Modelli a parametro singolo

Modello Normale e altri modelli standard, scelta della a priori

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DEAMS

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Agenda

Modelli a parametro singolo

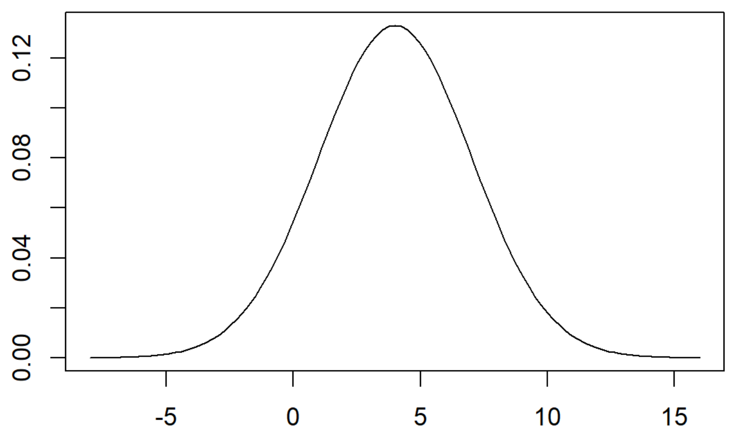
- Modello Normale: media incognita
- Modello Normale: varianza incognita
- Altre distribuzioni standard
  - (Binomiale)
  - Poisson (con appl)
  - Esponenziale
- Scelta della a priori

Dati normali

Normale / Gaussiana

- Osservazioni y a valori reali
- Media θ e varianza σ^2 (o deviazione standard σ) (dapprima assumiamo σ^2 noto)

$$p(y|\theta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$$
$$y \sim \mathrm{N}(\theta, \sigma^2)$$



Motivi per utilizzare la distribuzione normale

- Ampiamente usato nella modellazione statistica
- Fondamentale in quanto molti modelli e paradigmi di analisi avanzati si basano su o coinvolgono distribuzioni normali
- Distribuzione normale spesso giustificata in base al teorema del limite centrale
- Più spesso utilizzata per convenienza computazionale o per tradizione

Teorema del limite centrale

De Moivre, Laplace, Gauss, Chebysev, Liapounov, Markov, et al.

A determinate condizioni la somma (e la media) di variabili casuali si approssima alla distribuzione gaussiana al tendere di $n \to \infty$

I problemi

- non vale per tutte le distribuzioni, ad esempio, Cauchy
- potrebbe richiedere n grandi, per esempio con la Binomiale quando θ si avvicina a 0 o 1
- non vale se una delle variabili ha scala molto maggiore

Modello per una distribuzione normale con media non nota e varianza nota

Distribuzione normale (più osservazioni) con media non nota e varianza nota

Assume that observations $y = (y_1, \ldots, y_n)$ are normal (with unknown mean and known variance)

The joint distribution $p(\mathbf{y}) = p(y_1, \ldots, y_n)$ (multivariate distribution) is often (quite) difficult to specify directly

Then, it is easier to consider the joint conditional distribution $p(\mathbf{y}|\boldsymbol{\theta})$

A common assumption is that of **exchangeability** (beyond that of normality), i.e.,

$$y_1, \ldots, y_n|\theta \sim N(y_i|\theta, \sigma^2) \text{ iid}$$

or

$$p(\mathbf{y}|\theta) = \prod_i \underbrace{p(y_i|\theta)}_{\substack{\| \\ N(y_i|\theta,\sigma^2)}}$$

Note su scambiabilità

Non è necessario comprendere o utilizzare il termine scambiabilità prima del capitolo sui modelli gerarchici (capitolo 3).
A questo punto è sufficiente sapere che assumere scambiabilità

- riflette una convinzione, vale a dire che si ha un'opinione *simile* su ogni soggetto
- versione più debole di iid (iid è un caso speciale), (quindi)
  - l'indipendenza implica la scambiabilità,
  - la scambiabilità non implica indipendenza.
- Implica che possiamo modellare ogni x_i separatamente

De Finetti

> Diffidente nei confronti delle facili ipotesi di indipendenza, per lo più assunte da default, de Finetti avanzò l'idea che per la dipendenza stocastica (condizionata o meno) si potesse parlare di **dipendenza in senso** diretto e in senso indiretto
>
> – **diretto** quando un evento modifica (in modo significativo ai fini delle valutazioni di probabilità) le circostanze in cui si verifica un altro evento;
>
> – **indiretto** quando si manifesta un fatto H che influisce su tutta una collezione di eventi "separati" ($E_1, E_2, \ldots, E_k$).

Per quest'ultimo, tipico è il caso (ben noto agli attuari) di un inverno rigido che influisce sulla probabilità di morte di soggetti che non hanno tra loro nè relazioni nè vicinanze.

Likelihood

The likelihood is given by

$$p(y|\theta) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2\sigma^2} (y_i - \theta)^2 \right\} \right)$$

It is well known that

$$p(y|\theta) \propto \exp\left\{ -\frac{n}{2\sigma^2} (\bar{y} - \theta)^2 \right\}$$

where $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$ is the sample mean.

Data (y) influence through the sample mean $\bar{y}$

- The sample mean is a **sufficient statistic** for θ (μ)
- σ^2/n is the variance of the sample mean

The natural conjugate prior of an exponential distribution

The Gaussian distribution in exponential form is, for a single observation,

$$p(y_i|\theta) = \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y_i^2}{2\sigma^2}} \right) e^{-\frac{\theta^2}{2\sigma^2}} e^{\frac{\theta}{\sigma^2} y_i}$$

the likelihood is then, letting $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$

$$p(y|\theta) \propto e^{-n\frac{\theta^2}{2\sigma^2}} e^{\frac{\theta}{\sigma^2} n\bar{y}} = g(\theta)^n e^{\phi(\theta)^T t(y)}$$

and the conjugate prior is

$$p(\theta) \propto g(\theta)^\eta e^{\phi(\theta)^T \nu} = e^{-\eta\frac{\theta^2}{2\sigma^2}} e^{\frac{\theta}{\sigma^2}\nu} = \exp\left\{ -\frac{\eta}{2\sigma^2} \left(\theta^2 - 2\frac{\nu}{\eta}\theta \right) \right\}$$

that is, the conjugate family is the Gaussian family:

$$\theta \sim N(\mu_0, \sigma_0^2)$$

Distribuzione normale con media non nota θ
A priori coniugata per θ

Si assuma noto σ^2, la distribuzione normale è la famiglia coniugata per la media non nota di una distribuzione normale

$$\text{Likelihood} \quad p(y|\theta) \propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\theta)^2\right)$$

$$\text{Prior} \qquad p(\theta) \propto \exp\left(-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2\right)$$

$$\exp(a)\exp(b) = \exp(a+b)$$

$$\text{Posterior} \qquad p(\theta|y) \propto \exp\left(-\frac{1}{2}\left[\frac{(\bar{y}-\theta)^2}{\sigma^2/n} + \frac{(\theta-\mu_0)^2}{\sigma_0^2}\right]\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma_n^2}(\theta-\mu_n)^2\right)$$

