_2\right) = 0,$$

we easily deduce the differential inequality

$$E'(t) \lesssim \int |\nabla u_2(t,\cdot)||w(t,\cdot)|^2\,\mathrm{d}x \leqslant C_0 p\left(\|w(t,\cdot)\|_{L^\infty}^{\frac{2}{p-1}}\int |w|^2\,\mathrm{d}x\right)^{\frac{p-1}{p}} \leqslant Mp\,E(t)^{1-\frac{1}{p}}.$$

It is well known that the differential equality $f' = f^\alpha$, $\alpha \in (0,1)$ has no unique solution stemming from zero, but we know that $\bar{E}(t) = (Mt)^p$ si a solution that is maximal in the sense that $E(t) \leqslant \bar{E}(t)$. Set $t^\star$ s.t. $Mt^\star \leqslant 1/2$ and we obtain that

$$E(t) \leqslant 2^{-p} \xrightarrow{p\to\infty} 0, \quad \text{in } \left[0, t^\star\right],$$

concluding.

Chapter 5

The Vortex Patch Problem: Persistence of Regularity

The Yudovich theory guarantees the global existence and uniqueness of weak solutions for initial vorticity in $L^1 \cap L^\infty$. This framework is powerful enough to handle initial data that are characteristic functions of a domain, leading to a classic problem in fluid dynamics known as the *vortex patch problem*.

Let us consider an initial vorticity that is constant inside a bounded, simply connected domain $D_0 \subset \mathbb{R}^2$ and zero outside:

$$\omega_0(x) = \mathbb{1}_{D_0}(x).$$

We assume that the boundary of the initial domain, ∂D_0, is a smooth simple curve, for instance of class $\mathcal{C}^{1,\gamma}\left(\mathbb{T};\mathbb{R}^2\right)$ for some $\gamma \in (0,1)$.

According to the Yudovich theory, the vorticity at a later time t is simply the advection of the initial data by the unique flow map X:

$$\omega(t,x) = \omega_0\left(X^{-t}(x)\right) = \mathbb{1}_{D(t)}(x),$$

where $D(t) := X(t, D_0)$ is the image of the initial domain under the flow. While the velocity field u is only log-Lipschitz, a fundamental question arises concerning the boundary of the patch:

Question. If ∂D_0 is a $\mathcal{C}^{1,\gamma}\left(\mathbb{T};\mathbb{R}^2\right)$ curve, does $\partial D(t)$ remain $\mathcal{C}^{1,\gamma}\left(\mathbb{T};\mathbb{R}^2\right)$ for any, or all $t > 0$?

This question has a rich history. For some time, it was conjectured that the boundary could develop singularities in finite time, a phenomenon that would serve as a two-dimensional model for small-scale creation [10]. However, this was proven to be false in a landmark paper by J.-Y. Chemin [4], with a more geometric proof provided shortly after by Bertozzi and Constantin [1]. They showed that the boundary of the patch not only remains smooth but also that its curvature can grow at most as a double exponential in time.

To state the main result, it is convenient to track the boundary of the patch implicitly as the zero level-set of a function $\varphi(t,x)$. We start with a function $\varphi_0 \in \mathcal{C}^{1,\gamma}\left(\mathbb{R}^2\right)$ such that

$$D_0 = \left\{x \in \mathbb{R}^2 \mid \varphi_0(x) > 0\right\}, \qquad \partial D_0 = \left\{x \in \mathbb{R}^2 \mid \varphi_0(x) = 0\right\}.$$

Since the vorticity equation (3.49) is a transport equation, the function φ evolves according to the same law:

$$\partial_t \varphi + u \cdot \nabla \varphi = 0, \qquad \varphi\big|_{t=0} = \varphi_0. \tag{5.1}$$

The patch at time t is then given by $D(t) = \left\{x \mid \varphi(t,x) > 0\right\}$. The regularity of the boundary $\partial D(t)$ is determined by the regularity of $\varphi(t,\cdot)$.

Notation 5.1. ▷ Let us recall that the velocity flow of Yudovich solutions is L^∞, so that if $\omega_0 = \mathbb{1}_{D_0}$, and since the vorticity is transported by the velocity flow, then $D(t)$ is bounded set for any $t > 0$. We denote with

$$\mathsf{L} = \mathsf{L}(T) = \inf\left\{R > 0 \;\middle|\; D(t) \subset B(0,R),\ \forall\ t \in [0,T]\right\} < \infty,$$

▷ We use the notation $\left|\nabla\varphi\right|_\gamma = \sup_{x\neq y} \frac{\left|\nabla\varphi(x)-\nabla\varphi(y)\right|}{\left|x-y\right|^\gamma}$,

▷ Let (X, d) metric, $A, B \subset X$ and $c \in X$ we denote with

$$\mathsf{d}(c, A) = \inf_{a\in A} \mathsf{d}(c, a), \qquad \mathsf{d}(A, B) = \inf_{(a,b)\in A\times B} \mathsf{d}(a, b).$$

Theorem 5.2 (Global Regularity for Vortex Patches). *Let D_0 be a bounded domain with a $\mathcal{C}^{1,\gamma}$ boundary, for $\gamma \in (0,1)$, and let φ_0 be a corresponding level-set function that is non-degenerate on the boundary, i.e.,*

$$\inf_{x\in\partial D_0} \left|\nabla\varphi_0(x)\right| \geq m > 0.$$

Then the unique weak solution φ to (5.1), with u given by the Biot-Savart law for $\omega = \mathbb{1}_{D(t)}$, remains in $\mathcal{C}^{1,\gamma}$ for all time. More precisely, there exists a

$$C = C\left(\mathsf{L}, \left|\nabla\varphi_0\right|_\gamma, \left\|\nabla\varphi_0\right\|_{L^\infty}, \left|\nabla\varphi_0\right|_{\inf}\right) > 0,$$

and $C_0 > 0$ depending on the initial data such that the following bounds hold for all $t > 0$:

$$\begin{aligned}
\left\|\nabla u(t,\cdot)\right\|_{L^\infty} &\leqslant \left\|\nabla u(0,\cdot)\right\|_{L^\infty} e^{Ct}, \\
\left|\nabla\varphi(t,\cdot)\right|_\gamma &\leqslant \left|\nabla\varphi_0\right|_{\dot{\mathcal{C}}^\gamma} e^{(C_0+\gamma)e^{Ct}}, \\
\left\|\nabla\varphi(t,\cdot)\right\|_{L^\infty} &\leqslant \left\|\nabla\varphi_0\right\|_{L^\infty} e^{e^{Ct}}, \\
\inf_{x\in\partial D(t)} \left|\nabla\varphi(t,x)\right| &\geqslant \left(\inf_{x\in\partial D_0} \left|\nabla\varphi_0(x)\right|\right) e^{-e^{Ct}}.
\end{aligned}$$

5.1 The Gradient of the Velocity

Proposition 5.3 (Key Technical Estimate). *Let u be the velocity field generated by the vortex patch $D(t)$, which is the positive level set of a $\mathcal{C}^{1,\gamma}$ function $\varphi(t,\cdot)$. Assume $D(t) \subset B(0,\mathsf{L})$. Then the following bound holds:*

$$\left\|\nabla u\right\|_{L^\infty} \leqslant C\left[1 + \log\left(1 + \frac{\mathsf{L}\left|\nabla\varphi\right|_\gamma}{\inf_{x\in\partial D(t)}\left|\nabla\varphi(x)\right|}\right)\right], \tag{5.2}$$

where the constant C depends only on γ.

The proof of this proposition relies on a delicate geometric argument that analyzes the cancellation properties of the singular integral near the boundary. We will now lay the groundwork for this proof.

Compendium in harmonic analysis: Calderón-Zygmund theory A central quantity for understanding the evolution of the patch boundary is the velocity gradient, ∇u. Since $u = K \star \omega$ and the kernel K is homogeneous of degree -1, its gradient will be homogeneous of degree -2. For $\omega = \mathbb{1}_{D(t)}$, this results in a singular integral. To define it properly, we must use the concept of the *Cauchy principal value.*

Definition 5.4 (Cauchy Principal Value). Let $f \in \mathcal{C}^\infty\left(\mathbb{R}^\mathsf{d}\setminus\{0\};\mathbb{R}\right) \cap L^1_{\text{loc}}\left(\mathbb{R}^\mathsf{d};\mathbb{R}\right)$. The *Cauchy principal value integral* of f is defined as:

$$\text{p.v.}\int f(x)\,\mathrm{d}x := \lim_{\varepsilon\to0^+}\int_{|x|\geq\varepsilon} f(x)\,\mathrm{d}x.$$

Lemma 5.5 (Distributional Derivative of Homogeneous Kernels). *Let $\mathcal{K} \in \mathcal{C}^\infty\left(\mathbb{R}^\mathsf{d}\setminus\{0\};\mathbb{R}\right)$ be homogeneous of degree $1-\mathsf{d}$. For any test function $\phi \in \mathcal{C}^\infty_c\left(\mathbb{R}^\mathsf{d};\mathbb{R}\right)$, its distributional derivative is given by:*

$$\left\langle \partial_j\mathcal{K} \mid \phi\right\rangle = \text{p.v.}\int \partial_j\mathcal{K}(x)\,\phi(x)\,\mathrm{d}x + c_j\phi(0), \tag{5.3}$$

where the constant c_j is given by the integral over the unit sphere:

$$c_j := \int_{\mathbb{S}^{\mathsf{d}-1}} \mathcal{K}(z)\,z_j\,\mathrm{d}\mathcal{H}^{\mathsf{d}-1}(z).$$

Proof. The distributional derivative is defined by duality, $\langle \partial_j \mathcal{K} \mid \phi \rangle := -\langle \mathcal{K} \mid \partial_j \phi \rangle$. Since $\mathtt{K} \in L^1_{\text{loc}}(\mathbb{R}^{\mathtt{d}})$, we compute the integral by excising the singularity and applying Green's formula on the domain $\{x : |x| \geq \varepsilon\}$:

$$\begin{aligned}\langle \mathcal{K} \mid \partial_j \phi \rangle &= \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \mathcal{K}(x)\, \partial_j \phi(x)\, \mathrm{d}x \\ &= \lim_{\varepsilon \to 0^+} \left(-\int_{|x| \geq \varepsilon} \partial_j \mathcal{K}(x)\, \phi(x)\, \mathrm{d}x - \int_{|x| = \varepsilon} \mathcal{K}(x)\, \phi(x)\, n_j \mathrm{d}\mathcal{H}^{\mathtt{d}-1} \right),\end{aligned}$$

where $n_j = -x_j / |x|$ is the inward-pointing normal vector. The first term on the right is, by definition, $-\text{p.v.} \int \partial_j \mathcal{K} \phi \mathrm{d}x$. The boundary integral becomes:

$$\begin{aligned}\lim_{\varepsilon \to 0^+} \int_{|x| = \varepsilon} \mathcal{K}(x)\, \phi(x) \frac{x_j}{|x|} \mathrm{d}\mathcal{H}^{\mathtt{d}-1} &= \lim_{\varepsilon \to 0^+} \int_{\mathbb{S}^{\lceil -1}} \mathtt{K}(\varepsilon z)\, \phi(\varepsilon z)\, z_j \varepsilon^{\lceil -1} \mathrm{d}\mathcal{H}^{\mathtt{d}-1}(z) \\ &= \phi(0) \int_{\mathbb{S}^{\lceil -1}} \mathtt{K}(z)\, z_j \mathrm{d}\mathcal{H}^{\mathtt{d}-1}(z) = c_j \phi(0).\end{aligned}$$

Combining the terms gives the result stated in (5.3). □

Definition 5.6 (Calderón-Zygmund kernels)**.** Let $\mathtt{d} \in \mathbb{N} \setminus 0$, $\mathcal{K} \in \mathcal{C}^\infty(\mathbb{R}^{\mathtt{d}} \setminus 0; \mathbb{R})$ is a *Calderón-Zygmund kernel* if

1. **decay:** there exists $C > 0$ such that for all $x \in \mathbb{R}^{\mathtt{d}}$
$$|\mathcal{K}(x)| \leq C |x|^{-\mathtt{d}};$$
2. **zero average:** we have that
$$\int_{\mathbb{S}^{\mathtt{d}-1}} \mathcal{K}(x)\, \mathrm{d}x = 0.$$

Definition 5.7 (Singular Integral Operator)**.** We say that T is a *Singular Integral Operator (SIO)* if there exists a $\mathcal{K}$ CZ kernel such that

$$Tf = \mathcal{K} \star f.$$

Application to incompressible Euler equations Applying Lemma 5.5 to the Biot-Savart kernel and convolving with the vorticity $\omega = \mathbb{1}_{D(t)}$ yields the precise formula for the velocity gradient. The principal value integral defines a SIO as per Definition 5.7, and the Dirac delta term, after convolution, produces a jump discontinuity across the boundary $\partial D(t)$.

Proposition 5.8. *The velocity gradient for a vortex patch is given by:*

$$\nabla u(x) = \frac{1}{2\pi} \text{p.v.} \int_{D(t)} \frac{\sigma(x-y)}{|x-y|^2} \mathrm{d}y + \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbb{1}_{D(t)}(x). \tag{5.4}$$

Proof. The velocity field u is given by the Biot-Savart law as the convolution $u = K \star \omega$, where $K(x) := \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$ is the kernel and $\omega = \mathbb{1}_{D(t)}$ is the vorticity of the patch. The velocity gradient is therefore the convolution of the vorticity with the distributional derivative of the kernel:

$$\nabla u = (\nabla K) \star \omega.$$

The kernel K is homogeneous of degree $1 - \mathtt{d}$ (with $\mathtt{d} = 2$). We can therefore apply Lemma 5.5 to its derivative, which decomposes the operator ∇K into a principal value integral operator and a multiple of the Dirac delta distribution:

$$\nabla K(\cdot) = \text{p.v.} \nabla K(\cdot) + C\delta_0,$$

where the constant matrix C is determined by integrating the kernel K against the outward normal on the unit sphere. The j-th column of C is the vector $c_j = \int_{\mathbb{S}^1} K(z)\, z_j \mathrm{d}\mathcal{H}^1(z)$. A direct computation shows:

$$C = \frac{1}{2\pi} \int_{\mathbb{S}^1} \begin{pmatrix} -z_2 \\ z_1 \end{pmatrix} \begin{pmatrix} z_1 & z_2 \end{pmatrix} \mathrm{d}\mathcal{H}^1(z) = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Convolving this decomposition of ∇K with $\omega = \mathbb{1}_{D(t)}$ yields the two terms in (5.4). □

Remark 5.9 (Properties of the kernel σ)**.** The matrix kernel $\sigma(z)$ in (5.4) has the explicit form:

$$\sigma(z) = \frac{1}{|z|^2}\begin{pmatrix} 2z_1 z_2 & z_2^2 - z_1^2 \\ z_2^2 - z_1^2 & -2z_1 z_2 \end{pmatrix}.$$

The key properties of this kernel, which are essential for the analysis, are:

- It is a smooth function on $\mathbb{R}^2 \setminus \{0\}$ and is *homogeneous of degree zero.*
- It has a *zero mean value* on the unit circle $\mathbb{S}^1$.
- It is an *even function,* i.e., $\sigma(-z) = \sigma(z)$.

The main technical challenge is to control the L^∞ norm of the singular integral part. The following proposition provides the crucial estimate, linking the size of ∇u to the geometric properties of the boundary $\partial D(t)$, as described by the level-set function φ.

