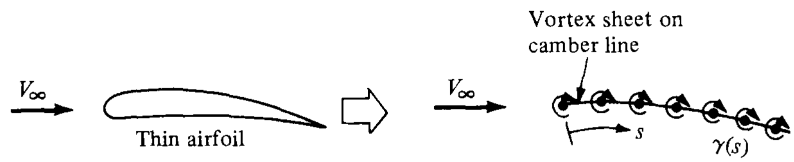


Figure 4.11 Thin airfoil approximation.

the surface and the airflow (see Figure 1.35). This boundary layer is a highly viscous region in which the large velocity gradients produce substantial vorticity; that is, $\nabla \times \mathbf{V}$ is finite within the boundary layer. (Review Section 2.12 for a discussion of vorticity.) Hence, in real life, there is a distribution of vorticity along the airfoil surface due to viscous effects, and our philosophy of replacing the airfoil surface with a vortex sheet (such as in Figure 4.10) can be construed as a way of modeling this effect in an inviscid flow.[3]

Imagine that the airfoil in Figure 4.10 is made very thin. If you were to stand back and look at such a thin airfoil from a distance, the portions of the vortex sheet on the top and bottom surface of the airfoil would almost coincide. This gives rise to a method of approximating a thin airfoil by replacing it with a single vortex sheet distributed over the camber line of the airfoil, as sketched in Figure 4.11. The strength of this vortex sheet $\gamma(s)$ is calculated such that, in combination with the freestream, the camber line becomes a streamline of the flow. Although the approach shown in Figure 4.11 is approximate in comparison with the case shown in Figure 4.10, it has the advantage of yielding a closed-form analytical solution. This philosophy of thin airfoil theory was first developed by Max Munk, a colleague of Prandtl, in 1922 (see Reference 12). It is discussed in Sections 4.7 and 4.8.

4.5 The Kutta Condition

The lifting flow over a circular cylinder was discussed in Section 3.15, where we observed that an infinite number of potential flow solutions were possible, corresponding to the infinite choice of Γ. For example, Figure 3.28 illustrates three different flows over the cylinder, corresponding to three different values of Γ. The same situation applies to the potential flow over an airfoil; for a given airfoil at a given angle of attack, there are an infinite number of valid theoretical solutions, corresponding to an

3 It is interesting to note that some recent research by NASA is hinting that even as complex a problem as flow separation, heretofore thought to be a completely viscous-dominated phenomenon, may in reality be an inviscid-dominated flow which requires only a rotational flow. For example, some inviscid flow-field numerical solutions for flow over a circular cylinder, when vorticity is introduced either by means of a nonuniform freestream or a curved shock wave, are accurately predicting the separated flow on the rearward side of the cylinder. However, as exciting as these results may be, they are too preliminary to be emphasized in this book. We continue to talk about flow separation in Chapters 15 to 20 as being a viscous-dominated effect, until definitely proved otherwise. This recent research is mentioned here only as another example of the physical connection between vorticity, vortex sheets, viscosity, and real life.

infinite choice of Γ. For example, Figure 4.12 illustrates two different flows over the same airfoil at the same angle of attack but with different values of Γ. At first, this may seem to pose a dilemma. We know from experience that a given airfoil at a given angle of attack produces a single value of lift (e.g., see Figure 4.5). So, although there is an infinite number of possible potential flow solutions, nature knows how to pick a particular solution. Clearly, the philosophy discussed in the previous section is not complete—we need an additional condition that *fixes* Γ for a given airfoil at a given α.

To attempt to find this condition, let us examine some experimental results for the development of the flow field around an airfoil which is set into motion from an initial state of rest. Figure 4.13 shows a series of classic photographs of the flow over an airfoil, taken from Prandtl and Tietjens (Reference 8). In Figure 4.13*a*, the flow has just started, and the flow pattern is just beginning to develop around the airfoil. In these early moments of development, the flow tries to curl around the sharp trailing edge from the bottom surface to the top surface, similar to the sketch shown at the left of Figure 4.12. However, more advanced considerations of inviscid, incompressible flow (see, e.g., Reference 9) show the theoretical result that the velocity becomes infinitely large at a sharp corner. Hence, the type of flow sketched at the left of Figure 4.12, and shown in Figure 4.13*a*, is not tolerated very long by nature. Rather, as the real flow develops over the airfoil, the stagnation point on the upper surface (point 2 in Figure 4.12) moves toward the trailing edge. Figure 4.13*b* shows this intermediate stage. Finally, after the initial transient process dies out, the steady flow shown in Figure 4.13*c* is reached. This photograph demonstrates that the flow is *smoothly leaving the top and the bottom surfaces of the airfoil at the trailing edge.* This flow pattern is sketched at the right of Figure 4.12 and represents the type of pattern to be expected for the steady flow over an airfoil.

Reflecting on Figures 4.12 and 4.13, we emphasize again that in establishing the steady flow over a given airfoil at a given angle of attack, nature adopts that particular value of circulation (Γ_2 in Figure 4.12) which results in the flow leaving smoothly at the trailing edge. This observation was first made and used in a theoretical analysis by the German mathematician M. Wilhelm Kutta in 1902. Therefore, it has become known as the *Kutta condition*.