Calcula

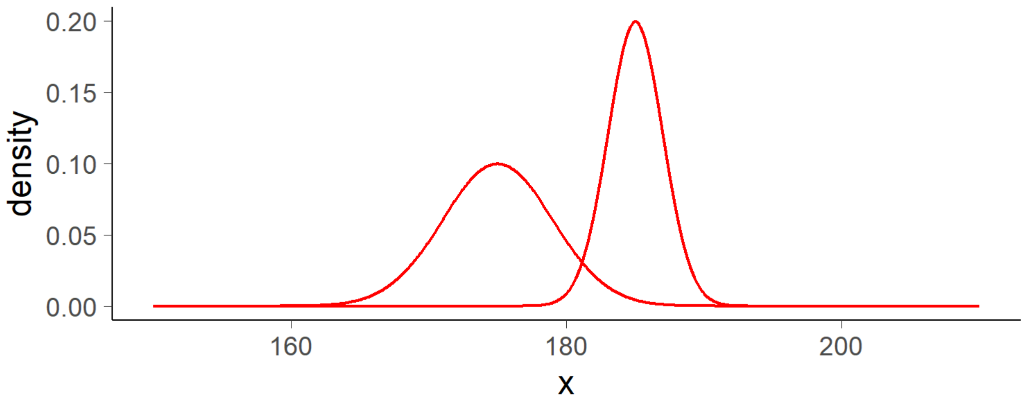
$$
\begin{aligned}
p(\theta|y) \propto p(y|\theta)\pi(\theta) &\propto \exp\left\{-\frac{n}{2\sigma^2}(\bar{y}-\theta)^2\right\}\exp\left\{-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2\right\} \\
&\propto \exp\left\{-\frac{n}{2\sigma^2}\theta^2 - \frac{1}{2\sigma_0^2}\theta^2 + \frac{\theta\bar{y}n}{\sigma^2} + \frac{\theta\mu_0}{\sigma_0^2}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)\theta^2 + \theta\left(\frac{n}{\sigma^2}\bar{y}+\frac{1}{\sigma_0^2}\mu_0\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)^{-1}}\left(\theta^2 - 2\theta\frac{\frac{n}{\sigma^2}\bar{y}+\frac{1}{\sigma_0^2}\mu_0}{\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)^{-1}}\left(\theta - \frac{\frac{n}{\sigma^2}\bar{y}+\frac{1}{\sigma_0^2}\mu_0}{\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}}\right)^2\right\}
\end{aligned}
$$

Posterior distribution (cont.)

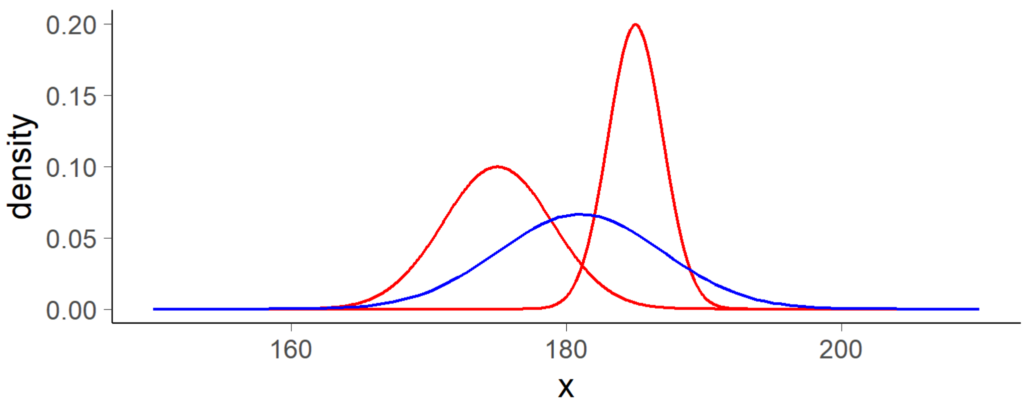
$$p(\theta|y) \propto \exp\left(-\frac{1}{2\sigma_n^2}(\theta-\mu_n)^2\right)$$

$$\theta|y \sim \mathrm{N}(\mu_n, \sigma_n^2), \quad \text{where} \quad \mu_n = \frac{\frac{1}{\sigma_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}} \quad \text{and} \quad \frac{1}{\sigma_n^2} = \frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}$$

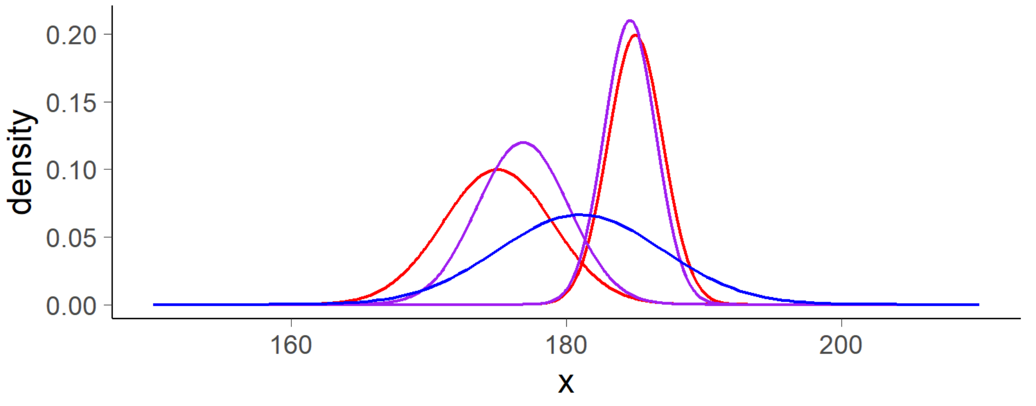
- posterior precision = prior precision + data precision
  (precision=1/variance, data precision is precision of the ML estimator $\bar{y}$)
- posterior mean is a weighted mean with precisions as weights
- $\mu_n = \mu_0 + (\bar{y}-\mu_0)\frac{\sigma_0^2}{\sigma_0^2+\sigma^2/n}$ or $\mu_n = \bar{y} - (\bar{y}-\mu_0)\frac{\sigma^2/n}{\sigma_0^2+\sigma^2/n}$
- $\mu_n = \mu_0$ if $\bar{y} = \mu_0$ or $\sigma_0^2 = 0$; in limit: $\mu_n \xrightarrow[\sigma_0\to 0]{} \mu_0$
  $\mu_n = \bar{y}$ if $\bar{y} = \mu_0$ or $\sigma^2 = 0$; in limit $\mu_n \xrightarrow[n\to\infty]{} \bar{y}$
- if $\sigma_0^2 = \sigma^2$, the prior corresponds to one virtual observation with value μ_0
- If $\sigma_0 \to \infty$ for a fixed n, or if $n \to \infty$ for fixed σ_0,
  $p(\theta|y) \approx \mathrm{N}(\theta|\bar{y}, \sigma^2/n)$ i.e. posterior approximates the likelihood
- Lastly, $\sigma_n \xrightarrow[n\to\infty]{} 0$ as well as $\sigma_n \xrightarrow[\sigma_0\to 0]{} 0$
  (but what the posterior is more concentrated around in the two cases?)



Supponi che vuoi indovinare quanto sono alte 2 persone e di una sei meno certo che dell'altra $\mu_1 = 175, \mu_2 = 185,$ $\sigma_1 = 4, \sigma_2 = 2$



Supponi di basare la priori aui parametri della popolazione maschile in Finlandia $\mu_{pop} = 181,$ $\sigma_{pop} = 6$



La posteriori combina priori e osservazione, se l'osservazione è più precisa la priori influenza meno, viceversa...