Proof of Proposition 6.7 Recall from Equation (5.4) that the velocity gradient consists of a singular integral term and a local term. The local term is bounded by $\frac{1}{2}$, so we only need to control the L^∞ norm of the principal value integral. Let us select a $x_0 \in \mathbb{R}^2 \setminus \partial D$, let us define

$$I(x_0) := \frac{1}{2\pi}\text{p.v.}\int_{D(t)} \frac{\sigma(x_0 - y)}{|x_0 - y|^2}\mathrm{d}y. \tag{5.5}$$

The core of the proof is to analyze this integral based on the distance of the point x_0 to the boundary $\partial D(t)$. Let us define the distance function

$$\mathsf{d}(x_0) := \mathsf{d}(x_0, \partial D), \tag{5.6}$$

and the crucial geometric length scale:

$$\delta := \left(\frac{\inf_{x\in\partial D(t)} |\nabla\varphi(x)|}{|\nabla\varphi|_\gamma}\right)^{1/\gamma}$$

This quantity δ represents the length scale below which the boundary can be considered approximately flat. Let us define the tubular neighborhood of ∂D

$$\mathcal{T}_{\delta/2} := \left\{x_0 \in \mathbb{R}^2 \,:\, \mathsf{d}(x_0) \leqslant \frac{\delta}{2}\right\},$$

and for all $x_0 \in \mathcal{T}_{\delta/2}$ and $\rho \geqslant \mathsf{d}(x_0)$ let us define the set

$$S_\rho(x_0) := \left\{s \in \mathbb{S}^1 \,:\, x_0 + \rho s \in D\right\},$$

i.e., the set of directions such that $x_0 + \rho s \in D$. Let now $\tilde{x} := \tilde{x}(x_0) \in \partial D$ be such that $|x_0 - \tilde{x}| = \mathsf{d}(x_0)$, and let us define the semicircle

$$\Sigma(x_0) := \left\{s \in \mathbb{S}^1 \,:\, \nabla\varphi(\tilde{x}) \cdot s \leqslant 0\right\},$$

and the symmetric difference

$$R_\rho(x_0) := \left[S_\rho(x_0) \setminus \Sigma(x_0)\right] \cup \left[\Sigma(x_0) \setminus S_\rho(x_0)\right]. \tag{5.7}$$

The key technical point is that, denoting with $\mathcal{H}^1$ the Lebesgue measure on $\mathbb{S}^1$, as $\mathsf{d}(x_0) \to 0$ the quantity $\mathcal{H}^1(R_\rho(x_0)) \to 0$ at a controlled rate, i.e.

Lemma 5.10 (Geometric Lemma)**.** *Let $x_0 \in \mathbb{R}^{\mathsf{d}}$ such that its distance to the boundary,* $\mathsf{d}(x_0)$ *defined in* (5.6)*, satisfies* $\mathsf{d}(x_0) < \delta/2$*. For any radius $\rho \geq \mathsf{d}(x_0)$, let $R_\rho(x_0)$ be as in* (5.7)*. Then there exists a $C := C_{\gamma,\mathsf{d}}$ such that*

$$\mathcal{H}^{\mathsf{d}-1}(R_\rho(x_0)) \leqslant C\left[\frac{d(x_0)}{\rho} + \left(\frac{\rho}{\delta}\right)^\gamma\right].$$

We postpone the proof of Lemma 5.10 and show how to use it to prove the technical estimate (5.2).

Recall that we want to provide a bound for $I(x_0)$ defined in (5.5) and we suppose $x_0 \in D$. We split the integration sets in points "close" to x_0 and far ones, i.e.

$$
\begin{aligned}
I(x_0) &:= I_\delta(x_0) + I^\delta(x_0),\\
I_\delta(x_0) &:= \frac{1}{2\pi}\text{p.v.}\int_{D\cap\{|x_0-y|\leqslant\delta\}} \frac{\sigma(x_0-y)}{|x_0-y|^2}\mathrm{d}y,\\
I^\delta(x_0) &:= \frac{1}{2\pi}\text{p.v.}\int_{D\cap\{|x_0-y|\geqslant\delta\}} \frac{\sigma(x_0-y)}{|x_0-y|^2}\mathrm{d}y.
\end{aligned}
$$

Step 1 (Bounds for $I^\delta(x_0)$)**.** Let us at first bound I^δ, recalling that we suppose that $D \subset B(0,\mathsf{L})$ we have, passing to polar coordinates

$$
\begin{aligned}
\left|I^\delta(x_0)\right| &= \frac{1}{2\pi}\left|\int_{D\cap\{|x_0-y|\geqslant\delta\}} \frac{\sigma(x_0-y)}{|x_0-y|^2}\mathrm{d}y\right|,\\
&\leqslant C\int_\delta^{\mathsf{L}}\frac{\mathrm{d}\rho}{\rho} \leqslant C\log\left(\frac{\mathsf{L}}{\delta}\right).
\end{aligned}
\tag{5.8}
$$

Step 2 (Bounds for $I_\delta(x_0)$)**.** We study now I_δ and we write it in polar coordinated centered in x_0 obtaining

$$
I_\delta(x_0) = \int_{\tilde{R}}\int_{S_\rho(x_0)} \frac{\sigma(\rho e^{is})}{\rho}\,\mathrm{d}\mathcal{H}^1(s)\,\mathrm{d}\rho,
$$

where the integration set in the radial direction is an unspecified and irrelevant set which is bounded in $B(0,\delta)$ since we are integrating in the tubular neighborhood $\mathcal{T}_{\delta/2}$. Let now be $\rho_0 \in (0, \mathsf{d}(x_0))$, so that $B(x_0,\rho_0) \subset D$. Since σ has zero average on $\mathbb{S}^1$ (cf. Remark 5.9) we have that

$$
\int_{\mathbb{S}^1}\sigma\left(\rho_0 e^{is}\right)\mathrm{d}\mathcal{H}^1(s) = 0,
$$

so that the integral contributions for $\rho < \mathsf{d}(x_0)$ are all nil, hence I_δ becomes

$$
I_\delta(x_0) = \int_{d(x_0)}^{\tilde{\rho}}\left[\int_{S_\rho(x_0)}\sigma\left(\rho e^{is}\right)\,\mathrm{d}\mathcal{H}^1(s)\right]\frac{\mathrm{d}\rho}{\rho},
\tag{5.9}
$$

where again $\tilde{\rho} < \delta$ is irrelevant in our context. The crucial feature of (5.9) is the fact that the integration set in the radial direction is bounded from below, this observation holds true also in the case $x_0 \notin D$. Next let us consider the semicircle $\Sigma(x_0)$, we have

$$
S_\rho(x_0) \subset \Sigma(x_0) \qquad \text{or} \qquad \Sigma(x_0) \subset S_\rho(x_0).
$$

In what follows we shall implicitly consider the case $S_\rho(x_0) \subset \Sigma(x_0)$ and provide bounds that are symmetric w.r.t. the inclusion, which can thus be applied to the other case as well. Since $\sigma(z) = \sigma(-z)$, the function σ is zero-homogeneous and Remark 5.9 we have that

$$
\int_{\Sigma(x_0)}\sigma\left(\rho e^{is}\right)\,\mathrm{d}\mathcal{H}^1(s) = \frac{1}{2}\int_{\Sigma(x_0)}\left(\sigma\left(\rho e^{\mathrm{i}s}\right)+\sigma\left(-\rho e^{\mathrm{i}s}\right)\right)\,\mathrm{d}\mathcal{H}^1(s) = \frac{1}{2}\int_{\mathbb{S}(0,\rho)}\sigma\mathrm{d}\mathcal{H}^1 = 0,
\tag{5.10}
$$

so that Eqs. (5.9) and (5.10) give

$$
I_\delta(x_0) = \int_{d(x_0)}^{\tilde{\rho}}\left[\int_{S_\rho(x_0)\setminus\Sigma(x_0)}\sigma\left(\rho e^{is}\right)\,\mathrm{d}\mathcal{H}^1(s)\right]\frac{\mathrm{d}\rho}{\rho}.
$$

Since σ is homogeneous of order zero we have that $|\sigma(z)| \leq 1$ for all z so that we obtain the bound

$$
|I_\delta(x_0)| \leqslant \int_{\mathsf{d}(x_0)}^{\delta}\frac{\mathcal{H}^1\left(S_\rho(x_0)\setminus\Sigma(x_0)\right)}{\rho}\,\mathrm{d}\rho.
\tag{5.11}
$$

The estimate (5.11) is true when $S_\rho(x_0) \subset \Sigma(x_0)$, if $\Sigma(x_0) \subset S_\rho(x_0)$ we have

$$|I_\delta(x_0)| \leqslant \int_{\mathsf{d}(x_0)}^{\delta} \frac{\mathcal{H}^1\left(\Sigma(x_0) \setminus S_\rho(x_0)\right)}{\rho}\, \mathrm{d}\rho,$$

so that the bound (cf. (5.7))

$$|I_\delta(x_0)| \leqslant \int_{\mathsf{d}(x_0)}^{\delta} \frac{\mathcal{H}^1\left(R_\rho(x_0)\right)}{\rho}\, \mathrm{d}\rho, \tag{5.12}$$

covers both cases. We use now the estimate provided in Lemma 5.10 in (5.12) and we have that

$$|I_\delta(x_0)| \leqslant \int_{\mathsf{d}(x_0)}^{\delta} \frac{1}{\rho}\left(\frac{\mathsf{d}(x_0)}{\rho} + \left(\frac{\rho}{\delta}\right)^\gamma\right) \mathrm{d}\rho \leqslant \frac{1}{2} + \frac{1}{\gamma}\left(1 - \frac{1}{2^\gamma}\right) \leqslant C.$$

which we combine it now with (5.8) and we obtain

$$|I(x_0)| \leqslant C\left[1 + \log\left(\frac{\mathsf{L}\left|\nabla\varphi\right|_\gamma}{\left|\nabla\varphi\right|_{\inf}}\right)\right].$$

Proof of Lemma 5.10 The proof hinges on quantifying the deviation of the true boundary from its tangent plane. Let $\tilde{x}$ be the unique projection of x_0 onto ∂D, so that $|x_0 - \tilde{x}| = \mathsf{d}(x_0)$. A direction $s \in \mathbb{S}^{\mathsf{d}-1}$ belongs to the set $R_\rho(x_0)$ if the sign of the defining function $\varphi\left(x_0 + \rho s\right)$ differs from the sign of its linear approximation at $\tilde{x}$.

Let $x := x_0 + \rho s$. The linear approximation is $L(x) = \nabla\varphi(\tilde{x}) \cdot (x - \tilde{x})$, since $\varphi(\tilde{x}) = 0$. A sign mismatch can only occur if the magnitude of the linear term is smaller than the Taylor remainder, $R(x, \tilde{x}) := \varphi(x) - L(x)$. Thus, any $s \in R_\rho(x_0)$ must satisfy the fundamental condition:

$$|L(x)| \leq |R(x, \tilde{x})|. \tag{5.13}$$

The proof proceeds in two main steps: first, we translate this condition into a constraint on the angle of the direction vector s, and second, we estimate the measure of the set of angles satisfying this constraint.

Part 1 (Deriving the Angular Constraint)**.** Let ψ be the angle between the direction vector s and the normal vector $\nabla\varphi(\tilde{x})$. The magnitude of the linear term is given by

$$|L(x)| = \left|\nabla\varphi(\tilde{x}) \cdot \left(x_0 - \tilde{x} + \rho s\right)\right| = \left|\nabla\varphi(\tilde{x})\right| \left|\rho\cos\psi \pm \mathsf{d}(x_0)\right|.$$

The remainder is bounded by the $\mathcal{C}^{1,\gamma}$ regularity of φ:

$$|R(x, \tilde{x})| \leq \frac{\left|\nabla\varphi\right|_\gamma}{1+\gamma} |x - \tilde{x}|^{1+\gamma} \leq \frac{\left|\nabla\varphi\right|_\gamma}{1+\gamma} \left(\rho + \mathsf{d}(x_0)\right)^{1+\gamma}.$$

Substituting these into our core condition (5.13) yields

$$\left|\nabla\varphi(\tilde{x})\right| \left|\rho\cos\psi \pm \mathsf{d}(x_0)\right| \leq \frac{\left|\nabla\varphi\right|_\gamma}{1+\gamma} \left(\rho + \mathsf{d}(x_0)\right)^{1+\gamma}.$$

To isolate $\cos\psi$, we use the triangle inequality $\left|\rho\cos\psi\right| \leq \left|\rho\cos\psi \pm \mathsf{d}(x_0)\right| + \mathsf{d}(x_0)$ and rearrange the terms:

$$\rho\left|\cos\psi\right| \leq \mathsf{d}(x_0) + \frac{\left|\nabla\varphi\right|_\gamma}{\left(1+\gamma\right)\left|\nabla\varphi(\tilde{x})\right|} \left(\rho + \mathsf{d}(x_0)\right)^{1+\gamma}.$$

Dividing by ρ and using $\left|\nabla\varphi(\tilde{x})\right| \geq \left|\nabla\varphi\right|_{\inf}$ and the hypothesis $\mathsf{d}(x_0) \leq \rho$ (so $\rho + \mathsf{d}(x_0) \leq 2\rho$), we get

$$\begin{aligned}\left|\cos\psi\right| &\leq \frac{\mathsf{d}(x_0)}{\rho} + \frac{\left|\nabla\varphi\right|_\gamma}{\left(1+\gamma\right)\left|\nabla\varphi\right|_{\inf}} \frac{\left(2\rho\right)^{1+\gamma}}{\rho} \\ &\leq \frac{\mathsf{d}(x_0)}{\rho} + \frac{2^{1+\gamma}}{1+\gamma} \frac{\left|\nabla\varphi\right|_\gamma}{\left|\nabla\varphi\right|_{\inf}} \rho^\gamma.\end{aligned}$$

Using the definition $\delta^\gamma = \left|\nabla\varphi\right|_{\inf} / \left|\nabla\varphi\right|_\gamma$ and absorbing all numerical factors into a single constant C, we arrive at the final constraint on the angle:

$$\left|\cos\psi\right| \leq C\left[\frac{\mathsf{d}(x_0)}{\rho} + \left(\frac{\rho}{\delta}\right)^\gamma\right]. \tag{5.14}$$

Part 2 (Estimating the Measure)**.** The condition (5.14) confines the direction vector s to a narrow band on the sphere $\mathbb{S}^{\mathrm{d}-1}$ around the "equator" defined by the plane orthogonal to the normal $\nabla\varphi(\tilde{x})$ (where $\psi \approx \pi/2$ and thus $\cos\psi \approx 0$).