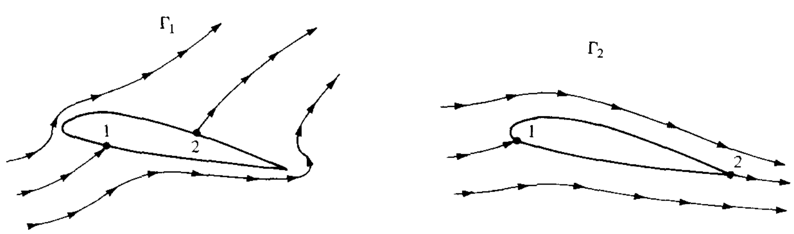
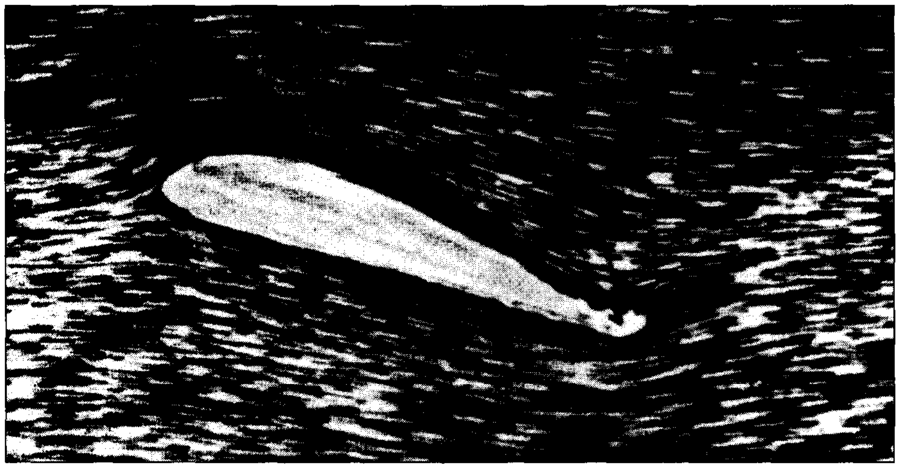
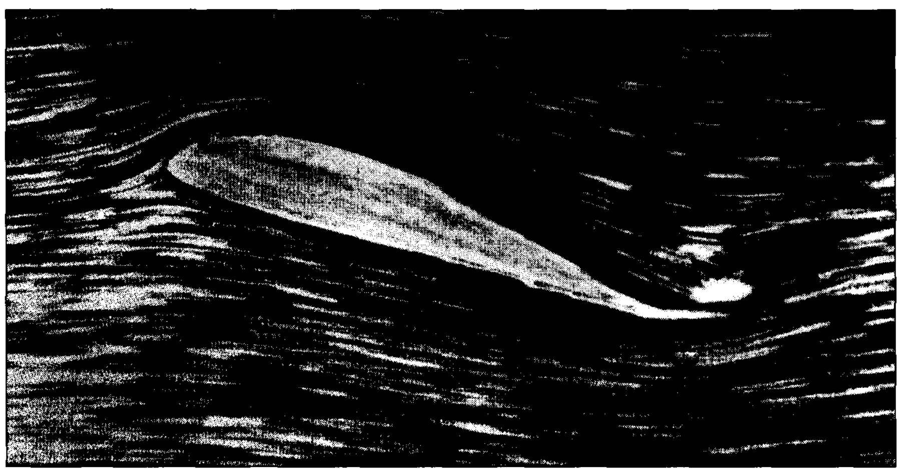


Figure 4.12 Effect of different values of circulation on the potential flow over a given airfoil at a given angle of attack. Points 1 and 2 are stagnation points.



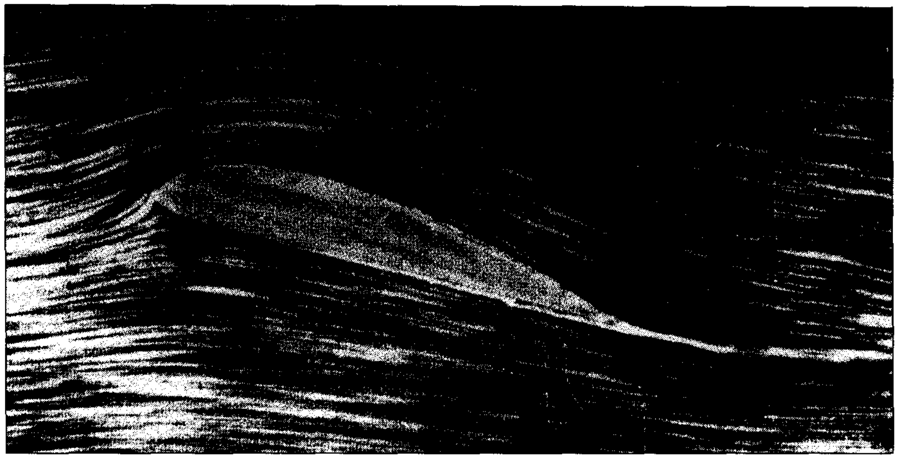
(a)