Parametrizzazione in termini della precisione

- ***Precision*** $\tau = 1/\sigma^2$ as the inverse of the variance
- $x \mid \mu, \sigma^2 \sim N(\mu, \sigma^2) \rightarrow x \mid \mu, \tau \sim N(\mu, \tau)$
- Conditional probability of the data

Precision of the data

$$x_i \mid \mu, \tau \sim N(\mu, \tau)$$

- Prior distribution

Precision of the prior

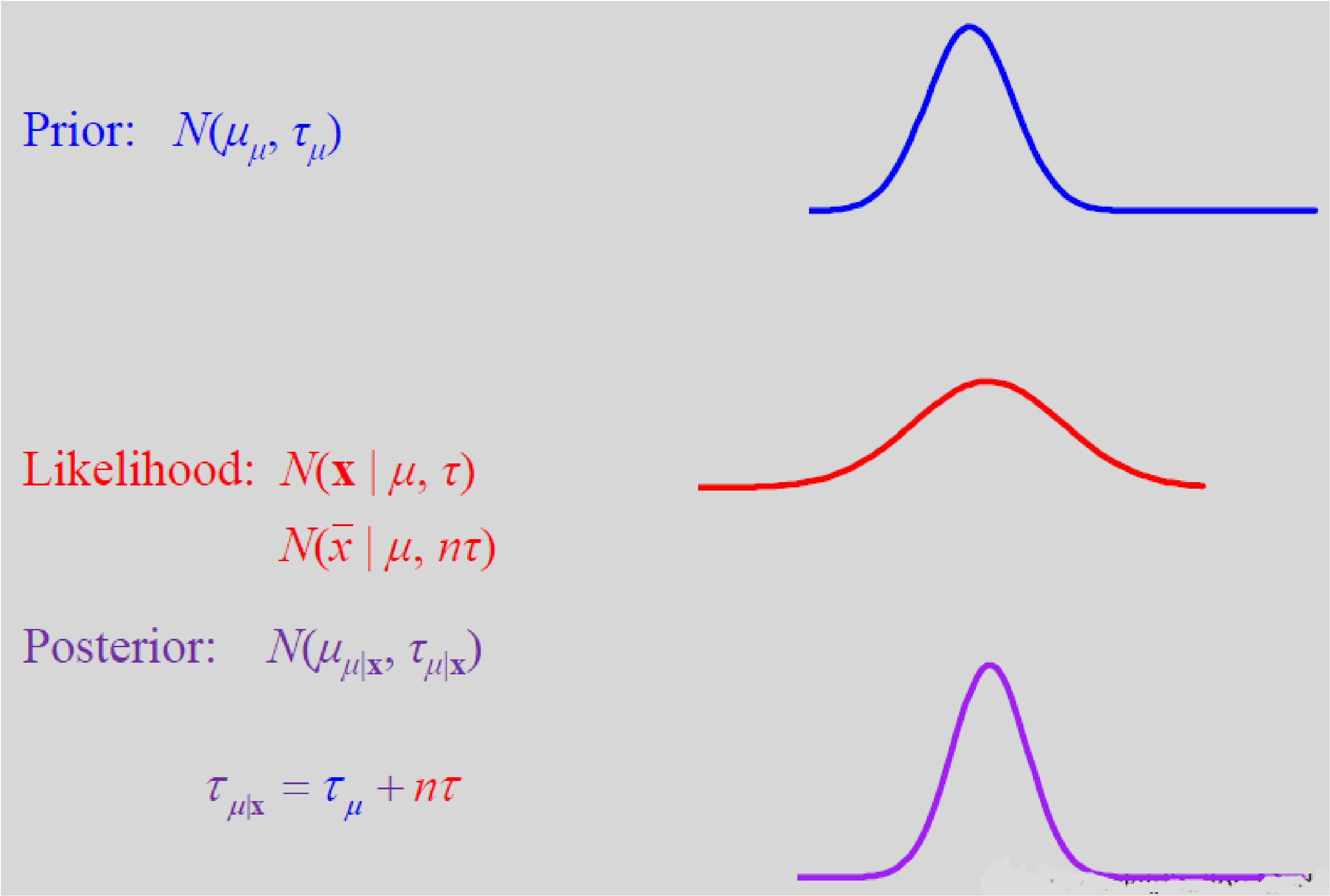
$$\mu \mid \mu_\mu, \tau_\mu \sim N(\mu_\mu, \tau_\mu)$$

- Posterior distribution

Precision of the posterior

$$\mu \mid \mathbf{x}, \tau \sim N(\mu_{\mu|\mathbf{x}}, \tau_{\mu|\mathbf{x}})$$

Accumulazione della precisione nella a posteriori



Media a posteriori

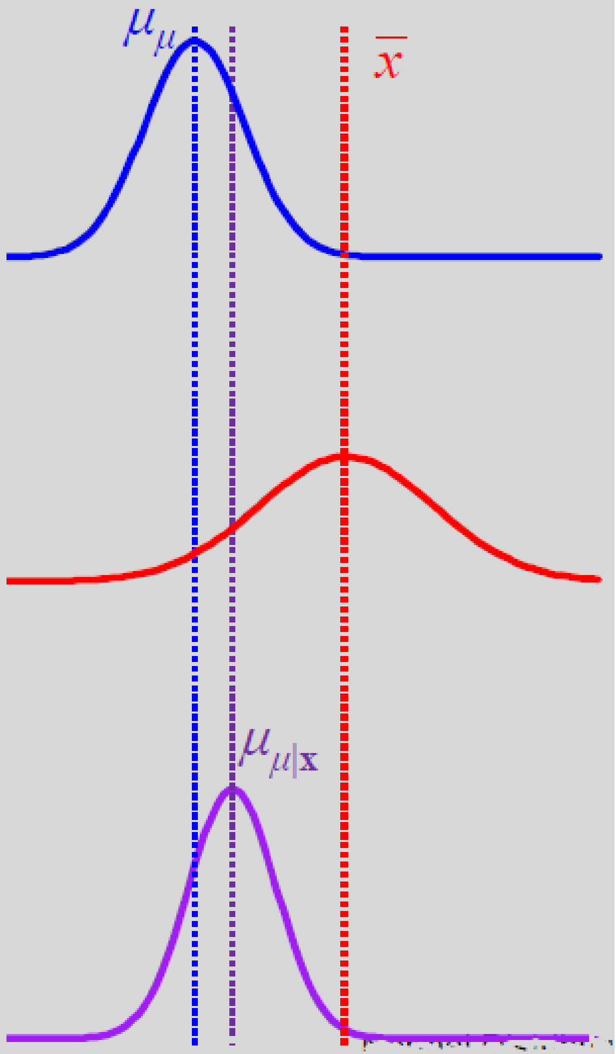
Prior: $N(\mu_\mu, \tau_\mu)$

Likelihood: $N(\mathbf{x} \mid \mu, \tau)$

$N(\bar{x} \mid \mu, n\tau)$

Posterior: $N(\mu_{\mu|\mathbf{x}}, \tau_{\mu|\mathbf{x}})$

$$\mu_{\mu|\mathbf{x}} = \frac{\tau_\mu}{\tau_\mu + n\tau}\mu_\mu + \frac{n\tau}{\tau_\mu + n\tau}\bar{x}$$