The $\mathcal{H}^{\mathrm{d}-1}$-measure of the set of points on the unit sphere satisfying $\left|\cos\psi\right| \leq A$ for some small $A > 0$ is bounded by $C_{\mathrm{d}} A$. Applying this geometric fact to our result from (5.14), we obtain the final estimate:

$$\mathcal{H}^{\mathrm{d}-1}\left(R_\rho(x_0)\right) \leq C_{\gamma,\mathrm{d}}\left[\frac{d(x_0)}{\rho} + \left(\frac{\rho}{\delta}\right)^\gamma\right],$$

which concludes the proof. □

5.2 Proof of Theorem 5.2

Proposition 5.11. *Let W be a divergence-free vector field tangent to ∂D and let u be give by the Biot-Savart law* (4.1)*, then*

$$\nabla u(x)\, W(x) = \frac{1}{2\pi}\text{p.v.}\int \frac{\sigma(x-y)}{\left|x-y\right|^2}\left(W(x) - W(y)\right)\mathrm{d}x.$$

Proof. Let us exploit the following identity

$$\nabla_y\left(\nabla_y^\perp \log\left|x-y\right|\right) = \frac{\sigma(x-y)}{\left|x-y\right|^2},$$

so that integration by parts give

$$\begin{aligned}
\frac{1}{2\pi}\text{p.v.}\int \nabla_y\left(\nabla_y^\perp \log\left|x-y\right|\right) W(y)\,\mathrm{d}x &= \frac{1}{2\pi}\lim_{\varepsilon\to0^+}\int_{D\cap\{|x-y|\geqslant\varepsilon\}} \partial_{y_i}\left(\partial_{y_j}^\perp \log\left|x-y\right|\right) W_i(y)\,\mathrm{d}x,\\
&= -\frac{1}{2\pi}\lim_{\varepsilon\to0^+}\int_{D\cap\{|x-y|=\varepsilon\}} \nabla_y^\perp \log\left|x-y\right| W(y)\cdot\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}S\\
&\quad -\frac{1}{2\pi}\lim_{\varepsilon\to0^+}\int_{D\cap\{|x-y|\geqslant\varepsilon\}} \partial_{y_j}^\perp \log\left|x-y\right| \underbrace{\partial_{y_i} W_i(y)}_{=0}\,\mathrm{d}x,
\end{aligned}$$

thus since W is tangent to ∂D we obtain that

$$-\frac{1}{2\pi}\lim_{\varepsilon\to0^+}\int_{D\cap\{|x-y|=\varepsilon\}} \nabla_y^\perp \log\left|x-y\right| W(y)\cdot\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}S = -\frac{1}{2}\,\mathbb{1}_D(x)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} W(x),$$

which in turn implies that

$$\begin{aligned}
\nabla u(x)\, W(x) &= \left(\frac{1}{2\pi}\text{p.v.}\int_D \frac{\sigma(x-y)}{\left|x-y\right|^2}\mathrm{d}x + \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbb{1}_D(x)\right) W(x)\\
&= \frac{1}{2\pi}\text{p.v.}\int \frac{\sigma(x-y)}{\left|x-y\right|^2}\left(W(x) - W(y)\right)\mathrm{d}x\\
&\quad \underbrace{+\frac{1}{2\pi}\text{p.v.}\int_D \frac{\sigma(x-y)}{\left|x-y\right|^2} W(y)\,\mathrm{d}x + \frac{1}{2}\,\mathbb{1}_D(x)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} W(x),}_{=0}
\end{aligned}$$

concluding. □

We need the following commutator estimates in Hölder spaces before proceeding.

Lemma 5.12. *Let $\psi \in L^\infty$, $f \in \dot{C}^\gamma$, $\mathcal{K}$ a Calderon-Zygmund kernel homogeneous of degree $-\mathtt{d}$ with zero mean on $\mathbb{S}^{\mathrm{d}-1}$ and such that $|\nabla\mathcal{K}| \lesssim |x|^{-(\mathrm{d}+1)}$. Let*

$$G(x) = \text{p.v.}\int \mathcal{K}(x-y)\left(f(x) - f(y)\right)\psi(y)\,\mathrm{d}x,$$

then

$$|G|_\gamma \leqslant C_0(\gamma,\mathrm{d})\left(\left\|\mathcal{K}\star\psi\right\|_{L^\infty} + \left\|\psi\right\|_{L^\infty}\right)\left|f\right|_\gamma.$$

An application of Lemma 5.12 gives the following result.

Corollary 5.13. *Let u and W be as in Proposition 5.11, then*

$$|\nabla u W|_\gamma \leqslant C_0 \|\nabla u\|_{L^\infty} |W|_\gamma . \tag{5.15}$$

We can now finally prove the required global bounds.

Proposition 5.14. *Let φ be the solution of* (5.1) *in* $[0,T]$, $T<\infty$, *then we have*

$$\left|\nabla^\perp \varphi(t,\cdot)\right|_\gamma \leqslant \left|\nabla^\perp \varphi_0\right|_\gamma \exp\left\{ (C_0+\gamma)\int_0^t \left\|\nabla u\left(t',\cdot\right)\right\|_{L^\infty} \mathrm{d}t' \right\}, \tag{5.16}$$

$$\left\|\nabla^\perp \varphi(t,\cdot)\right\|_{L^\infty} \leqslant \left\|\nabla^\perp \varphi_0\right\|_{L^\infty} \exp\left\{ \int_0^t \left\|\nabla u\left(t',\cdot\right)\right\|_{L^\infty} \mathrm{d}t' \right\}, \tag{5.17}$$

$$\inf_{x\in\partial D(t)} \left|\nabla\varphi(t,x)\right| \geqslant \left(\inf_{x\in\partial D_0} \left|\nabla\varphi_0(x)\right|\right) \exp\left\{ -\int_0^t \left\|\nabla u\left(t',\cdot\right)\right\|_{L^\infty} \mathrm{d}t' \right\}. \tag{5.18}$$

Once Proposition 5.14 is proved, Theorem 5.2 follows; we plug the estimates (5.16) and (5.18) in (5.2) and apply Gronwall's lemma, deducing the bound

$$\|\nabla u(t,\cdot)\|_{L^\infty} \leqslant \|\nabla u_0\|_{L^\infty} e^{Ct},$$

which we apply to (5.16)–(5.18) obtaining that

$$\left\|\nabla^\perp \varphi(t,\cdot)\right\|_{C^\gamma} \leqslant \left\|\nabla^\perp \varphi_0\right\|_{C^\gamma} e^{(C_0+\gamma)e^{Ct}}, \qquad \inf_{x\in\partial D(t)} \left|\nabla\varphi(t,x)\right| \geqslant \left(\inf_{x\in\partial D_0} \left|\nabla\varphi_0(x)\right|\right) e^{-e^{Ct}}.$$

□

Proof of Proposition 5.14. Let us denote with $w(t,\alpha) = W(t,X(t,\alpha)) = \nabla^\perp\varphi(t,X(t,\alpha))$ and we rewrite the evolution equation for W in Lagrangian coordinates we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t,\alpha) = \nabla u(t,X(t,\alpha))\; w(t,\alpha),$$

thus multiplying the above equation by $w/|w|$ and applying Gronwall's inequality we obtain that

$$\exp\left\{ -\int_0^t \left|\nabla u\left(t',X\left(t',\alpha\right)\right)\right| \mathrm{d}t' \right\} \leqslant \frac{|w(t,\alpha)|}{|w(0,\alpha)|} \leqslant \exp\left\{ \int_0^t \left|\nabla u\left(t',X\left(t',\alpha\right)\right)\right| \mathrm{d}t' \right\},$$

proving (5.17) and (5.18) by converting back to Eulerian coordinates and taking the supremum/infimum over the domain/boundary.

We have to prove now only (5.16). The solution to the transport equation for W can be expressed via Duhamel's formula in Eulerian coordinates:

$$W(t,x) = W_0\left(X^{-t}(x)\right) + \int_0^t (\nabla u W)\left(t', X\left(t', X^{-t}(x)\right)\right) \mathrm{d}t',$$

so that if $x\neq y$ so that for $x\neq y$

$$\begin{aligned} \left|W(t,x) - W(t,y)\right| \leqslant{}& \left|W_0\left(X^{-t}(x)\right) - W_0\left(X^{-t}(y)\right)\right| \\ &+ \int_0^t \left|(\nabla u W)\left(t', X^{-(t-t')}(x)\right) - (\nabla u W)\left(t', X^{-(t-t')}(y)\right)\right| \mathrm{d}t'. \end{aligned}$$

Taking the Hölder quotient, this becomes

$$|W(t,\cdot)|_\gamma \leqslant |W_0|_\gamma \left\|\nabla X^{-t}(\cdot)\right\|_{L^\infty}^\gamma + \int_0^t \left|\nabla u W\left(t',\cdot\right)\right|_\gamma \left\|\nabla X^{-(t-t')}(\cdot)\right\|_{L^\infty}^\gamma \mathrm{d}t'.$$

A standard application of Gronwall's inequality to the ODE for the flow gradient gives

$$\left\|\nabla X^{-t}(\cdot)\right\|_{L^\infty} \leqslant \exp\left\{ \int_0^t \|\nabla u(s,\cdot)\|_{L^\infty} \mathrm{d}s \right\},$$

$$\left\|\nabla X^{-(t-t')}(\cdot)\right\|_{L^\infty} \leqslant \exp\left\{\int_0^{t-t'} \|\nabla u(s,\cdot)\|_{L^\infty} \,\mathrm{d}s\right\}.$$

Let us denote $Q(s) = \|\nabla u(s,\cdot)\|_{L^\infty}$. Using the commutator estimate (5.15), we obtain

$$|W(t,\cdot)|_\gamma \leqslant |W_0|_\gamma \exp\left\{\gamma \int_0^t Q(s)\,\mathrm{d}s\right\} + C_0 \int_0^t Q(t')\left|W(t',\cdot)\right|_\gamma \exp\left\{\gamma \int_0^{t-t'} Q(s)\,\mathrm{d}s\right\} \mathrm{d}t'.$$

Define $G(t) = |W(t,\cdot)|_\gamma \exp\left\{-\gamma \int_0^t Q(s)\,\mathrm{d}s\right\}$. Differentiating and using the previous inequality shows that G satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t) \leq C_0 Q(t)\, G(t),$$

so that another application of Gronwall's lemma concludes the proof of (5.16). □

Chapter 6

The Flexibility of Weak Solutions: Convex Integration

The theory developed by Yudovich, while powerful, applies to a "tame" class of weak solutions that retain uniqueness. We now venture into the more "wild" aspects of weak solutions by introducing the powerful method of *convex integration*. This geometric technique, first developed in its modern form by M. Gromov [6], was adapted by De Lellis and Székelyhidi [5] to the context of fluid dynamics. It reveals a surprising flexibility in the Euler equations, allowing for the construction of infinitely many weak solutions that can exhibit physically startling behavior.

Our presentation is largely based on the original work [5], focusing on a simplified approach to make the core ideas as accessible as possible. For a broader perspective on convex integration methods, we refer the interested reader to the online notes [**?**]. It is also worth mentioning that the theory has evolved significantly since 2009, with remarkable results concerning the regularity of weak solutions, as seen in [**?**, 3, 8].

6.1 The Paradox of Non-uniqueness

We consider the incompressible Euler equations, which we write here in conservative form:

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \tag{6.1}$$

Recall from Remark 3.12, that for a divergence-free field, the convective term can be written as $\operatorname{div}(u \otimes u) = u \cdot \nabla u$.

Definition 6.1 (Weak Solution)**.** A pair (u, p) is a *weak solution of* (6.1) if $u \in L^2_{\mathrm{loc}}(\mathbb{R}^{\mathtt{d}+1}; \mathbb{R}^{\mathtt{d}})$ and $p \in L^1_{\mathrm{loc}}(\mathbb{R}^{\mathtt{d}+1}; \mathbb{R})$, and the pair satisfies (6.1) in the sense of distributions.

The requirement $u \in L^2_{\mathrm{loc}}$ is physically natural, as it ensures that the kinetic energy, $\frac{1}{2}|u|^2$, is locally integrable. However, this definition is too broad to single out the physically relevant solutions. In fact, it allows for solutions that blatantly violate the conservation of energy. A groundbreaking and deeply counter-intuitive result in this direction was established by Scheffer [13] and Shnirelman [14].

Theorem 6.2. · *For* $\mathtt{d} = 2$*, there exists a non-trivial weak solution* (u, p) *to the Euler equations* (6.1) *with compact support in spacetime.*

The solution constructed in Theorem 6.2 describes a fluid that is initially at rest, spontaneously starts moving without any external force, and then returns to a state of rest after a finite time. Such behavior is clearly paradoxical and underscores the profound challenge of defining a proper solution concept for turbulent flows. The convex integration framework provides a constructive proof of this flexibility.

6.2 The De Lellis-Székelyhidi Theorem

The main result we aim to explain is the celebrated theorem by De Lellis and Székelyhidi, which shows that one can construct weak solutions with a precisely prescribed kinetic energy profile.

Theorem 6.3 (Weak solutions via convex integration, [5]). *Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ be a bounded open set and let $T > 0$. Let $\bar{e} : \Omega \times [0,T] \to \mathbb{R}_+$ be a strictly positive and uniformly continuous function such that*

$$\bar{e} \in L^{\infty}\left([0,T]; L^1(\Omega)\right).$$

Then there exist a weak solution (u,p) to the Euler equations (6.1) *such that:*

- **i** $u \in L^{\infty}\left(\mathbb{R}^{\mathrm{d}}\right)$,
- **ii** *u has compact support in $\Omega \times [0,T]$, and in particular $u(\cdot,0) = u(\cdot,T) \equiv 0$,*
- **iii** *The kinetic energy and pressure are prescribed by $\bar{e}$ almost everywhere in $\Omega \times [0,T]$:*

$$\frac{1}{2}|u(x,t)|^2 = \bar{e}(x,t) \quad \text{and} \quad p(x,t) = -\frac{2}{\mathrm{d}}\bar{e}(x,t),$$

- **iv** *The pair (u,p) is obtained as an L^2-limit of a sequence $(u_k,p_k)_{k\in\mathbb{N}} \subset C_c^{\infty}\left(\mathbb{R}^{\mathrm{d}}\right)$. For any k the pair (u_k,p_k) is a classical solution to the forced Euler equations*

$$\begin{cases} \partial_t u_k + \operatorname{div}(u_k \otimes u_k) + \nabla p_k = f_k \\ \operatorname{div} u_k = 0 \end{cases}$$

where the forcing term $f_k \in C_c^{\infty}$ vanishes in the limit, in the sense that for any $N > 0$, $\|f_k\|_{H^{-N}} \to 0$ as $k \to \infty$.

Remark 6.4. ▷ The third point of Theorem 6.3 implies that one can prescribe the kinetic energy of the fluid arbitrarily. For instance, if $\bar{e}$ is the characteristic function of $\Omega \times [0,T]$, the resulting velocity is identically zero outside this spacetime cylinder, despite the absence of any physical boundary. The velocity is at rest at both the initial and final times. This describes a fluid that spontaneously generates motion and then comes to a complete stop, which is a physically implausible scenario.

▷ The final point of the theorem is crucial. It ensures that these "pathological" solutions are not isolated phenomena. They can be approximated in the L^2 topology by a sequence of smooth solutions to the Euler equations with a well-behaved forcing term that vanishes in the limit. In other words, the closure of the space of "well-behaved" solutions in a weak topology is large enough to contain these physically paradoxical flows.

6.3 Plane Wave Analysis and a Reformulation of the Euler Equations

Notation 6.5. From now on we use the notation $\mathrm{Id} := \mathrm{Id}_{\mathbb{R}^{\mathrm{d}}}$.

The core idea of convex integration is to first solve a related, but simpler, linear system and then show that solutions to this system can be made to satisfy the original nonlinear equation through an iterative process.

Consider a general first-order linear system of PDEs:

$$\sum_{i=1}^{\mathrm{d}+1} A_i \partial_{y_i} z = 0, \qquad z : \Omega \subset \mathbb{R}^{\mathrm{d}+1} \to \mathbb{R}^N, \tag{6.2}$$

where the solution z is constrained to lie within a specific set, known as the *constitutive relation*:

$$z(y) \in K \Subset \mathbb{R}^N \text{ a.e. in } \Omega. \tag{6.3}$$

A special class of solutions for such systems are *plane waves*, of the form $z(y) = \lambda h(\xi \cdot y)$ for some vector $\lambda \in \mathbb{R}^N$, direction $\xi \in \mathbb{S}^{\mathrm{d}}$, and profile $h \in L^1_{\mathrm{loc}}(\mathbb{R})$. Substituting this into (6.2) transforms the PDE into an algebraic condition on λ and ξ. The set of admissible directions λ forms a cone.