(b)

Figure 4.13 The development of steady flow over an airfoil; the airfoil is impulsively started from rest and attains a steady velocity through the fluid. (a) A moment just after starting. (b) An intermediate time. *(Source: Prandtl and Tiejens, Reference 8.)*

In order to apply the Kutta condition in a theoretical analysis, we need to be more precise about the nature of the flow at the trailing edge. The trailing edge can have a finite angle, as shown in Figures 4.12 and 4.13 and as sketched at the left of Figure 4.14, or it can be cusped, as shown at the right of Figure 4.14. First, consider the trailing edge with a finite angle, as shown at the left of Figure 4.14. Denote the velocities along the top surface and the bottom surface as V_1 and V_2, respectively. V_1



(c)

Figure 4.13 (continued) The development of steady flow over an airfoil; the airfoil is impulsively started from rest and attains a steady velocity through the fluid. (c) The final steady flow. *(Source: Prandtl and Tiejens, Reference 8.)*

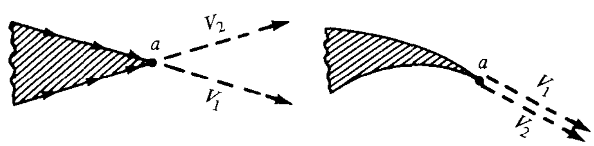


Figure 4.14 Different possible shapes of the trailing edge and their relation to the Kutta condition.

is parallel to the top surface at point a, and V_2 is parallel to the bottom surface at point a. For the finite-angle trailing edge, if these velocities were finite at point a, then we would have two velocities in two different directions at the same point, as shown at the left of Figure 4.14. However, this is not physically possible, and the only recourse is for both V_1 and V_2 to be zero at point a. That is, for the finite trailing edge, point a is a stagnation point, where $V_1 = V_2 = 0$. In contrast, for the cusped trailing edge shown at the right of Figure 4.14, V_1 and V_2 are in the same direction at point a, and hence both V_1 and V_2 can be finite. However, the pressure at point a, p_2, is a single, unique value, and Bernoulli's equation applied at both the top and bottom surfaces immediately adjacent to point a yields

$$p_a + \frac{1}{2}\rho V_1^2 = p_a + \frac{1}{2}\rho V_2^2$$

or

$$V_1 = V_2$$

Hence, for the cusped trailing edge, we see that the velocities leaving the top and bottom surfaces of the airfoil at the trailing edge are finite and equal in magnitude and direction.

We can summarize the statement of the Kutta condition as follows:

1. For a given airfoil at a given angle of attack, the value of Γ around the airfoil is such that the flow leaves the trailing edge smoothly.
2. If the trailing-edge angle is finite, then the trailing edge is a stagnation point.
3. If the trailing edge is cusped, then the velocities leaving the top and bottom surfaces at the trailing edge are finite and equal in magnitude and direction.

Consider again the philosophy of simulating the airfoil with vortex sheets placed either on the surface or on the camber line, as discussed in Section 4.4. The strength of such a vortex sheet is variable along the sheet and is denoted by $\gamma(s)$. The statement of the Kutta condition in terms of the vortex sheet is as follows. At the trailing edge (TE), from Equation (4.8), we have

$$\gamma(\text{TE}) = \gamma(a) = V_1 - V_2 \qquad \textbf{[4.9]}$$

However, for the finite-angle trailing edge, $V_1 = V_2 = 0$; hence, from Equation (4.9), $\gamma(\text{TE}) = 0$. For the cusped trailing edge, $V_1 = V_2 \neq 0$; hence, from Equation (4.9), we again obtain the result that $\gamma(\text{TE}) = 0$. Therefore, the Kutta condition expressed in terms of the strength of the vortex sheet is

$$\boxed{\gamma(\text{TE}) = 0} \qquad \textbf{[4.10]}$$

4.5.1 WITHOUT FRICTION COULD WE HAVE LIFT?

In Section 1.5 we emphasized that the resultant aerodynamic force on a body immersed in a flow is due to the net integrated effect of the pressure and shear stress distributions over the body surface. Moreover, in Section 4.1 we noted that lift on an airfoil is primarily due to the surface pressure distribution, and that shear stress has virtually no effect on lift. It is easy to see why. Look at the airfoil shapes in Figures 4.12 and 4.13, for example. Recall that pressure acts *normal* to the surface, and for these airfoils the direction of this normal pressure is essentially in the vertical direction, that is, the lift direction. In contrast the shear stress acts *tangential* to the surface, and for these airfoils the direction of this tangential shear stress is mainly in the horizontal direction, that is, the drag direction. Hence, pressure is the dominant player in the generation of lift, and shear stress has a negligible effect on lift. It is for this reason that the lift on an airfoil below the stall can be accurately predicted by *inviscid* theories such as that discussed in this chapter.

However, if we lived in a perfectly inviscid world, an airfoil could not produce lift. Indeed, the presence of friction is the very reason why we have lift. These sound like strange, even contradictory statements to our discussion in the preceding