Shrinkage toward the prior

- La a posteriori è una *sintesi* della a priori e della verosimiglianza
- Può essere vista come un **aggiornamento della a priori** alla luce dei dati (così come riflessi nella verosimiglianza)
- Può essere vista come un ***augmentation*** (aumento) **delle informazioni nei dati** (così come riflesse nella verosimiglianza) grazie alle informazioni nella a priori
    - In questo modo, la a posteriori non appare esattamente come la verosimiglianza, viene "tirata" verso la a priori
    - La a posteriori mostra un ***shrinkage*** (contrazione) verso la a priori
    - La misura nella quale la a posteriori si restringe verso la a priori dipende dalla quantità relativa di informazioni nella a priori e nei dati

Poca informazione nei dati (n piccolo)

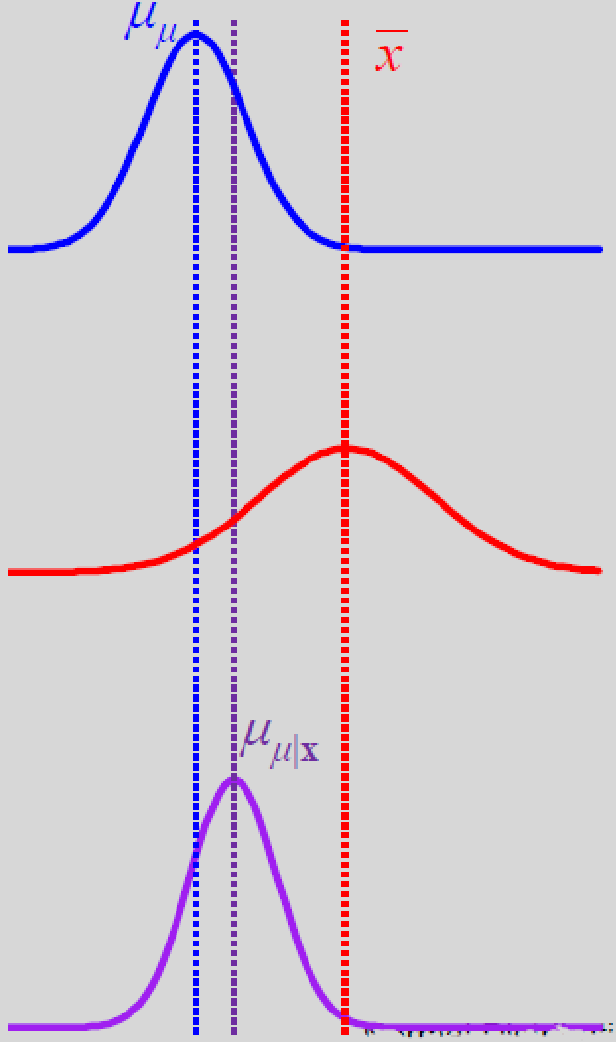
Prior: $N(\mu_\mu, \tau_\mu)$

Likelihood: $N(\mathbf{x} \mid \mu, \tau)$

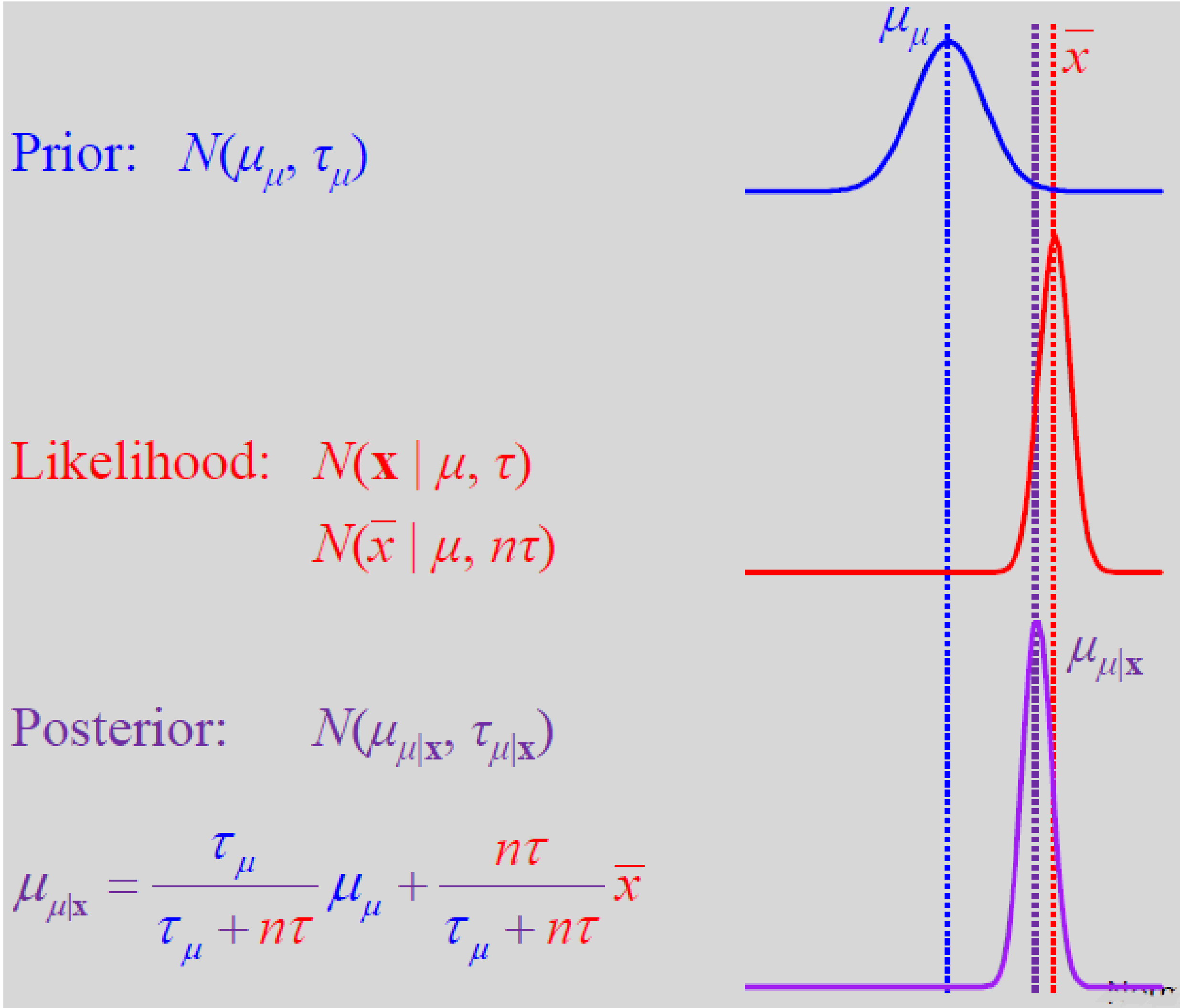
$N(\overline{x} \mid \mu, n\tau)$

Posterior: $N(\mu_{\mu|\mathbf{x}}, \tau_{\mu|\mathbf{x}})$

$$\mu_{\mu|\mathbf{x}} = \frac{\tau_\mu}{\tau_\mu + n\tau}\mu_\mu + \frac{n\tau}{\tau_\mu + n\tau}\overline{x}$$