Definition 6.6 (The Λ-cone). The Λ*-cone* associated with the system (6.2) is the set

$$\Lambda := \left\{\lambda \in \mathbb{R}^N \mid \exists \xi \in \mathbb{S}^{\mathrm{d}} \text{ such that } \lambda h(\xi \cdot y) \text{ is a solution of (6.2) for all } h \in L^1_{\mathrm{loc}}(\mathbb{R})\right\}.$$

The Euler equations are nonlinear, so they do not directly fit this framework. The first crucial step is to rewrite them as a linear system subject to a nonlinear algebraic constraint.

Definition 6.7. Let $\mathcal{S}_0^{\mathtt{d}}$ denote the space of symmetric, trace-free $\mathtt{d}\times\mathtt{d}$ matrices, i.e.

$$\mathcal{S}_0^{\mathtt{d}} := \left\{M \in \mathbb{R}^{n\times n} \middle|\, M_{i,j} = M_{j,i},\ \operatorname{tr} M = 0\right\}.$$

Lemma 6.8 (Relaxation of the Euler Equations)**.** *Let*

$$u \in L^\infty\left(\mathbb{R}^{\mathtt{d}}\times\mathbb{R};\mathbb{R}^{\mathtt{d}}\right) \qquad R \in L^\infty\left(\mathbb{R}^{\mathtt{d}}\times\mathbb{R};\mathcal{S}_0^{\mathtt{d}}\right) \qquad q \in L^\infty\left(\mathbb{R}^{\mathtt{d}}\times\mathbb{R};\mathbb{R}\right)$$

The triple (u,R,q) is a distributional solution to the relaxed Euler system*:*

$$\begin{cases}\partial_t u + \operatorname{div} R + \nabla q = 0,\\ \operatorname{div} u = 0,\end{cases} \tag{6.4}$$

and satisfies the constitutive relation

$$R = u\otimes u - \frac{|u|^2}{\mathtt{d}}\mathrm{Id} \quad \text{a.e. in } \mathbb{R}^{\mathtt{d}}\times\mathbb{R}, \tag{6.5}$$

if and only if the pair (u,p) with $p := q - \frac{|u|^2}{\mathtt{d}}$ is a weak solution to the incompressible Euler equations (6.1)*.*

Proof of Lemma 6.8. We need to prove the equivalence in both directions.

$\Longrightarrow$ Assume that the triple (u,R,q) is a distributional solution to the relaxed Euler system (6.4) and satisfies the constitutive relation (6.5). We want to show that (u,p) with $p := q - \frac{|u|^2}{\mathtt{d}}$ is a weak solution to the Euler equations (6.1).

The second equation of the relaxed system is $\operatorname{div} u = 0$, which is the incompressibility condition. We substitute the constitutive relation (6.5) into the first equation of (6.4):

$$\partial_t u + \operatorname{div}\left(u\otimes u - \frac{|u|^2}{\mathtt{d}}\mathrm{Id}\right) + \nabla q = 0.$$

We can expand the divergence term. Since $\operatorname{div} u = 0$, we have the identity $\operatorname{div}(u\otimes u) = (u\cdot\nabla)\,u$. The divergence of the second term is simply a gradient: $\operatorname{div}\left(\frac{|u|^2}{\mathtt{d}}\mathrm{Id}\right) = \nabla\left(\frac{|u|^2}{\mathtt{d}}\right)$. The equation thus becomes:

$$\partial_t u + (u\cdot\nabla)\,u - \nabla\left(\frac{|u|^2}{\mathtt{d}}\right) + \nabla q = 0.$$

Combining the gradient terms, we get:

$$\partial_t u + (u\cdot\nabla)\,u + \nabla\left(q - \frac{|u|^2}{\mathtt{d}}\right) = 0.$$

By defining $p := q - \frac{|u|^2}{\mathtt{d}}$, we recover the Euler momentum equation:

$$\partial_t u + (u\cdot\nabla)\,u + \nabla p = 0.$$

This equation, together with $\operatorname{div} u = 0$, confirms that (u,p) is a weak solution to the Euler equations.

$\Longleftarrow$ Assume that (u,p) is a weak solution to the Euler equations (6.1). We define R and q as follows:

$$\begin{aligned} R &:= u\otimes u - \frac{|u|^2}{\mathtt{d}}\mathrm{Id},\\ q &:= p + \frac{|u|^2}{\mathtt{d}}.\end{aligned}$$

By its definition, R is a symmetric matrix. Its trace is $\operatorname{tr}(R) = \operatorname{tr}(u \otimes u) - \operatorname{tr}\left(\frac{|u|^2}{\mathrm{d}}\mathrm{Id}\right) = |u|^2 - \mathrm{d}\frac{|u|^2}{\mathrm{d}} = 0$, so $R \in \mathcal{S}_0^{\mathrm{d}}$. The constitutive relation (6.5) is satisfied by definition.

Now we verify that (u, R, q) solves the relaxed system (6.4). The condition $\operatorname{div} u = 0$ is given. We start with the Euler momentum equation:

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0.$$

We substitute $u \otimes u = R + \frac{|u|^2}{\mathrm{d}}\mathrm{Id}$ and $p = q - \frac{|u|^2}{\mathrm{d}}$ into this equation:

$$\partial_t u + \operatorname{div}\left(R + \frac{|u|^2}{\mathrm{d}}\mathrm{Id}\right) + \nabla\left(q - \frac{|u|^2}{\mathrm{d}}\right) = 0.$$

Using the linearity of the divergence and gradient operators, we get:

$$\partial_t u + \operatorname{div} R + \nabla\left(\frac{|u|^2}{\mathrm{d}}\right) + \nabla q - \nabla\left(\frac{|u|^2}{\mathrm{d}}\right) = 0.$$

The gradient terms cancel, leaving us with the first equation of the relaxed system:

$$\partial_t u + \operatorname{div} R + \nabla q = 0.$$

This completes the proof.

□

The system (6.4) is linear in the state variable $z = (u, R, q)$, while the nonlinearity of the Euler equations is encoded entirely in the algebraic constraint (6.5). This reformulation can be written even more compactly. Let $y := (x, t)$ and define the symmetric $(\mathrm{d}+1) \times (\mathrm{d}+1)$ matrix field

$$U := \begin{bmatrix} R + q\mathrm{Id} & u^\top \\ u & 0 \end{bmatrix}.$$

Then the linear system (6.4) is equivalent to the single divergence-free condition in spacetime:

$$\operatorname{div}_y U = 0. \tag{6.6}$$

Proposition 6.9 (The Λ-cone for Euler)**.** *The Λ-cone associated with the relaxed Euler system* (6.4) *is the set:*

$$\Lambda = \left\{ (u, R, q) \in \mathbb{R}^{\mathrm{d}} \times \mathcal{S}_0^{\mathrm{d}} \times \mathbb{R} \;\middle|\; \det \begin{bmatrix} R + q\mathrm{Id} & u^\top \\ u & 0 \end{bmatrix} = 0 \right\}. \tag{6.7}$$

Furthermore,

a *For any pair $(u, R) \in \mathbb{R}^{\mathrm{d}} \times \mathcal{S}_0^{\mathrm{d}}$, there exists a scalar $q \in \mathbb{R}$ such that $(u, R, q) \in \Lambda$.*

b *For any velocity $u_0 \in \mathbb{R}^{\mathrm{d}}$, there exists a pressure $p_0 \in \mathbb{R}$ and a spacetime direction $(\xi, \tau) \in \mathbb{S}^{\mathrm{d}}$ such that for any profile $h \in L^1_{\mathrm{loc}}(\mathbb{R})$, and $\varepsilon > 0$ the plane wave*

$$(u, p)(x, t) = (u_0, p_0)\, h\left(\frac{\xi \cdot x + \tau t}{\varepsilon}\right)$$

is a weak solution to the incompressible Euler equations (6.1).

Proof. We select a constant state $z_0 = (u_0, R_0, q_0)$ and a spacetime direction $(\xi, \tau) \in \mathbb{R}^{\mathrm{d}+1}$. The corresponding spacetime matrix is

$$U_0 = \begin{bmatrix} R_0 + q_0\mathrm{Id} & u_0^\top \\ u_0 & 0 \end{bmatrix}.$$

The plane wave $z(x, t) = z_0 h(\xi \cdot x + \tau t)$ is a solution to $\operatorname{div}_y U = 0$ if

$$U_0 \begin{bmatrix} \xi \\ \tau \end{bmatrix} h'(\xi \cdot x + \tau t) = 0.$$

This equation admits non-trivial solutions for (ξ, τ) if and only if $\det(U_0) = 0$, which proves the characterization of Λ.

a) Let $(u,R) \in \mathbb{R}^d \times \mathcal{S}_0^d$ be given. We look for a purely spatial plane wave, setting $\tau = 0$. The condition becomes finding a non-zero $\xi \in \mathbb{R}^d$ and a $q \in \mathbb{R}$ such that

$$\begin{cases} R\xi + q\xi = 0, \\ u \cdot \xi = 0. \end{cases}$$

The second equation requires ξ to be in the orthogonal complement of u, denoted $\langle u\rangle^\perp$. Let $P_u := \mathrm{Id} - \frac{u\otimes u}{|u|^2}$ be the orthogonal projector onto $\langle u\rangle^\perp$. Applying this projector to the first equation gives $P_u R\xi + qP_u\xi = 0$. Since $\xi \in \langle u\rangle^\perp$, $P_u\xi = \xi$, and we can write the problem as finding an eigenvector for the projected matrix:

$$P_u R P_u \xi = -q\xi.$$

The matrix $P_u R P_u$ is symmetric and acts on the space $\langle u\rangle^\perp$. By the spectral theorem, it has a complete set of real eigenvalues and eigenvectors. Thus, a solution pair (q,ξ) always exists.

b) This follows from part (a). Given u_0, we define $R_0 := u_0 \otimes u_0 - \frac{|u_0|^2}{d}\mathrm{Id}$. By part (a), we can find a q_0 and a purely spatial direction ξ (so $\tau = 0$) to form a plane wave solution for the relaxed system. By Lemma 6.8, the pair (u,p) with $p = q_0 - \frac{|u_0|^2}{d}$ is a weak solution to the Euler equations.

□

The proposition shows that the set of possible oscillatory directions for the relaxed system is very rich. The following lemma identifies a particularly useful family of such directions, which will serve as the building blocks for our construction.

Lemma 6.10 (Pressureless plane-waves)**.** *Let $a, b \in \mathbb{R}^d$ such that $|a| = |b|$ and $a \neq b$. Define*

$$W(a,b) := (a-b, a\otimes a - b\otimes b) \in \mathbb{R}^d \times \mathcal{S}_0^d.$$

Then the state vector $\lambda = (W(a,b), 0)$ belongs to the Λ-cone.

Proof. Since $|a| = |b|$, the matrix $a\otimes a - b\otimes b$ is trace-free. We must show that $\det(U) = 0$ for the state $\lambda = (u,R,q) = (a-b, a\otimes a - b\otimes b, 0)$. It is a direct algebraic verification to show that for any vector $\xi \in (a-b)^\perp$, we have $a\cdot\xi = b\cdot\xi$. Then the vector $(\xi^\intercal, -(a\cdot\xi))^\intercal$ is in the kernel of the matrix

$$U = \begin{pmatrix} a\otimes a - b\otimes b & a-b \\ (a-b)^\intercal & 0 \end{pmatrix},$$

which proves the claim by Proposition 6.9, Eq. (6.7). □

6.4 The Λ-convex Hull of K

Definition 6.11 (Constitutive set)**.** We define the *constitutive set* K as the set of states satisfying the relation (6.5):

$$K = \left\{ (u,R) \in \mathbb{R}^d \times \mathcal{S}_0^d \mid R = u\otimes u - \frac{|u|^2}{d}\mathrm{Id} \right\}.$$

For any $r > 0$, we define the constant-energy subset:

$$K_r = K \cap \{|u| = r\}.$$

Thanks to Lemma 6.8, it is clear that it suffices to study the linear problem (6.4) and verify if the state variables satisfy the constitutive relation (6.5) almost everywhere, i.e., if $(u,R) \in K$ a.e. in $\Omega \times [0,T]$.

Definition 6.12 (Convex hull)**.** Let $A \subset \mathbb{R}^d$. The *convex hull* of A, denoted A^{co}, is defined as the intersection of all convex sets containing A:

$$A^{co} = \bigcap_{\substack{C \text{ is convex} \\ A\subset C}} C.$$

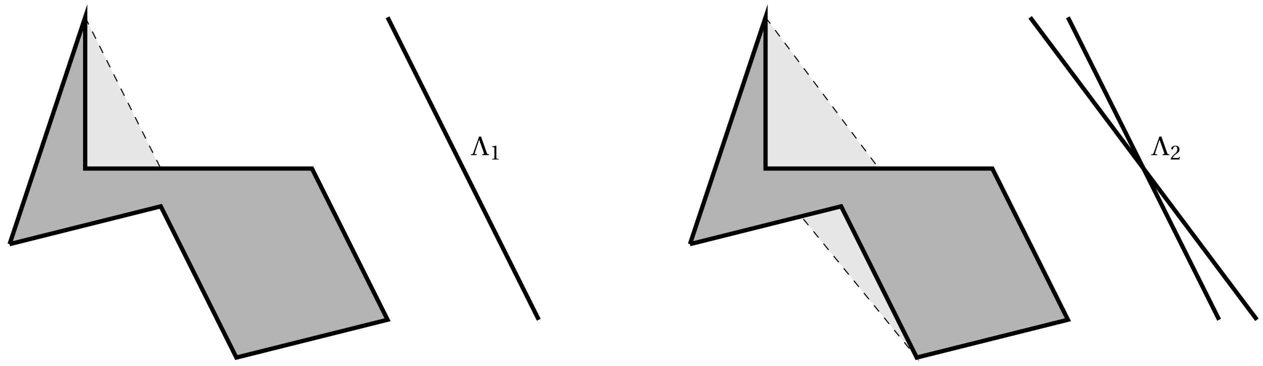

Figure 6.1: Λ_1 vs. Λ_2 convexity.

Definition 6.13 (Λ-convexity)**.** Given a set $C \subset \mathbb{R}^d$ and a closed cone $\Lambda \subset \mathbb{R}^d$, we say that C is Λ*-convex* if for every $x, y \in C$ such that $x - y \in \Lambda$, the line segment $[x, y]$ is contained in C.

Definition 6.14 (Λ-convex hull)**.** Given a set $A \subset \mathbb{R}^d$, we define its Λ*-convex hull*, denoted A^Λ, as the smallest Λ-convex set containing A.

Remark 6.15. Suppose $\Lambda_1 \subset \Lambda_2$. Then the following implication holds:

$$\Lambda_2\text{-convexity} \implies \Lambda_1\text{-convexity}.$$

Indeed, suppose a set $\mathcal{K}$ is Λ_2-convex. This means that

$$\text{for all } x, y \in \mathcal{K} \text{ such that } x - y \in \Lambda_2 \implies [x, y] \subset \mathcal{K}. \tag{6.8}$$

If the logical proposition (6.8) holds for all differences $x - y$ in the larger set Λ_2, it must also hold for all differences in the smaller set $\Lambda_1 \subset \Lambda_2$.

An immediate implication of this fact is the following: given a set $C \subset \mathbb{R}^d$, its Λ_2-convex hull is

$$C^{\Lambda_2} = \bigcap_{\substack{C' \supset C \\ C' \text{ is } \Lambda_2\text{-convex}}} C'.$$

However, since every Λ_2-convex set is also Λ_1-convex, the collection of sets we intersect over to obtain C^{Λ_2} is a subset of the collection for C^{Λ_1}. This leads to the following chain of inclusions:

$$C^{\Lambda_2} = \bigcap_{\substack{C' \supset C \\ C' \text{ is } \Lambda_2\text{-convex}}} C' \supset \bigcap_{\substack{C' \supset C \\ C' \text{ is } \Lambda_1\text{-convex}}} C' = C^{\Lambda_1}.$$

Therefore, for any $\Lambda_1 \subset \Lambda_2$, we have that

$$C^{\Lambda_1} \subset C^{\Lambda_2} \subset C^{\mathbb{R}^d} = C^{\text{co}}.$$

Definition 6.16 (Extreme Point)**.** A point z in a set $A \subset \mathbb{R}^d$ is called an *extreme point* of A if it is not an interior point of any line segment whose endpoints belong to A.

Theorem (Krein-Milman)**.** *Let $\mathcal{K}$ be a compact convex set in a locally convex topological vector space X. Then $\mathcal{K}$ is the convex hull of its extreme points.*

Notation 6.17 (Eigenvalues)**.** Given a diagonalizable matrix $M \in \mathbb{R}^{d\times d}$, we denote its eigenvalues by $(\sigma_i(M))_{i=1}^d$ and define

$$\sigma_{\max}(M) := \max_i \sigma_i(M), \qquad \sigma_{\min}(M) := \min_i \sigma_i(M).$$

Proposition 6.18. *For any $r > 0$, the Λ-convex hull of K_r coincides with its standard convex hull, and it has the explicit characterization:*

$$K_r^{\text{co}} = \left\{(u,R) \in \mathbb{R}^{\mathrm{d}} \times \mathcal{S}_0^{\mathrm{d}} \mid |u| \leqslant r \text{ and } u \otimes u - R \leqslant \frac{r^2}{\mathrm{d}}\mathrm{Id}\right\}.$$

The inequality $M \leq C\mathrm{Id}$ for a symmetric matrix M means that $\sigma_{\max}(M) \leq C$.

Proof. Let us define the set

$$C_r = \left\{(u,R) \in \mathbb{R}^{\mathrm{d}} \times \mathcal{S}_0^{\mathrm{d}} \mid |u| \leqslant r,\ u \otimes u - R \leqslant \frac{r^2}{\mathrm{d}}\mathrm{Id}\right\},$$

we will prove that:

a) C_r is convex,

b) C_r is compact,

c) K_r contains all of the extreme points of C_r.

The Krein-Milman theorem will then imply that $K_r^{\text{co}} = C_r$.

a) For any $(u,R) \in \mathbb{R}^{\mathrm{d}} \times \mathcal{S}_0^{\mathrm{d}}$, we define the *generalized kinetic energy*[1] as

$$e(u,R) = \max_{\xi \in \mathbb{S}^{\mathrm{d}-1}} \left(|u \cdot \xi|^2 - \xi \cdot R\xi\right).$$

The function e is the maximum of a family of convex functions, and is therefore convex itself. Moreover, C_r can be written as the intersection of two convex sets:

$$C_r = e^{-1}\left(\left(-\infty, \frac{r^2}{\mathrm{d}}\right]\right) \cap \{|u| \leqslant r\},$$

and is therefore convex.

b) By its definition, C_r is a closed subset of $\mathbb{R}^{\mathrm{d}} \times \mathcal{S}_0^{\mathrm{d}}$. To prove compactness, it remains to show that the operator norm of R is uniformly bounded for any $(u,R) \in C_r$. We can immediately deduce the matrix inequality:

$$R \geqslant u \otimes u - \frac{r^2}{\mathrm{d}}\mathrm{Id} \geqslant -\frac{r^2}{\mathrm{d}}\mathrm{Id},$$

where the second inequality holds since $u \otimes u$ is positive semi-definite. This implies that the minimum eigenvalue of R is bounded below: $\sigma_{\min}(R) \geqslant -\frac{r^2}{\mathrm{d}}$. Since R is trace-free, $\sum_{i=1}^{\mathrm{d}} \sigma_i(R) = 0$. We can use this to bound the maximum eigenvalue:

$$\sigma_{\max}(R) = -\sum_{i=2}^{\mathrm{d}} \sigma_i(R) \leq -(\mathrm{d}-1)\,\sigma_{\min}(R) \leq \frac{\mathrm{d}-1}{\mathrm{d}} r^2.$$

In terms of the associated quadratic form, this means:

$$-\frac{r^2}{\mathrm{d}}|\xi|^2 \leqslant \xi \cdot R\xi \leqslant \left(1 - \frac{1}{\mathrm{d}}\right) r^2 |\xi|^2. \tag{6.9}$$

We can now bound the operator norm $\|R\| = \sup_{\xi,\eta \in \mathbb{S}^{\mathrm{d}-1}} \eta \cdot R\xi$. Using the polarization identity and the bounds from (6.9):

$$\eta \cdot R\xi = \frac{1}{4}\left((\eta+\xi)\cdot R(\eta+\xi) - (\eta-\xi)\cdot R(\eta-\xi)\right) \leqslant r^2,$$

uniformly for all $\eta, \xi \in \mathbb{S}^{\mathrm{d}-1}$.

[1] Equivalently, $e(u,R)$ represents the maximum eigenvalue of the symmetric matrix $u \otimes u - R$.

c) We now prove that the extreme points of C_r are contained in K_r. Without loss of generality[2], we assume that $u\otimes u - R = \operatorname{diag}(\lambda_1,\dots,\lambda_{\mathtt{d}})$ and that its eigenvalues are ordered such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\mathtt{d}}$. The proof is divided into three sub-cases:

- ▷ Suppose $|u| = r$. Since $(u,R)\in C_r$, it follows that

$$u\otimes u - R \leqslant \frac{|u|^2}{\mathtt{d}}\mathrm{Id}.$$

Let us define the matrix

$$M = u\otimes u - R - \frac{|u|^2}{\mathtt{d}}\mathrm{Id},$$

which is symmetric, diagonalizable, and negative semi-definite ($M \leqslant 0$). Taking the trace of M, we find

$$\operatorname{tr}(M) = \operatorname{tr}(u\otimes u) - \operatorname{tr}(R) - \operatorname{tr}\left(\frac{|u|^2}{\mathtt{d}}\mathrm{Id}\right) = |u|^2 - 0 - \mathtt{d}\frac{r^2}{\mathtt{d}} = r^2 - r^2 = 0.$$

A symmetric, negative semi-definite matrix with zero trace must be the zero matrix. Therefore, $M \equiv 0$, which implies

$$R = u\otimes u - \frac{|u|^2}{\mathtt{d}}\mathrm{Id}.$$

This shows that $(u,R)\in K_r$.

- ▷ Suppose a point $(u,R)\in C_r$ is such that the matrix inequality is saturated, i.e., $\sigma_{\max}(u\otimes u - R) = \lambda_1 = \frac{r^2}{\mathtt{d}}$. Taking the trace of the matrix $u\otimes u - R$, we have $\operatorname{tr}(u\otimes u - R) = |u|^2 \le r^2$. Since the trace is the sum of the eigenvalues and λ_1 is the largest, we must have $|u|^2 = \sum_i \lambda_i \le \mathtt{d}\lambda_1 = \mathtt{d}\frac{r^2}{\mathtt{d}} = r^2$. If $|u| < r$, the point cannot be extreme. Thus, we must have $|u| = r$, which reduces to the previous case, implying that $(u,R)\in K_r$.

- ▷ Finally, suppose $(u,R)\in C_r$ is a point in the strict interior of the set, meaning $|u| < r$ and the matrix inequality is not saturated, i.e., $\sigma_{\max}(u\otimes u - R) = \lambda_1 < \frac{r^2}{\mathtt{d}}$. Let $\{e_i\}_{i=1}^{\mathtt{d}}$ be the orthonormal eigenvectors of the symmetric matrix $u\otimes u - R$, and we write $u = \sum u_i e_i$ in this basis. We define a perturbation direction:

$$\bar u = e_{\mathtt{d}}, \qquad \bar R = u\otimes e_{\mathtt{d}} + e_{\mathtt{d}}\otimes u - 2u_{\mathtt{d}} e_{\mathtt{d}}\otimes e_{\mathtt{d}}.$$

For any $t\in\mathbb{R}$, we consider the matrix for the perturbed state $\left(u + t\bar u, R + t\bar R\right)$:

$$M_t = (u + t\bar u)\otimes(u + t\bar u) - \left(R + t\bar R\right) = (u\otimes u - R) + \left(2tu_{\mathtt{d}} + t^2\right) e_{\mathtt{d}}\otimes e_{\mathtt{d}}.$$

A direct calculation shows that the eigenvalues of M_t are $\lambda_1,\dots,\lambda_{\mathtt{d}-1}$ and a shifted final eigenvalue:

$$\tilde\lambda_{\mathtt{d}} = \lambda_{\mathtt{d}} + \left(2tu_{\mathtt{d}} + t^2\right).$$

Since $|u| < r$ and $\lambda_1 < r^2/\mathtt{d}$, we can choose a sufficiently small $t \neq 0$ such that both conditions for being in C_r are still strictly satisfied for the perturbed state: $|u + t\bar u| < r$ and $\sigma_{\max}(M_t) < \frac{r^2}{\mathtt{d}}$. This means that the points $\left(u \pm t\bar u, R \pm t\bar R\right)$ are both in C_r. Thus, (u,R) is the midpoint of a segment contained in C_r and cannot be an extreme point.

Since, thanks to point **a)**, we know that C_r is a convex set containing K_r, we have

$$K_r^{\mathrm{co}} \subset C_r.$$

On the other hand, thanks to point **c)**, we deduce that every extreme point of C_r is an element of K_r, which implies

$$C_r \subset K_r^{\mathrm{co}},$$

and therefore $K_r^{\mathrm{co}} = C_r$.

[2]The matrix $u\otimes u - R$ is symmetric and therefore diagonalizable.

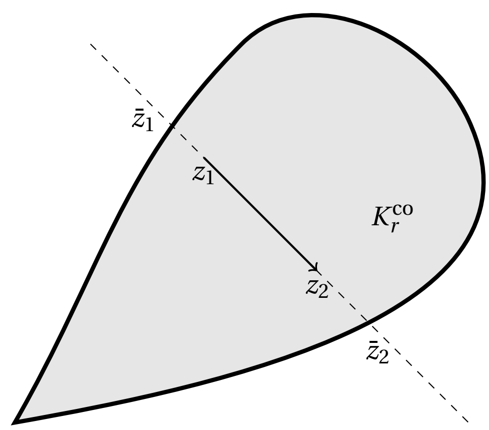

Figure 6.2: By extending the line segment between z_2 and z_1, we intersect the set $\partial K_r^{\mathrm{co}}$; these points of intersection determine $\bar{z}_1$ and $\bar{z}_2$.

We have thus proven that $K_r^{\mathrm{co}} = C_r$. Finally, we want to prove that

$$K_r^{\Lambda} = K_r^{\mathrm{co}}.$$

By definition, $K_r^{\Lambda} \subset K_r^{\mathrm{co}}$. We want to prove that in this particular case, Λ-convexity and standard convexity are equivalent. Consider two points $z_1, z_2 \in K_r^{\mathrm{co}}$. Since K_r^{co} is bounded, there exist two scalars $\mu, \nu > 0$ and two extreme points of K_r^{co}, which we denote by $\bar{z}_1, \bar{z}_2$, such that

$$\bar{z}_1 = \left(\bar{u}_1, \bar{R}_1\right) = z_1 - \mu\left(z_2 - z_1\right),$$
$$\bar{z}_2 = \left(\bar{u}_2, \bar{R}_2\right) = z_2 + \nu\left(z_2 - z_1\right).$$

Since every extreme point of K_r^{co} is an element of K_r, we deduce that

$$\bar{R}_i = \bar{u}_i \otimes \bar{u}_i - \frac{r^2}{\mathrm{d}}\mathrm{Id}.$$

Let us now consider the direction vector $\bar{z}_2 - \bar{z}_1$:

$$\bar{z}_2 - \bar{z}_1 = \left(\bar{u}_2 - \bar{u}_1, \bar{u}_2 \otimes \bar{u}_2 - \bar{u}_1 \otimes \bar{u}_1\right) = W\left(\bar{u}_2, \bar{u}_1\right),$$

and since $\left(W\left(\bar{u}_2, \bar{u}_1\right), 0\right) \in \Lambda$ thanks to the result of Lemma 6.10, we deduce that

$$z_2 - z_1 = \frac{1}{1+\mu+\nu} W\left(\bar{u}_2, \bar{u}_1\right),$$

which proves that the segment connecting any pair of states in K_r^{co} has a direction collinear to an element in the Λ-cone, which concludes the proof.

□

Lemma 6.19 (Directional Oscillation)**.** *There exists a dimensional constant $C > 0$ such that for every $z = (u, R) \in \mathring{K}_r^{\mathrm{co}}$, there exists a perturbation $\bar{\lambda} = \left(\bar{u}, \bar{R}\right) \in \mathbb{R}^{\mathrm{d}} \times \mathcal{S}_0^{\mathrm{d}}$ such that $\left(\bar{\lambda}, 0\right) \in \Lambda$ and the segment $\sigma = \left[z - \bar{\lambda}, z + \bar{\lambda}\right]$ is contained in $\mathring{K}_r^{\mathrm{co}}$. Furthermore, the following estimates hold:*

$$|\bar{u}| \geq \frac{C}{r}\left(r^2 - |u|^2\right), \qquad \operatorname{dist}\left(\sigma, \partial \mathring{K}_r^{\mathrm{co}}\right) \geq \frac{1}{2}\operatorname{dist}\left(z, \partial \mathring{K}_r^{\mathrm{co}}\right).$$

Theorem (Carathéodory)**.** *Let $\mathrm{D} \geq 1$ and let P be a subset of $\mathbb{R}^{\mathrm{D}}$. For any point $x \in P^{\mathrm{co}}$, x can be written as a convex combination of at most $\mathrm{D}+1$ elements of P. That is, there exist scalars $\lambda_j \in [0,1]$ and points $x_j \in P$ for $j = 1, \ldots, \mathrm{D}+1$ such that*

$$\sum_{j=1}^{\mathrm{D}+1} \lambda_j = 1, \qquad x = \sum_{j=1}^{\mathrm{D}+1} \lambda_j x_j.$$

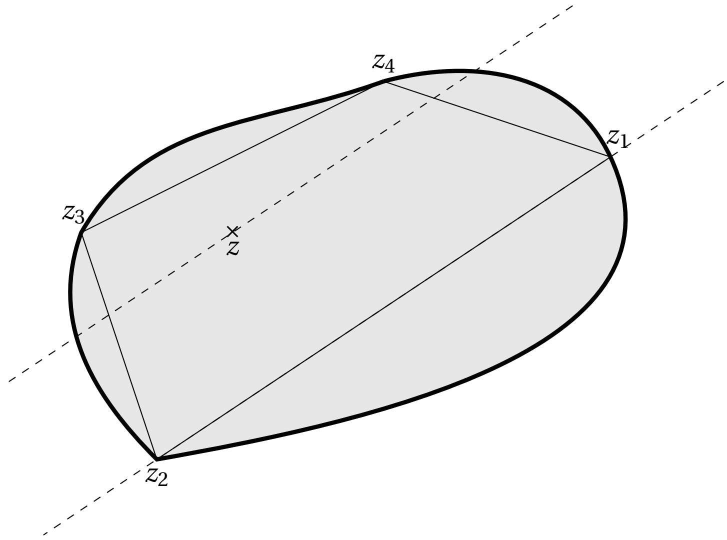

Figure 6.3: Oscillazione con direzione $z_2 - z_1$ passante per z.