Molta informazione nei dati (n grande)



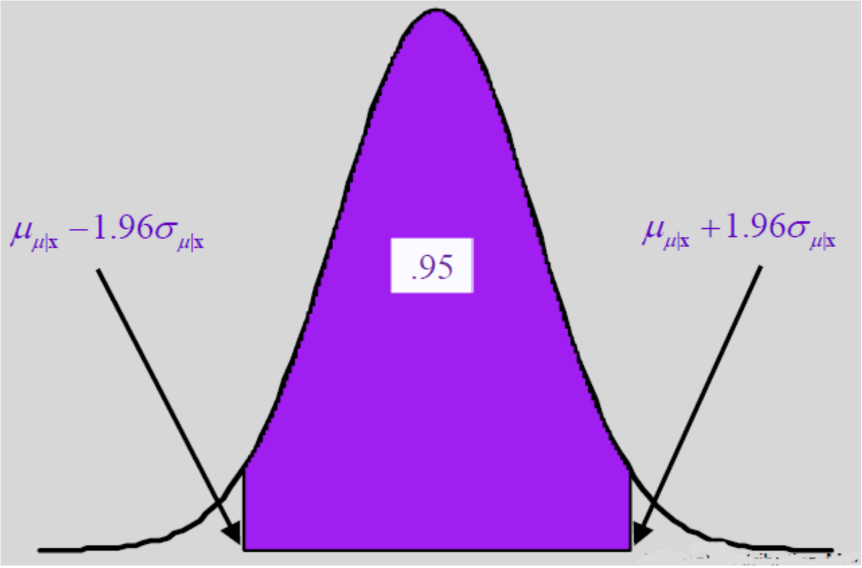
Inferenza intervallare

In una distribuzione a posteriori normale (gaussiana) $p(\mu|\mathbf{x}) = N(\mu_{\mu|\mathbf{x}}, \sigma^2_{\mu|\mathbf{x}})$ un $100(1-\alpha)$% intervallo *highest posterior* (e *central*) è

$$\mu_{\mu|\mathbf{x}} \pm z_{1-\alpha/2}\sigma_{\mu|\mathbf{x}}$$

dove $z_{1-\alpha/2}$ è il $100(1-\alpha/2)$-esimo percentile di una normale standard (i.e, $p(Z > z_{1-\alpha/2}) = \alpha/2$).

$$p(\mu_{\mu|\mathbf{x}} - 1.96\sigma_{\mu|\mathbf{x}} \leq \mu \leq \mu_{\mu|\mathbf{x}} + 1.96\sigma_{\mu|\mathbf{x}}) = 0.95$$



Esempio

- Supponiamo di avere un test con punteggio da 0-100
- essere interessati alla distribuzione dei punteggi dei test per gli studenti
- avere i punteggi dei test di n studenti
- Sia x_i il punteggio del test per lo studente i
- Postuliamo una distribuzione per i punteggi dei test degli studenti $x_i|\mu,\sigma^2 \sim N(\mu,\sigma^2)$
  - μ: la media della popolazione dei punteggi
  - σ^2: la variabilità nella distribuzione dei punteggi
- La varianza nota è $\sigma^2 = 25$
- $\mathbf{x} = (91, 85, 72, 87, 71, 77, 88, 94, 84, 92)$, $\bar{x} = 84.1$
- A priori per μ: $N(\mu_\mu, \sigma_\mu^2)$
- $\mu_\mu = 75, \sigma_\mu^2 = 50$

Esempio: distribuzione a posteriori

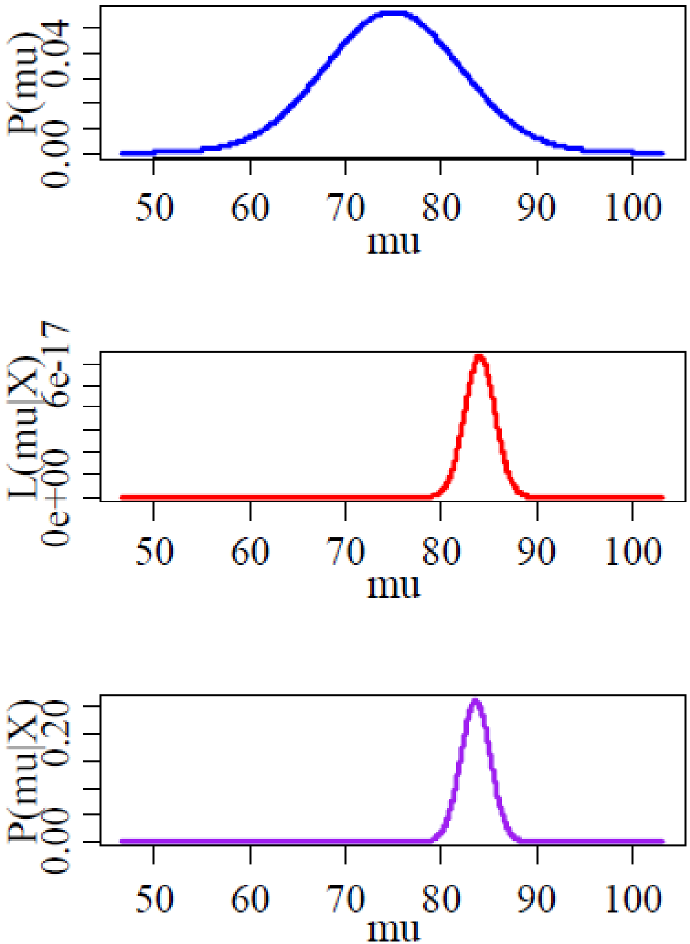
$$p(\mu \mid \mathbf{x}, \sigma^2) \propto p(\mathbf{x} \mid \mu, \sigma^2)\, p(\mu) \; = \Pi_i\, N(x_i \mid \mu, \sigma^2)\, N(\mu_\mu, \sigma_\mu{}^2)$$

$$p(\mu \mid \mathbf{x}, \sigma^2) = N(\mu_{\mu|\mathbf{x}}, \sigma_{\mu|\mathbf{x}}{}^2) \; = N(\mu_{\mu|\mathbf{x}} = 83.67, \sigma_{\mu|\mathbf{x}}{}^2 = 2.38)$$

$$\mu_{\mu|\mathbf{x}} = \left(\frac{\mu_\mu}{\sigma_\mu^2} + \frac{n\bar{x}}{\sigma^2}\right) \Bigg/ \left(\frac{1}{\sigma_\mu^2} + \frac{n}{\sigma^2}\right) = \left(\frac{75}{50} + \frac{(10)(84.1)}{25}\right) \Bigg/ \left(\frac{1}{50} + \frac{10}{25}\right) \approx 83.67$$

$$\sigma_{\mu|\mathbf{x}}^2 = 1 \Bigg/ \left(\frac{1}{\sigma_\mu^2} + \frac{n}{\sigma^2}\right) = 1 \Bigg/ \left(\frac{1}{50} + \frac{10}{25}\right) \approx 2.38$$