Proof of Lemma 6.19. Let $z = (u, R) \in \mathring{K}_r^{co}$. By Carathéodory's theorem, z lies within a simplex whose vertices, $\{z_i\}_{i=1}^{D+1}$, are elements of K_r. We can therefore write z as a convex combination:

$$z = \sum_{i=1}^{D+1} \lambda_i z_i, \qquad \lambda_i \in (0,1)\,,\; z_i \in K_r, \qquad \sum_{i=1}^{D+1} \lambda_i = 1,$$

where $D = \frac{\mathrm{d}(\mathrm{d}+3)}{2} - 1$ is the dimension of the space $\mathbb{R}^{\mathrm{d}} \times \mathcal{S}_0^{\mathrm{d}}$. Note that the coefficients λ_i belong to the open interval $(0,1)$ because we are considering z as an *interior point* of K_r^{co}.

Without loss of generality, assume that λ_1 is the largest coefficient. Then for any $j > 1$, the points $z \pm \frac{\lambda_j}{2}\left(z_j - z_1\right)$ remain in $\mathring{K}_r^{co}$. Indeed, for the positive case:

$$\begin{aligned} z + \frac{\lambda_j}{2}\left(z_j - z_1\right) &= \sum_{i=1}^{D+1} \lambda_i z_i + \frac{\lambda_j}{2} z_j - \frac{\lambda_j}{2} z_1 \\ &= \underbrace{\left(\lambda_1 - \frac{\lambda_j}{2}\right)}_{>0} z_1 + \underbrace{\left(\lambda_j + \frac{\lambda_j}{2}\right)}_{>0} z_j + \sum_{i \neq 1, j} \underbrace{\lambda_i}_{>0} z_i. \end{aligned}$$

This is a new convex combination of the points z_i with strictly positive coefficients, and its sum is $\sum \lambda_i - \frac{\lambda_j}{2} + \frac{\lambda_j}{2} = 1$. A similar argument holds for the negative case.

The perturbation direction is chosen from the segments $\left\{z_j - z_1\right\}$. We define our final perturbation $\bar{\lambda} = \left(\bar{u}, \bar{R}\right)$ to be a scaled version of one of these segments, for instance by setting $\bar{\lambda} := \frac{\lambda_2}{2}\left(z_2 - z_1\right)$.

Claim: The open segment

$$2\mathring{\sigma} := \left(z - 2\bar{\lambda}, z + 2\bar{\lambda}\right)$$

is contained in $\mathring{K}_r^{co}$. †

Proof. The proof is a direct calculation. Any element of $2\mathring{\sigma}$ can be written as $z + \xi\left(2\bar{\lambda}\right)$ for some $\xi \in (-1,1)$. Recalling that $z = \sum_i \lambda_i z_i$ and $2\bar{\lambda} = \lambda_2\left(z_2 - z_1\right)$, we have:

$$\begin{aligned} z + 2\xi\bar{\lambda} &= \left(\sum_i \lambda_i z_i\right) + \xi\lambda_2\left(z_2 - z_1\right) \\ &= \left(\lambda_1 - \xi\lambda_2\right) z_1 + \left(\lambda_2 + \xi\lambda_2\right) z_2 + \sum_{i=3}^{D+1} \lambda_i z_i. \end{aligned}$$

This is a new convex combination of the points $\{z_i\}$. Since we assumed $\lambda_1 \geq \lambda_2$ and we know $|\xi| < 1$, all coefficients are strictly positive. The sum of the coefficients is $\sum_i \lambda_i = 1$. Thus, the point lies in the interior of the simplex, and therefore in $\mathring{K}_r^{co}$. □

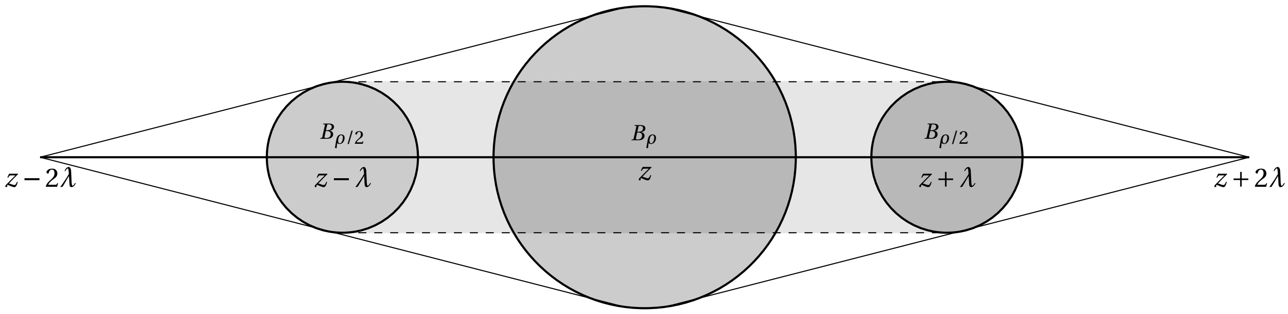

Figure 6.4: Questa figura rappresenta il guscio convesso di $[z-2\lambda, z+2\lambda] \cup B_\rho(z)$. La regione ombreggiata che unisce $B_{\rho/2}(z-\lambda)$ a $B_{\rho/2}(z+\lambda)$ è esattamente $\bigcup_{\zeta\in[z-\lambda,z+\lambda]} B_{\rho/2}(\zeta)$ ed è interamente contenuta in $\left\{[z-2\lambda, z+2\lambda] \cup B_\rho(z)\right\}^{\text{co}}$

Let us define the *closed* segment

$$\sigma := \left[z-\bar{\lambda}, z+\bar{\lambda}\right].$$

Therefore, for any ρ such that

$$0 < \rho \leqslant \operatorname{dist}\left(z, \partial \mathring{K}_r^{\text{co}}\right),$$

and for any $\zeta \in \sigma$, the following inclusion holds by convexity:

$$B_{\rho/2}(\zeta) \subset \left\{2\mathring{\sigma} \cup B_\rho(z)\right\}^{\text{co}} \subset \mathring{K}_r^{\text{co}}.$$

Thus, if we set

$$\bar{\rho} := \operatorname{dist}\left(z, \partial \mathring{K}_r^{\text{co}}\right),$$

we deduce the relation

$$\bigcup_{\zeta\in\sigma} B_{\bar{\rho}/2}(\zeta) \subset \left\{2\mathring{\sigma} \cup B_{\bar{\rho}}(z)\right\}^{\text{co}} \subset \mathring{K}_r^{\text{co}}. \tag{6.10}$$

The inclusion (6.10) therefore implies that the segment σ is uniformly separated from the boundary:

$$\operatorname{dist}\left(\sigma, \partial \mathring{K}_r^{\text{co}}\right) \geqslant \frac{1}{2}\operatorname{dist}\left(z, \partial \mathring{K}_r^{\text{co}}\right).$$

Furthermore, we can establish the quantitative estimate on the size of the velocity perturbation:

$$\frac{C}{r}\left(r^2 - |u|^2\right) \approx \frac{r-|u|}{N} = \frac{|u_1|-|u|}{N} \leqslant \frac{|u_1-u|}{N} \approx |\bar{u}|,$$

where the inequalities follow from the reverse triangle inequality and the definition of z as a convex combination of the points $\{z_i\}$.

Finally, it remains to prove that the constructed perturbation $\bar{\lambda} = \left(\bar{u}, \bar{R}\right)$ corresponds to an element in the Λ-cone. Recall that

$$\left(\bar{u}, \bar{R}\right) = \frac{\lambda_2}{2}(z_2 - z_1) = \frac{\lambda_2}{2}(u_2-u_1, R_2-R_1),$$

where $z_i = (u_i, R_i) \in K_r$ for $i = 1, 2$. By the definition of K_r, this implies that

$$R_i = u_i \otimes u_i - \frac{r^2}{\mathrm{d}}\mathrm{Id}.$$

Therefore, the difference is

$$(u_2-u_1, R_2-R_1) = (u_2-u_1, u_2\otimes u_2 - u_1\otimes u_1) = W(u_2, u_1).$$

Since $(W(u_2,u_1), 0) \in \Lambda$ by Lemma 6.10, our constructed perturbation $\bar{\lambda}$ is indeed collinear to a direction in the Λ-cone. □

6.5 Oscillatory Solutions with Compact Spacetime Support

The geometric construction in the previous section identified admissible directions within the Λ-cone. These directions allow us to perturb elements of $\mathring{K}_r^{co}$ while remaining inside this set and controlling the distance to the boundary. However, simple plane waves of the form $\lambda h(\xi \cdot x)$ do not have compact support.

The goal of this section is to construct localized "plane waves". The quotation marks are intentional: imposing compact support means the resulting waves will not oscillate perfectly along a line segment between two states, but rather their image will be contained within an arbitrarily small tubular neighborhood of it.

Constructing such vector waves directly is a difficult task. A more manageable approach is to define a *potential operator* that maps smooth scalar functions to solutions of the relaxed system (6.4).

Henceforth, we will denote by λ a generic direction in the Λ-cone of the form $\lambda = (W(a,b), 0)$, where $W(a,b)$ is defined as in Lemma 6.10.

Proposition 6.20 (Potential Operator for Localized Waves)**.** *For each $\lambda \in \Lambda$ of the form specified above, there exists a third-order differential operator*

$$A(\partial) : C_c^\infty\left(\mathbb{R}^{d+1};\mathbb{R}\right) \to C_c^\infty\left(\mathbb{R}^{d+1};\mathbb{R}^{(d+1)\times(d+1)}\right),$$

and a spacetime vector $(\xi,\tau) \in \mathbb{R}^{d+1}$ with $\xi \neq 0$ such that:

a) *For any smooth, compactly supported scalar function ϕ, the matrix field $U_\phi := A(\partial)\phi$ corresponds to a solution $\left(u, R, q\right)$ of the relaxed system* (6.4) *with $q \equiv 0$.*

b) *If ϕ has a plane-wave profile, i.e., $\phi(x,t) = \psi(\xi \cdot x + \tau t)$, then the operator recovers the original direction λ:*

$$A(\partial)\phi(x,t) = \lambda\, \psi'''(\xi \cdot x + \tau t).$$

Proof. Let the vectors $a, b \in \mathbb{R}^d$ be those defining the direction λ. We define the following component operators:

$$A_u^i(\partial) := \sum_{k,l=1}^{d} (a_i b_k - b_i a_k)\, \partial_{kll},$$

$$A_R^{ij}(\partial) := \sum_{k=1}^{d} (b_i a_k - a_i b_k)\, \partial_{tkj} + \sum_{k=1}^{d} \left(b_j a_k - a_j b_k\right) \partial_{tki},$$

and assemble them into the spacetime matrix operator $A(\partial)$:

$$A(\partial) := \begin{pmatrix} \left(A_R^{ij}(\partial)\right)_{i,j=1}^{d} & \left(A_u^{j}(\partial)\right)_{j=1}^{d} \\ \left(A_u^{i}(\partial)\right)_{i=1}^{d} & 0 \end{pmatrix}.$$

For any $\phi \in C_c^\infty\left(\mathbb{R}^{d+1};\mathbb{R}\right)$, the operator $A(\partial)$ canonically defines the solution triple $\left(u_\phi, R_\phi, q_\phi\right)$ where

$$u_\phi := A_u(\partial)\phi, \qquad R_\phi := A_R(\partial)\phi, \qquad q_\phi := 0.$$

We now verify part (a) of the proposition. A direct computation shows that the divergence and trace properties hold:

$$\operatorname{tr}\left(R_\phi\right) = \sum_{i,k=1}^{\lceil} 2\,(a_k b_i - a_i b_k)\, \partial_{tki}\phi = 0, \quad \text{so } R_\phi \in C_c^\infty\left(\mathbb{R}^{d+1}; \mathcal{S}_0^{d}\right),$$

$$\operatorname{div} u_\phi = \sum_{i,k,l=1}^{d} (a_i b_k - b_i a_k)\, \partial_{klli}\phi = 0,$$

where the second line vanishes due to the symmetry of ∂_{klli} in indices k, i and the antisymmetry of $(a_i b_k - b_i a_k)$ in the same indices. Finally, we check the momentum equation:

$$\partial_t\left(u_\phi\right)_i + \left(\operatorname{div} R_\phi\right)_i = \underbrace{\sum_{k,l} (a_i b_k - a_k b_i)\, \partial_{kllt}\phi}_{=\alpha_i} + \underbrace{\sum_{j,k} (b_i a_k - a_i b_k)\, \partial_{tkjj}\phi}_{=-\alpha_i} + \underbrace{\sum_{j,k} \left(b_j a_k - a_j b_k\right) \partial_{tkij}\phi}_{=0} = 0.$$

This proves that $U_\phi = A(\partial)\phi$ corresponds to a solution of (6.4).

To prove part (b), we consider a plane wave profile $\phi(x,t) = \psi\left(\frac{(a+b)\cdot x+kt}{\varepsilon}\right)$ for some parameters k,ε. By explicitly applying the differential operators, we obtain

$$\left(u_\phi, R_\phi\right)(x,t) = \varepsilon^{-3}\left(2(a-b)s^2, -2(a\otimes a - b\otimes b)sk\right)\psi'''\left(\frac{(a+b)\cdot x+kt}{\varepsilon}\right),$$

where for brevity we have set $s = \frac{|a+b|^2}{2} = r^2 + a\cdot b$. If we choose the parameters

$$k = -s, \qquad \varepsilon = \left(2s^2\right)^{1/3},$$

and define the spacetime direction $(\xi,\tau) := \left(\frac{a+b}{\varepsilon}, \frac{k}{\varepsilon}\right)$, we obtain the stated result:

$$\left(u_\phi, R_\phi, q_\phi\right) = \lambda\psi'''(\xi\cdot x + \tau t).$$

□

We can therefore say that the operator $A(\partial)$ is a *potential operator* that maps smooth, compactly supported functions to compactly supported vector waves that are collinear with a prescribed direction λ.

The next step is the following: not only do we want to have compactly supported waves collinear to $\lambda\in\Lambda$ that are solutions of (6.4), but we also want them to be oscillatory with frequency ε^{-1} and with zero mean, so that they converge weakly to zero as $\varepsilon\to 0$.

Lemma 6.21. *Let $\mathcal{O}$ be a non-empty open set of $\mathbb{R}^{d+1}$, and let $\lambda = (W(a,b), 0)\in\Lambda$ for $a,b\in r\mathbb{S}^{d-1}$, $a\neq b$. Furthermore, consider $\mathcal{O}'\subset\mathcal{O}$ a second open set of $\mathbb{R}^{d+1}$. For every neighborhood $\mathcal{V}$ of the segment $[-\lambda,\lambda]$, there exists a state $(u,R,0)\in C_c^\infty(\mathcal{O};\mathcal{V})$ such that*

a) *$(u,R,0)$ is a solution of (6.4),*

b) *if $\bar{u} = a-b$ is the velocity component of the oscillation in the direction $\lambda\in\Lambda$, then*

$$\frac{1}{|\mathcal{O}'|}\int_{\mathcal{O}'}|u(x,t)|\,\mathrm{d}x\mathrm{d}t > \alpha|\bar{u}|, \tag{6.11}$$

for some $\alpha > 0$.