Esempio: inferenza



$$\mu \sim N(\mu_\mu = 75, \sigma_\mu^2 = 50)$$

$$L(\mu \mid \mathbf{x}, \sigma^2) = p(\mathbf{x} \mid \mu, \sigma^2) = \Pi_i \, p(x_i \mid \mu, \sigma^2) = \Pi_i \, N(x_i \mid \mu, \sigma^2 = 25)$$

$$\mu \mid \mathbf{x}, \sigma^2 \sim N(\mu_{\mu|\mathbf{x}}, \sigma_{\mu|\mathbf{x}}^2)$$

$$\sim N(\mu_{\mu|\mathbf{x}} = 83.67, \sigma_{\mu|\mathbf{x}}^2 = 2.38)$$

Riassunto: distr a post $\mu|\mathbf{x}, \sigma^2 \sim N(\mu_{\mu|\mathbf{x}} = 83.67, \sigma^2_{\mu|\mathbf{x}} = 2.38)$, media (anche mediana e moda) a post è $= 83.67$, sd a post è $= 1.54$, 95% HPDI $(80.64, 86.69)$

Asintotiche & connessioni con l'inferenza frequentista

A posteriori asintotica

$$\mu_{\mu|\mathbf{x}} = \left(\frac{\mu_\mu}{\sigma_\mu^2} + \frac{n\bar{x}}{\sigma^2}\right) \Bigg/ \left(\frac{1}{\sigma_\mu^2} + \frac{n}{\sigma^2}\right) \qquad \sigma^2_{\mu|\mathbf{x}} = 1 \Bigg/ \left(\frac{1}{\sigma_\mu^2} + \frac{n}{\sigma^2}\right)$$

- Asymptotically, as $n \rightarrow \infty$, $\sigma^2_{\mu|\mathbf{x}} \rightarrow \frac{\sigma^2}{n}$, $\mu_{\mu|\mathbf{x}} \rightarrow \bar{x}$
- Asymptotically, as $n \rightarrow \infty$, $p(\mu \,|\, \mathbf{x}) \rightarrow N\left(\bar{x}, \frac{\sigma^2}{n}\right)$
- Likewise if $\sigma_\mu{}^2 \rightarrow \infty$

($\sigma_0^2 \to \infty$) A uniform "distribution'' for the mean

Consider the inference for the mean in a Gaussian sample starting from a prior $\theta \sim N(\mu_0, \sigma_0^2)$ the posterior is

$$N\left(\frac{\mu_0\sigma^2 + \bar{y}n\sigma_0^2}{\sigma^2 + n\sigma_0^2}, \frac{\sigma^2\sigma_0^2}{\sigma^2 + n\sigma_0^2}\right)$$

if σ_0^2 is big relative to σ^2/n this is approximately

$$N\left(\bar{y}, \frac{\sigma^2}{n}\right)$$

which is the same as we would obtain by assuming

$$p(\theta) \propto k$$

improper priors

Connessioni con l'inferenza frequentista

	Bayes, as $n \rightarrow \infty$ or $\sigma_\mu^2 \rightarrow \infty$	Frequentist, $n \rightarrow \infty$
Distribution	$p(\mu \mid \mathbf{x}) \rightarrow N\left(\bar{x}, \frac{\sigma^2}{n}\right)$	$p(\bar{X}) \rightarrow N\left(\mu, \frac{\sigma^2}{n}\right)$
Point estimate	$\mu_{\mu\mid\mathbf{x}} \rightarrow \bar{x}$	$\bar{x}$
Variability	$\sigma_{\mu\mid\mathbf{x}} \rightarrow \frac{\sigma}{\sqrt{n}}$	$\sigma_{\bar{X}} \rightarrow \frac{\sigma}{\sqrt{n}}$
Interval	$\bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$	$\bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

As $n \rightarrow \infty$ or $\sigma_\mu^2 \rightarrow \infty$, will be similar ***numerically***, but very different ***conceptually***

Critiche e giustificazioni

- Asintoticamente, la a priori diventa irrilevante
    - *Principio di stima stabile*
    - La critica al “ruolo dell'a priori” perde forza
- “Vi garantiamo l'approccio bayesiano asintoticamente, la nostra denuncia è per campioni finiti”
- Ma l'approccio frequentista è giustificato solo in modo asintotico!
    - Stima dei parametri, errori standard, adattamento del modello di dati
- L'approccio bayesiano non ha bisogno di invocare gli argomenti asintotici
    - E se li concedi al frequentista, non dovresti concederli al bayesiano?

The principle of stable estimation, or precise measurement, specifies that when a likelihood function is sharply peaked in an interval over which a prior density is relatively flat, the posterior density does not differ much from the normed likelihood function.

Riassunto del modello di distribuzione normale con media non nota

- Scambiabilità comporta fattorizzazione
- Sufficienza della media campionaria
- A priori Normale come coniugata
- Precisione come inversa della varianza
  - Questo è ciò che utilizzano alcuni software (BUGS, JAGS)
- A posteriori come sintesi di a priori e dati
  - *Shrinkage to the prior*
- Asintotiche e connessioni con l'approccio frequentista

Modello normale: distribuzione predittiva

Spesso la distribuzione predittiva è più interessante della distribuzione a posteriori. La distribuzione a posteriori descrive l'incertezza nei parametri (come la proporzione di "orsetti" rossi nel sacchetto), mentre la distribuzione predittiva descrive anche l'incertezza sull'evento futuro (come quale colore viene scelto all'estrazione successiva).

Nel caso di distribuzione Gaussiana con varianza nota σ^2 il modello è

$$y \sim N(\theta, \sigma^2),$$

ove σ^2 descrive l'incertezza aleatoria.
Con l'a priori uniforme l'a posteriori è

$$p(\theta|y) \sim N(\theta|\bar{y}, \sigma^2/n),$$

dove σ^2/n descrive l'incertezza epistemica connessa a θ, mentre la distribuzione predittiva a posteriori per un nuovo $\tilde{y}$ è

$$p(\tilde{y}|y) \sim N(\tilde{y}|\bar{y}, \sigma^2 + \sigma^2/n),$$

dove l'incertezza è la somma dell'incertezza aleatoria (σ^2) e dell'incertezza epistemica (σ^2/n).

Modello normale: distribuzione predittiva - a priori coniugata per la media

Nel caso di a priori normale, la distribuzione predittiva a posteriori è

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta$$
$$p(\tilde{y}|y) \propto \int \exp\left(-\frac{1}{2\sigma^2}(\tilde{y}-\theta)^2\right)\exp\left(-\frac{1}{2\sigma_n^2}(\theta-\mu_n)^2\right)d\theta$$

Quindi,

$$\tilde{y}|y \sim \mathrm{N}(\mu_n, \sigma^2 + \sigma_n^2)$$

(go to theorem)

Possiamo anche derivare i momenti di ordine primo e secondo come segue

$$\begin{aligned} E(\tilde{y}|y) &= E(E(\tilde{y}|y,\theta)|y) = E(\theta|y) = \mu_n \\ V(\tilde{y}|y) &= E(V(\tilde{y}|y,\theta)|y) + V(E(\tilde{y}|y,\theta)|y) \\ &= E(\sigma^2|y) + V(\theta|y) = \sigma^2 + \sigma_n^2 \end{aligned}$$

Di nuovo, la varianza predittiva è = varianza *del* modello osservazionale σ^2 + varianza a posteriori σ_n^2 (varianza *sul* modello).

Model for a normal distribution with known mean and unknown variance

Likelihood

Although not a realistic situation, this is relevant both as

- an example of inference for a scale parameter
- a building block for the model on Gaussian data with both the mean and the variance unknown

Let then θ be known and

$$y_1, \ldots, y_n | \sigma^2 \sim N(y_i | \theta, \sigma^2) \text{ iid}$$

so the likelihood is

$$\begin{aligned} p(y|\sigma^2) &\propto \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2\sigma^2} (y_i - \theta)^2 \right\} \right) \\ &\propto (\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \theta)^2 \right\} \\ &\propto (\sigma^2)^{-n/2} \exp\left\{ -\frac{n}{2\sigma^2} v \right\} \end{aligned}$$

where the sufficient statistic is $v = \frac{1}{n} \sum_{i=1}^{n} (y_i - \theta)^2$, which is the $\hat{\sigma}^2_{MLE}$.

Conjugate prior

The corresponding conjugate prior is the **inverse-gamma** (IG)

$$\sigma^2 \sim \text{Inv-gamma}(\alpha, \beta)$$

$$p(\sigma^2) \propto (\sigma^2)^{-(\alpha+1)} e^{-\beta/\sigma^2}$$

which is the same as saying that

$$\frac{1}{\sigma^2} \sim \text{Gamma}(\alpha, \beta)$$

with the same hyperparametrers α and β.

Note that

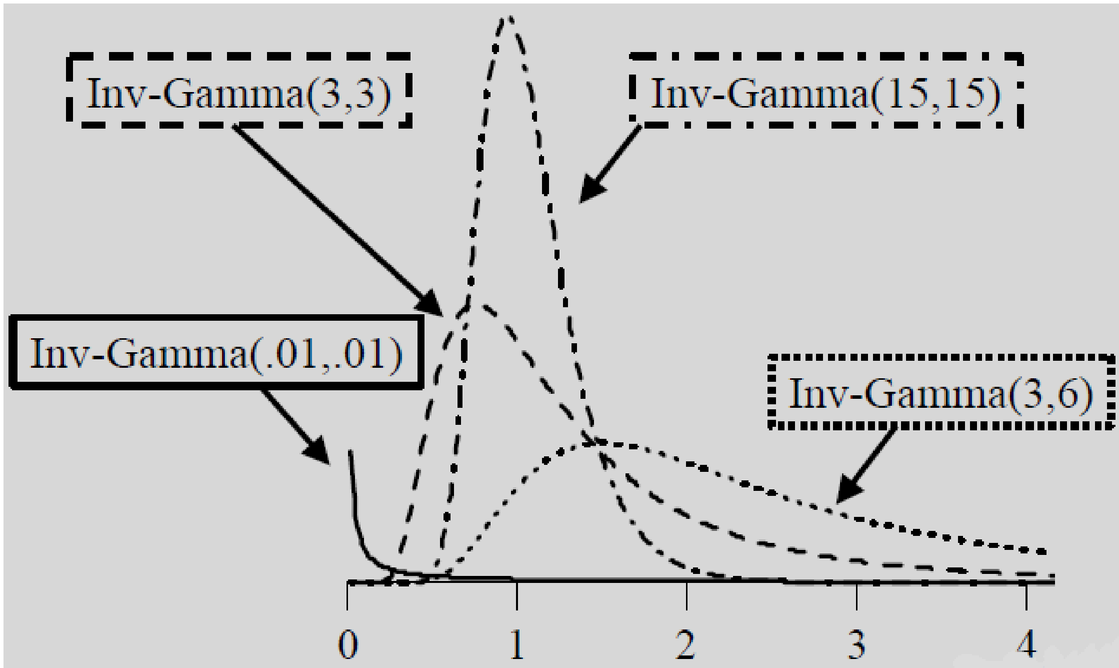
$E(\sigma^2) = \frac{\beta}{\alpha-1}$ for $\alpha > 1$,

$V(\sigma^2) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$,

$\text{Mode}(\sigma^2) = \frac{\beta}{\alpha+1}$

A priori gamma inversa per la varianza

$p(\sigma^2) = \mathrm{IG}(\alpha, \beta)$, con $\sigma^2 \geq 0, \alpha > 0, \beta > 0$



L'integrale è finito se $\alpha > 0$, la densità è finita se $\alpha \geq 1$

Non-informativa se $\alpha, \beta \to 0$

Posterior

The posterior distribution is then

$$p(\sigma^2|y) \propto p(y|\sigma^2)p(\sigma^2)$$

$$\begin{aligned} p(\sigma^2|y) &\propto (\sigma^2)^{-n/2} \exp\left\{-\frac{n}{2\sigma^2}v\right\}(\sigma^2)^{-\alpha-1}e^{-\beta/\sigma^2} \\ &\propto (\sigma^2)^{-n/2-\alpha-1} \exp\left\{-\frac{1}{\sigma^2}\left[\frac{n}{2}v+\beta\right]\right\} \end{aligned}$$

that is,

$$\sigma^2|y \sim \text{Inv-gamma}\left(\frac{n}{2}+\alpha, \frac{n}{2}v+\beta\right)$$

The posterior mean is $E(\sigma^2|y) = \frac{2\beta+nv}{2\alpha+n-2}$

The posterior mode is $\text{Mode}(\sigma^2|y) = \frac{2\beta+nv}{2\alpha+n+2}$

Reparametrization 1

It is convenient to reparametrize the model with the *precision* $\tau = 1/\sigma^2$, so the prior assumption is

$$\tau \sim \text{Gamma}(\alpha, \beta),$$

the likelihood is

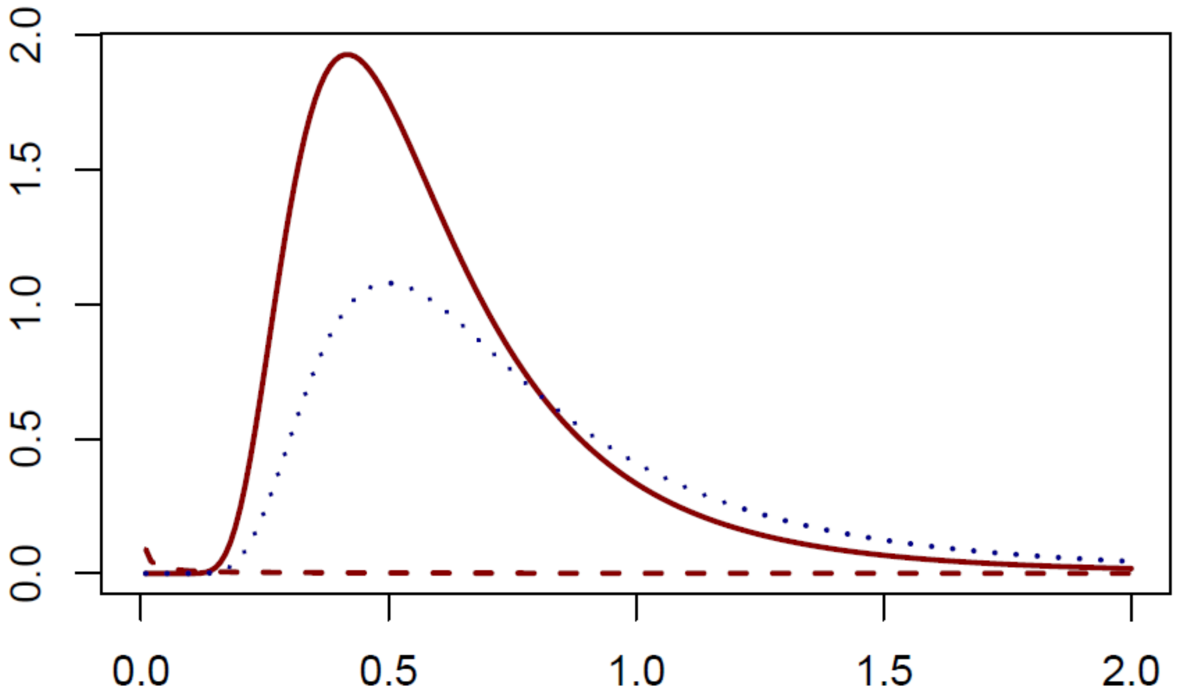
$$p(y|\tau) \propto (\tau)^{n/2} \exp\left\{-\frac{nv}{2}\tau\right\}$$