Proof. Let λ, A, ξ, τ be as in Proposition 6.20, and let $\phi\in C_c^\infty\left(\mathbb{R}^{d+1};\mathbb{R}\right)$ be such that $\operatorname{supp}(\phi)\subset\mathcal{O}$, $\phi\equiv 1$ in $\mathcal{O}'$, and $\phi(x,t)\in(0,1)$ for every $(x,t)\in\mathcal{O}\setminus\mathcal{O}'$. We define the following compactly supported perturbation:

$$\begin{aligned} z_\varepsilon(x,t) &= (u_\varepsilon, R_\varepsilon, 0)(x,t) \\ &= A(\partial)\left[\varepsilon^3\phi(x,t)\cos\left(\frac{\xi\cdot x+\tau t}{\varepsilon}\right)\right]. \end{aligned}$$

By the result proven in Proposition 6.20 part (a), we can say that z_ε is a solution of (6.4) for every $\varepsilon > 0$.

Thanks to standard considerations on the product rule for derivatives, we can say that

$$z_\varepsilon(x,t) = \varepsilon^3\phi(x,t)A(\partial)\cos\left(\frac{\xi\cdot x+\tau t}{\varepsilon}\right) + \sum_k \varepsilon^3 B_k(\partial)\phi(x,t)C_k(\partial)\cos\left(\frac{\xi\cdot x+\tau t}{\varepsilon}\right),$$

where the operators B_k, C_k are matrix differential operators, and the C_k are of order zero, one, or two. Applying part (b) of Proposition 6.20, we can deduce that

$$A(\partial)\cos\left(\frac{\xi\cdot x+\tau t}{\varepsilon}\right) = \frac{\lambda}{\varepsilon^3}\sin\left(\frac{\xi\cdot x+\tau t}{\varepsilon}\right),$$

which in turn implies

$$\left\| z_\varepsilon(x,t) - \lambda\phi(x,t)\sin\left(\frac{\xi\cdot x+\tau t}{\varepsilon}\right)\right\|_{L^\infty(\mathcal{O})} = \mathcal{O}(\varepsilon),$$

that is, z_ε can get arbitrarily close to the segment $[-\lambda,\lambda]$ as $\varepsilon\to 0$.

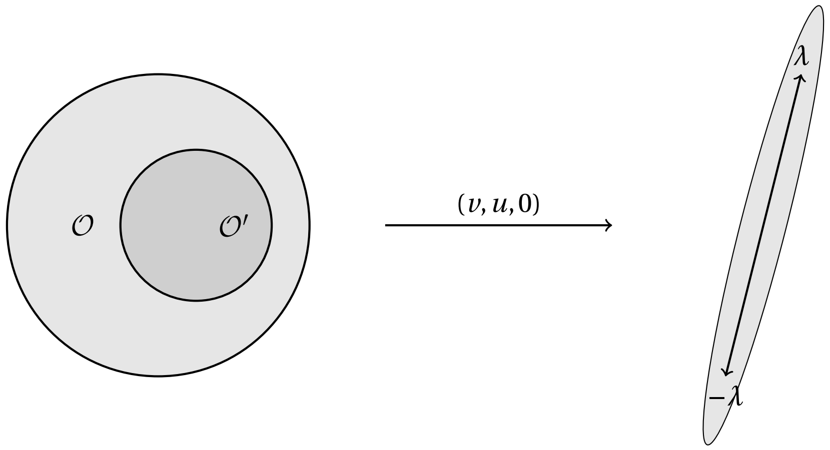

Figure 6.5: L'onda localizzata $(v, u, 0)$ mappa un aperto $\mathcal{O}$ di $\mathbb{R}^d$ in un aperto "quasi colineare" a λ.

To prove Item b, it is sufficient to note that in $\mathcal{O}'$,

$$z_\varepsilon(x,t) = \lambda \sin\left(\frac{\xi \cdot x + \tau t}{\varepsilon}\right),$$

and therefore

$$\frac{1}{|\mathcal{O}'|}\int_{\mathcal{O}'} |u_\varepsilon(x,t)|\,\mathrm{d}x\mathrm{d}t = \frac{|\bar{u}|}{|\mathcal{O}'|}\int_{\mathcal{O}'} \left|\sin\left(\frac{\xi\cdot x + \tau t}{\varepsilon}\right)\right| \mathrm{d}x\mathrm{d}t \xrightarrow{\varepsilon\to 0} \gamma|\bar{u}|,$$

for some $\gamma > 0$. It is then sufficient to select $\alpha < \gamma/2$ to prove the claim. □

6.6 Proof of Theorem 6.3

Definition 6.22 (Subsolution Space)**.** Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ be a bounded open set and $\bar{e} \in L^\infty\left([0,T]; L^1(\Omega; \mathbb{R}_+ \setminus 0)\right)$. We define the *subsolution space* X_0 for the Euler equations (E) as the set of triples

$$\left(u, R, q\right) \in \mathcal{C}^\infty\left(\mathbb{R}^{\mathrm{d}+1}; \mathbb{R}^{\mathrm{d}} \times \mathcal{S}^{\mathrm{d}}_0 \times \mathbb{R}\right),$$

such that

- $\operatorname{supp}\left(u, R, q\right) \subset \Omega \times [0,T]$,
- $\left(u, R, q\right)$ is a solution of (6.4) in $\mathbb{R}^{\mathrm{d}+1}$,
- $(u(x,t), R(x,t)) \in \mathring{K}^{\mathrm{co}}_{\sqrt{\bar{e}(x,t)}}$ a.e. in $\Omega \times [0,T]$.

Remark 6.23 (Characterization of Subsolutions)**.** The third condition in Definition 6.22 is undoubtedly the most difficult to interpret and verify. Fortunately, Proposition 6.18 allows us to characterize this condition through the simpler matrix inequality:

$$u(x,t) \otimes u(x,t) - R(x,t) < \frac{\bar{e}(x,t)}{\mathrm{d}}\mathrm{Id} \quad \text{a.e. in } \Omega \times [0,T].$$

Taking the trace of the inequality above, we obtain an important and intuitive property of subsolutions: their kinetic energy is strictly less than the prescribed energy profile.

$$|u(x,t)|^2 < \bar{e}(x,t). \tag{6.12}$$

Remark 6.24 (Non-emptiness of the Subsolution Space)**.** The subsolution space X_0 is non-empty. Since $\bar{e}$ is strictly positive in $\Omega \times [0,T]$ by hypothesis, the trivial state $\left(u, R, q\right) \equiv 0$ is a valid subsolution, as it trivially satisfies all three conditions.

Definition 6.25 (Subsolution Closure)**.** We define the space X as the closure of the subsolution space X_0 with respect to the weak-* topology of L^∞:

$$X := \overline{X_0}^{L^\infty_w - \star}.$$

Lemma 6.26 (Properties of the Subsolution Closure)**.** *The topological space $(X, L^\infty_w - \star)$ is metrizable, non-empty, and compact. Furthermore, if a state $(u, R, q) \in X$ saturates the energy inequality, i.e.,*

$$|u(x,t)|^2 = \bar{e}(x,t) \quad \textit{a.e. in}\, \Omega \times [0,T],$$

then the pair (u, p) with $p := q - \frac{1}{\mathrm{d}}|u|^2$ is a weak solution to the Euler equations (6.1) *and is identically zero outside of $\Omega \times [0,T]$.*

Proof. i) Since $0 \in X_0$, we immediately obtain that $X \neq \emptyset$.

ii) Since X is a closed and bounded subset of $L^\infty(\Omega \times [0,T])$, by the Banach-Alaoglu theorem it is compact and metrizable with respect to the $L^\infty_w - \star$ topology.

iii) Suppose $(u, R, q) \in X$ is such that $|u|^2 = \bar{e}$ a.e. in $\Omega \times [0,T]$. We now define the matrix

$$M := u \otimes u - R - \frac{\bar{e}}{\mathrm{d}}\mathrm{Id}.$$

Since (u, R, q) is a weak-* limit of elements in X_0, and for every element in X_0 the corresponding matrix M is negative semi-definite, this property is preserved in the limit. Thus, $M \leqslant 0$.

Furthermore, M is symmetric and therefore diagonalizable, with real eigenvalues μ_k and corresponding eigenvectors w_k. The trace of M is

$$\mathrm{tr}(M) = \mathrm{tr}(u \otimes u) - \mathrm{tr}(R) - \mathrm{tr}\left(\frac{\bar{e}}{\mathrm{d}}\mathrm{Id}\right) = |u|^2 - 0 - \mathrm{d}\frac{|u|^2}{\mathrm{d}} = 0.$$

Since the trace is the sum of the eigenvalues, $\sum_k \mu_k = 0$. If any eigenvalue μ_j were non-zero, then since all $\mu_k \leq 0$, there would have to be another eigenvalue $\mu_i < 0$. But this would make the sum strictly negative, a contradiction. The only possibility is that all eigenvalues are zero, which implies M is the zero matrix. We deduce that

$$R = u \otimes u - \frac{|u|^2}{\mathrm{d}}\mathrm{Id} \quad \text{a.e. in}\, \Omega \times [0,T],$$

which, by Lemma 6.8, implies that the pair (u, p) is a weak solution to (6.1), concluding the proof.

□

The kinetic energy profile $\bar{e}$ therefore completely distinguishes solutions from subsolutions. The methodology we will use to prove the main theorem is as follows:

- We consider a subsolution $z_0 = (u_0, R_0, q_0) \in X_0$. Since z_0 is a subsolution, by applying the inequality (6.12), we deduce in particular that $|u_0|^2 < \bar{e}$. We therefore want to "add kinetic energy" to the subsolution u_0 in the following manner: for each $k \in \mathbb{N}$, we define

$$u_k = u_0 + \text{localized oscillations with frequency } k,$$

  such that $u_k \xrightarrow{k\to\infty} u_0$ and

$$\liminf_{k\to\infty} \|u_k\|^2_{L^2(\Omega\times[0,T])} > \|u_0\|^2_{L^2(\Omega\times[0,T])}.$$

- We add kinetic energy to u_0 while always remaining inside the subsolution space X_0. Geometrically, this membership can be interpreted, thanks to the result of Proposition 6.18, by ensuring that for each k,

$$(u_k(x,t), R_k(x,t)) \in \mathring{K}^{\mathrm{co}}_{\sqrt{\bar{e}(x,t)}}.$$

  We can therefore perturb a subsolution, but not so much that the addition of an oscillatory element causes the state to no longer be a subsolution.

◇ The final goal is thus to construct a sequence of subsolutions $(u_k)_k$ such that

$$|u_k(x,t)|^2 \xrightarrow{k\to\infty} \bar{e}(x,t) \quad \text{a.e. in } \Omega\times[0,T].$$

Thanks to the result of Lemma 6.26, we conclude that the limit points of the sequence $(u_k)_k$ (if they exist) are weak solutions to (6.1).

Definition 6.27 (Weak-* Metric)**.** We define $\mathsf{d}^\star_\infty$ as the metric induced by the $L^\infty_w - \star$ topology on X, such that $\left(X,\mathsf{d}^\star_\infty\right)$ is a complete metric space.

Definition 6.28 (Baire Class 1 Function)**.** Let X, Y be two Banach spaces. A map $J : X \to Y$ is a *Baire class 1 function* if it is the pointwise limit of a sequence of continuous functions.

Lemma 6.29. *If J is a Baire class 1 function, then its set of points of continuity is a residual set (a countable intersection of open dense sets).*

Lemma 6.30 (The Identity Map is Baire-1)**.** *The identity map*

$$I : \left(X,\mathsf{d}^\star_\infty\right) \to L^2\left(\mathbb{R}^{\mathrm{d}+1}\right),$$

is a Baire class 1 function.

Proof. For each $\varepsilon > 0$, let η_ε be a standard spacetime mollifier, $\eta_\varepsilon(x,t) = \varepsilon^{-(\mathrm{d}+1)}\eta(x/\varepsilon, t/\varepsilon)$. For any state $z = \left(u,R,q\right) \in X$, we have by standard properties of mollification that

$$\eta_\varepsilon \star z \xrightarrow[\varepsilon\to 0]{L^2(\mathbb{R}^{\mathrm{d}+1})} z.$$

Let us consider a sequence $z_k = \left(u_k, R_k, q_k\right)$ such that $z_k \overset{\star}{\rightharpoonup} z$ in L^∞. For any fixed $\varepsilon > 0$, convolution is a continuous operation with respect to this topology, so

$$\eta_\varepsilon \star z_k \xrightarrow[k\to\infty]{L^2} \eta_\varepsilon \star z.$$

Therefore, the family of continuous maps

$$\begin{aligned} I_\varepsilon : \left(X,\mathsf{d}^\star_\infty\right) &\to L^2\left(\mathbb{R}^{\mathrm{d}+1}\right) \\ z &\mapsto \eta_\varepsilon \star z \end{aligned}$$

approximates the identity map I pointwise as $\varepsilon \to 0$. □

Claim: If $z = \left(u,R,q\right) \in X$ is a point of continuity of the identity map I, then

$$|u(x,t)|^2 = \bar{e}(x,t) \quad \text{a.e. in } \Omega\times[0,T].$$

†

If the Claim is therefore true, then by Lemma 6.26, every point of continuity of the identity map $I : \left(X,\mathsf{d}^\star_\infty\right) \to L^2\left(\mathbb{R}^{\mathrm{d}+1}\right)$ will be a solution to (6.1). We also know that X is non-empty, and thanks to Lemma 6.29, we deduce that the set of continuity points is a residual set. This implies that almost every element of X is a weak solution to (6.1), thus concluding the proof of the Main Theorem.

Let us now prove the Claim. Suppose that z is a point of continuity of I. This implies that there exists a sequence $(z_k)_k \subset X_0$ such that

$$z_k \overset{\star}{\rightharpoonup} z \text{ in } L^\infty, \qquad \text{and} \qquad I(z_k) \to I(z) \text{ in } L^2.$$

By definition, each $z_k = \left(u_k, R_k, q_k\right) \in C^\infty_c$ and is a solution to the relaxed system (6.4), thus

$$\partial_t u_k + \operatorname{div} R_k + \nabla q_k = 0.$$

With some algebraic manipulation, this equation is equivalent to

$$\partial_t u_k + \operatorname{div}(u_k \otimes u_k) + \nabla \underbrace{\left(q_k - \frac{|u_k|^2}{\mathtt{d}}\right)}_{=p_k} = \operatorname{div}\underbrace{\left(u_k \otimes u_k - R_k - \frac{|u_k|^2}{\mathtt{d}}\mathrm{Id}\right)}_{=f_k}.$$

By hypothesis, $z_k \to z$ in L^2. Since $z = (u, R, q)$ is a solution to (6.1), we know from Lemma 6.26 that $(u, R) \in K$, i.e., $R = u \otimes u - \frac{|u|^2}{\mathtt{d}}\mathrm{Id}$. The strong L^2 convergence therefore implies that the term

$$u_k \otimes u_k - R_k - \frac{|u_k|^2}{\mathtt{d}}\mathrm{Id} \to u \otimes u - R - \frac{|u|^2}{\mathtt{d}}\mathrm{Id} = 0$$

in the sense of distributions. Thus, the forcing term $f_k \to 0$ in H^{-N} for any sufficiently large N.

The following result, which we will refer to as the "oscillatory lemma," is a technical result that is essential for the proof of the Claim. We postpone its proof for now.

Lemma 6.31 (Oscillatory Lemma). *There exists a constant $\beta > 0$ such that for every subsolution $z_0 = (u_0, R_0, q_0) \in X_0$, there exists a sequence $(z_k)_k = (u_k, R_k, q_k)_k \subset X_0$ with the properties that $z_k \overset{\star}{\rightharpoonup} z_0$ in L^∞ and*

$$\|u_k\|_{L^2}^2 \geqslant \|u_0\|_{L^2}^2 + \beta\left(\|\bar{e}\|_{L^1} - \|u_0\|_{L^2}^2\right)^2.$$

Remark 6.32. Lemma 6.31 states that every strict subsolution $z_0 \in X_0$ can be perturbed by a weakly converging sequence that increases the kinetic energy.

Assuming Lemma 6.31 holds, we use it to prove the Claim, which in turn will complete the proof of the Main Theorem. Suppose for now that $z = (u, R, q)$ is a point of continuity for the identity map I. Let $(z_k)_k \subset X_0$ be a sequence that converges weak-* to z. For each k, we can apply the oscillatory lemma to find a sequence $z_{k,j} = (u_{k,j}, R_{k,j}, q_{k,j})$ that converges weak-* to z_k as $j \to \infty$. The lemma guarantees that

$$\left\|u_{k,j}\right\|_{L^2}^2 \geqslant \|u_k\|_{L^2}^2 + \beta\left(\|\bar{e}\|_{L^1} - \|u_k\|_{L^2}^2\right)^2, \tag{6.13}$$

for every $j \in \mathbb{N}$. We now define a diagonal sequence $(\tilde{z}_k)_k$ by setting $\tilde{z}_k := z_{k,k}$. This sequence converges weak-* to z as $k \to \infty$. Using the inequality (6.13) with $j = k$, we obtain

$$\|\tilde{u}_k\|_{L^2}^2 \geqslant \|u_k\|_{L^2}^2 + \beta\left(\|\bar{e}\|_{L^1} - \|u_k\|_{L^2}^2\right)^2. \tag{6.14}$$

By hypothesis, $z \in X$ is a point of continuity for the identity map I. Since both sequences $(\tilde{z}_k)_k$ and $(z_k)_k$ converge weak-* to z, their images under I must converge in L^2 to $I(z)$. We can therefore take the limit as $k \to \infty$ in (6.14) to get

$$\|u\|_{L^2}^2 \geqslant \|u\|_{L^2}^2 + \beta\left(\|\bar{e}\|_{L^1} - \|u\|_{L^2}^2\right)^2. \tag{6.15}$$

This inequality holds if and only if $\|\bar{e}\|_{L^1} = \|u\|_{L^2}^2$. However, since u is the weak-* limit of elements from X_0, and every element of X_0 must satisfy the inequality (6.12), we also know that $|u|^2 \leq \bar{e}$ a.e. in $\Omega \times [0, T]$. Combining these two conditions implies that $|u|^2 = \bar{e}$ a.e., and the proof of Theorem 6.3 is complete, pending the proof of the oscillatory lemma.

6.6.1 Proof of the Oscillatory Lemma (Lemma 6.31)

This proof is the main technical block of the theory developed in these notes. We note that the theory developed in **??**, Section 6.4, and **??** will be used exclusively in the present proof, which in turn forms the basis for the proof of the Main Theorem, as we have seen in Section 6.6.

Let us therefore define

$$m := \inf_{(x,t)\in\Omega\times[0,T]} \sqrt{\bar{e}(x,t)}.$$

By the hypotheses of the Main Theorem, we have $m > 0$. This technical assumption will be important later.

Step 1. Let $z_0 = \left(u_0, R_0, q_0\right) \in X_0$. By the result of Lemma 6.19, for every $(x,t) \in \Omega \times [0,T]$ there exists a direction

$$\lambda(x,t) = \left(\bar{u}(x,t), \bar{R}(x,t), 0\right),$$

in the Λ-cone for which

$$z_0(x,t) \pm \lambda(x,t) \in \mathring{K}^{co}_{\sqrt{\bar{e}(x,t)}},$$

and furthermore

$$|\bar{u}(x,t)| \geqslant \frac{C}{\sqrt{\bar{e}(x,t)}}\left(\bar{e}(x,t) - |u_0(x,t)|^2\right) \geqslant \frac{C}{m}\left(\bar{e}(x,t) - |u_0(x,t)|^2\right). \tag{6.16}$$

Step 2. We want to prove that there exists a $\delta > 0$ such that

$$\sup_{(x,t)\in\Omega\times[0,T]} \operatorname{dist}\left(z_0(x,t), \partial\mathring{K}^{co}_{\sqrt{\bar{e}(x,t)}}\right) \geqslant \delta > 0. \tag{6.17}$$

Since $z_0 = \left(u_0, R_0, q_0\right) \in X_0$ is a subsolution, we know it has compact support in $\Omega \times [0,T]$. In particular, z_0 is identically zero on the boundary $\partial(\Omega \times [0,T])$. Furthermore, for any $y = (x,t) \in \partial(\Omega \times [0,T])$, we have $z_0(y) = 0$, and since $\bar{e}(y) > m^2 > 0$, the origin is in the strict interior of the admissible set, i.e., $0 \in \mathring{K}^{co}_{\sqrt{\bar{e}(y)}}$.

By the uniform continuity of z_0 and $\bar{e}$, there exists a small neighborhood of the boundary, $\mathcal{V}_{\bar{\rho}} = \bigcup_{y\in\partial(\Omega\times[0,T])} B_{\bar{\rho}}(y)$, such that for all points inside this neighborhood, z_0 remains close to the origin and thus uniformly away from the boundary of the admissible set.

We now consider the compact set $\omega := \overline{\Omega \times [0,T]} \setminus \mathcal{V}_{\bar{\rho}}$. The image $z_0(\omega)$ is a compact set. By the definition of a subsolution, this image is contained within the open set $\mathring{K}^{co} := \bigcup_{(x,t)\in\omega} \mathring{K}^{co}_{\sqrt{\bar{e}(x,t)}}$. A compact set contained in an open set has a strictly positive distance from the boundary of the open set. Therefore, there must exist a $\delta > 0$ such that

$$\operatorname{dist}\left(z_0(\omega), \partial\mathring{K}^{co}\right) \geqslant \delta.$$

This uniform lower bound holds on ω and, by construction, also on $\mathcal{V}_{\bar{\rho}}$, which proves (6.17).

Step 3. We select a point $(x_0, t_0) \in \Omega \times [0,T]$ and choose a direction $\lambda(x_0,t_0) \in \Lambda$ in the canonical manner as was done in Step 1.
For a fixed point $(x_0,t_0) \in \Omega \times [0,T]$, we can say that

$$z_0(x_0,t_0) + \lambda(x_0,t_0) \in \mathring{K}^{co}_{\sqrt{\bar{e}(x_0,t_0)}},$$

and furthermore, thanks to the result of Lemma 6.19 and the inequality (6.17);

$$\operatorname{dist}\left(z_0(x_0,t_0) + \lambda(x_0,t_0), \partial\mathring{K}^{co}_{\sqrt{\bar{e}(x_0,t_0)}}\right) \geqslant \frac{1}{2}\operatorname{dist}\left(z_0(x_0,t_0), \partial\mathring{K}^{co}_{\sqrt{\bar{e}(x_0,t_0)}}\right) \geqslant \frac{\delta}{2}. \tag{6.18}$$

We now note that, since z_0 and $\bar{e}$ are uniformly continuous in $\Omega \times [0,T]$, the map

$$(x,t) \mapsto \operatorname{dist}\left(z_0(x,t) + \lambda(x_0,t_0), \partial\mathring{K}^{co}_{\sqrt{\bar{e}(x,t)}}\right),$$

is also uniformly continuous in $\Omega \times [0,T]$. We can therefore deduce that there exists an $\varepsilon > 0$ such that for every (x,t) with $|x - x_0| + |t - t_0| < \varepsilon$,

$$\operatorname{dist}\left(z_0(x,t) + \lambda(x_0,t_0), \partial\mathring{K}^{co}_{\sqrt{\bar{e}(x,t)}}\right) \geqslant \frac{\delta}{4}.$$

This means we can *locally* perturb a subsolution in a single direction from the Λ-cone, independent of the point (x,t) in a small neighborhood, and still remain a subsolution.

Let us consider a radius $r > 0$ sufficiently small so that $B_r(x_0,t_0) \subset \Omega \times [0,T]$. Thanks to the result of Lemma 6.21, there exists a state $z_r = \left(u_r, R_r, q_r\right)$, which is a solution of (6.4) with support in $B_r(x_0,t_0)$. Furthermore, the image $z_r(B_r(x_0,t_0))$ is contained in an ε-neighborhood of the segment $[-\lambda(x_0,t_0), \lambda(x_0,t_0)]$. Therefore, if ε is sufficiently small, I can deduce that

$$\operatorname{dist}\left(z_0(x,t) + z_r(x,t), \partial\mathring{K}^{co}_{\sqrt{\bar{e}(x,t)}}\right) \geqslant \frac{\delta}{8},$$

and applying the inequality (6.11), we obtain

$$\int_{B_r(x_0,t_0)} |u_r(x,t)|\,\mathrm{d}x\mathrm{d}t > \alpha\,|\bar{u}(x_0,t_0)|\,|B_r(x_0,t_0)|. \tag{6.19}$$

Step 4. Since u_0 is a subsolution in $\Omega \times [0, T]$, applying the inequality (6.12) we obtain

$$\bar{e}(x,t) - |u_0(x,t)|^2 > 0,$$

for almost every $(x,t) \in \Omega \times [0,T]$. **Claim:** There exists an $\eta > 0$ such that the energy gap is uniformly bounded below:

$$\sup_{(x,t)\in\Omega\times[0,T]} \left\{\bar{e}(x,t) - |u_0(x,t)|^2\right\} \geqslant \eta > 0. \tag{6.20}$$

†

Proof. Suppose this is not true. Then there exists a sequence $(y_k)_k \subset \Omega \times [0,T]$, converging to a point $\bar{y} \in \overline{\Omega \times [0,T]}$, such that

$$\bar{e}(y_k) - |u_0(y_k)|^2 \to 0^+.$$

Since u_0 is a subsolution, it is uniformly continuous and identically zero on the boundary $\partial(\Omega \times [0,T])$. Furthermore, the prescribed energy is strictly positive, $\bar{e}(y) \geqslant m^2 > 0$. If the limit point $\bar{y}$ were on the boundary, we would have $u_0(\bar{y}) = 0$, and the limit would be $\bar{e}(\bar{y}) \geq m^2 > 0$, a contradiction. Therefore, the limit point must be in the interior, $\bar{y} \in \Omega \times [0,T]$.

By continuity, this implies that at the limit point, $\bar{e}(\bar{y}) = |u_0(\bar{y})|^2$. This contradicts the strict inequality required for a subsolution in X_0. □

Since $\bar{e}$ and u_0 are uniformly continuous, we can consider r_0 to be the modulus of continuity of the function $\bar{e} - |u_0|^2$ associated with the constant η from the claim. There exists a finite covering of $\Omega \times [0,T]$ by balls

$$\left(B_{r_j}(x_j,t_j)\right)_{j=1}^N, \qquad \text{with } (x_j,t_j) \in \Omega \times [0,T] \text{ and } r_j \in (0, r_0).$$

Furthermore, using Equation (6.20) we can deduce the following inequality, valid for every $(x,t) \in B_{r_j}(x_j,t_j)$:

$$\begin{aligned}\bar{e}(x,t) - |u_0(x,t)|^2 &< \bar{e}(x_j,t_j) - |u_0(x_j,t_j)|^2 + \eta \\ &< 2\left(\bar{e}(x_j,t_j) - |u_0(x_j,t_j)|^2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\Omega\times[0,T]} \left(\bar{e}(x,t) - |u_0(x,t)|^2\right) \mathrm{d}x\mathrm{d}t &\leqslant \sum_{j=1}^N \int_{B_{r_j}(x_j,t_j)} \left(\bar{e}(x,t) - |u_0(x,t)|^2\right) \mathrm{d}x\mathrm{d}t \\ &< 2\sum_{j=1}^N \left(\bar{e}(x_j,t_j) - |u_0(x_j,t_j)|^2\right) |B_{r_j}(x_j,t_j)|.\end{aligned} \tag{6.21}$$

At this point, we fix $k \in \mathbb{N}$ large enough such that $1/k < \min\{r_0, \varepsilon\}$ and select a family of balls $\{B_{r_{k,j}}(x_{k,j}, t_{k,j})\}_j$ with radii $r_{k,j} < 1/k$ for which (6.21) holds. Then, for each ball $B_{r_{k,j}}(x_{k,j}, t_{k,j})$, we consider

$$z_{k,j} = (u_{k,j}, R_{k,j}, q_{k,j}),$$

a solution to (6.4) localized in $B_{r_{k,j}}(x_{k,j}, t_{k,j})$, with an oscillation collinear to the direction $\lambda(x_{k,j}, t_{k,j})$, as was done in the previous Step. We define

$$z_k = (u_k, R_k, q_k) := z_0 + \sum_j z_{k,j},$$

which is an element of X_0 since the supports of $z_{k,j}$ are disjoint. Thus,

$$\begin{aligned}\int_{\Omega\times[0,T]} |u_k(x,t) - u_0(x,t)|\,\mathrm{d}x\mathrm{d}t &= \sum_j \int_{B_{r_{k,j}}(x_{k,j},t_{k,j})} |u_{k,j}(x,t)|\,\mathrm{d}x\mathrm{d}t \\ &\overset{(6.11)}{>} \alpha \sum_j |\bar{u}(x_{k,j},t_{k,j})|\,|B_{r_{k,j}}(x_{k,j},t_{k,j})| \\ &\overset{(6.16)}{\geqslant} \frac{C\alpha}{m} \sum_j \left(\bar{e}(x_{k,j},t_{k,j}) - |u_0(x_{k,j},t_{k,j})|^2\right) |B_{r_{k,j}}(x_{k,j},t_{k,j})| \\ &\overset{(6.21)}{\geqslant} \frac{C\alpha}{2m} \int_{\Omega\times[0,T]} \left(\bar{e}(x,t) - |u_0(x,t)|^2\right) \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{6.22}$$

We note that $z_k \overset{\star}{\rightharpoonup} z_0$, since the perturbations $z_{k,j}$ have zero mean. Therefore,

$$
\begin{aligned}
\liminf_{k\to\infty} \|u_k\|^2_{L^2(\Omega\times[0,T])} &= \|u_0\|^2_{L^2} + \liminf_{k\to\infty}\left(2\langle u_0 \mid u_k - u_0\rangle_{L^2} + \|u_k - u_0\|^2_{L^2}\right) \\
&= \|u_0\|^2_{L^2} + \liminf_{k\to\infty} \|u_k - u_0\|^2_{L^2} \\
&\geqslant \|u_0\|^2_{L^2} + |\Omega\times[0,T]|^{-1}\left(\liminf_{k\to\infty}\|u_k - u_0\|_{L^1}\right)^2 .
\end{aligned}
\tag{6.23}
$$

Combining the inequalities (6.22) and (6.23), we obtain

$$
\liminf_{k\to\infty} \|u_k\|^2_{L^2} \geqslant \|u_0\|^2_{L^2} + \frac{C^2\alpha^2}{4m^2\,|\Omega\times[0,T]|}\left(\|\bar{e}\|_{L^1} - \|u_0\|^2_{L^2}\right)^2 .
$$

By defining

$$
\beta := \frac{C^2\alpha^2}{4m^2\,|\Omega\times[0,T]|} > 0,
$$

we conclude the proof. □

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