and the posterior is $\text{Gamma}(n/2 + \alpha, nv/2 + \beta)$:

$$\begin{aligned} p(\tau|y) &\propto \tau^{n/2} \exp\left\{-\frac{nv}{2}\tau\right\}\tau^{\alpha-1}e^{-\beta\tau} \\ &\propto \tau^{n/2+\alpha-1} \exp\left\{-\tau\left[\frac{nv}{2} + \beta\right]\right\} \end{aligned}$$

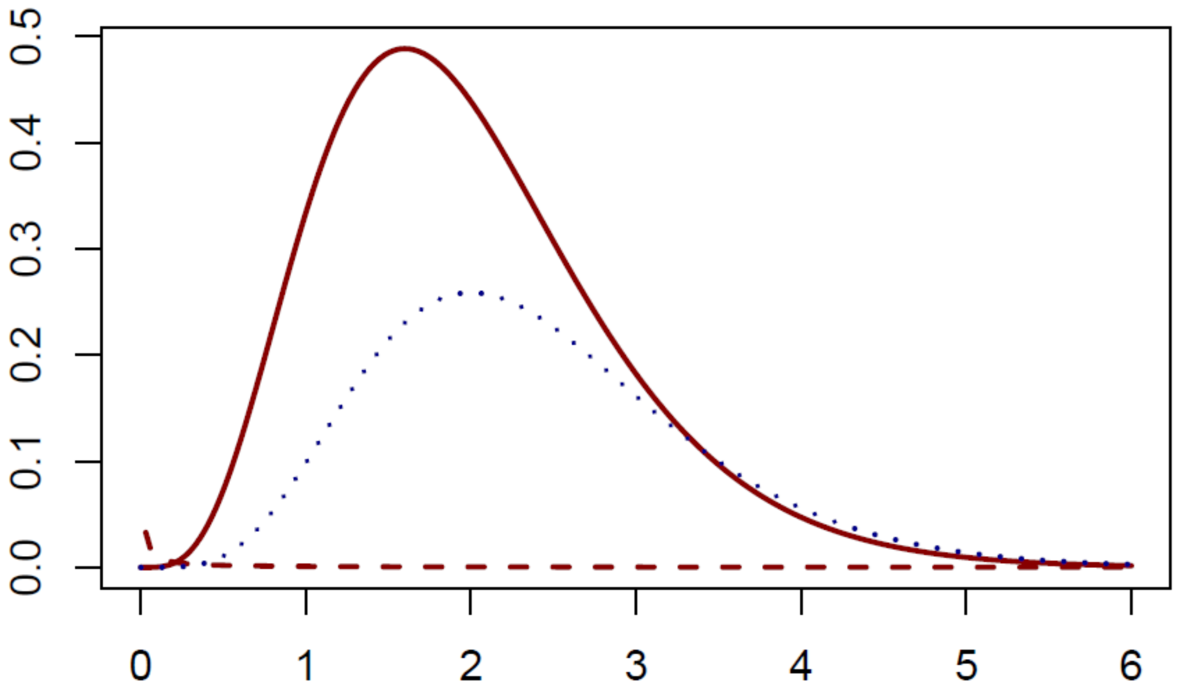
Inference for σ^2

Prior is an inverse gamma with parameters $\alpha = \beta = 10^{-3}$, sample variance is 0.5, $n = 10$



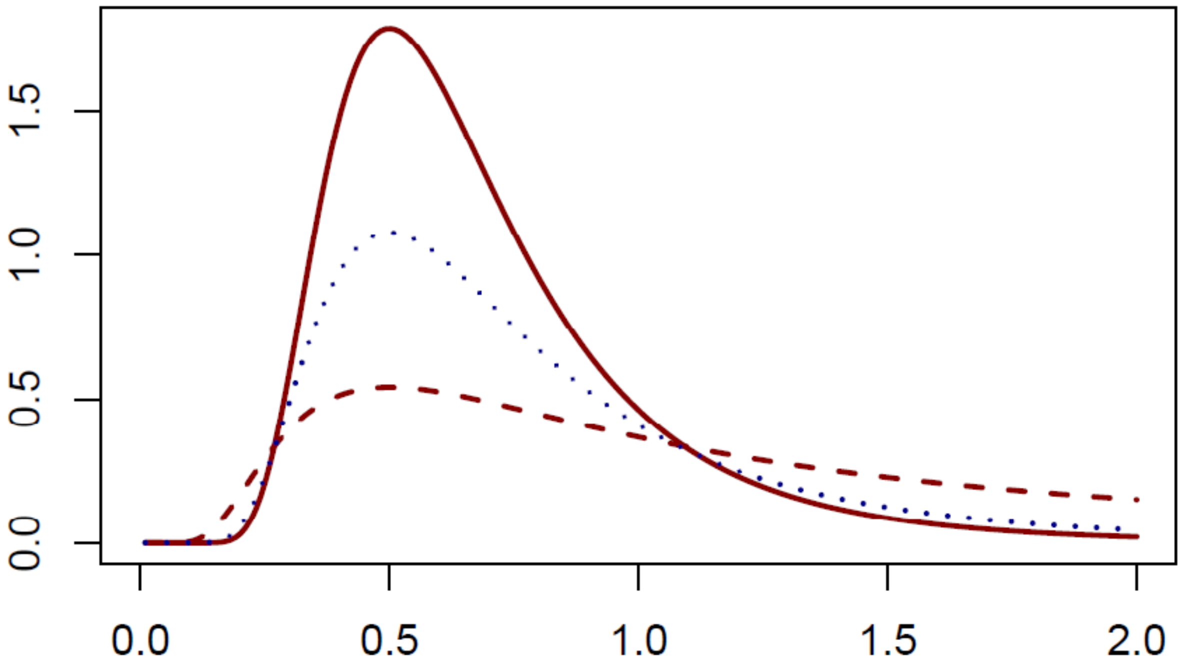
Inference for $\tau = 1/\sigma^2$

Prior is a gamma with parameters $\alpha = \beta = 10^{-3}$, sample variance is 0.5, $n = 10$



Inference for σ^2

Prior is an inverse gamma with parameters $\alpha = \beta = 1$, sample variance is 0.5, $n = 10$



Inference for $\tau = 1/\sigma^2$

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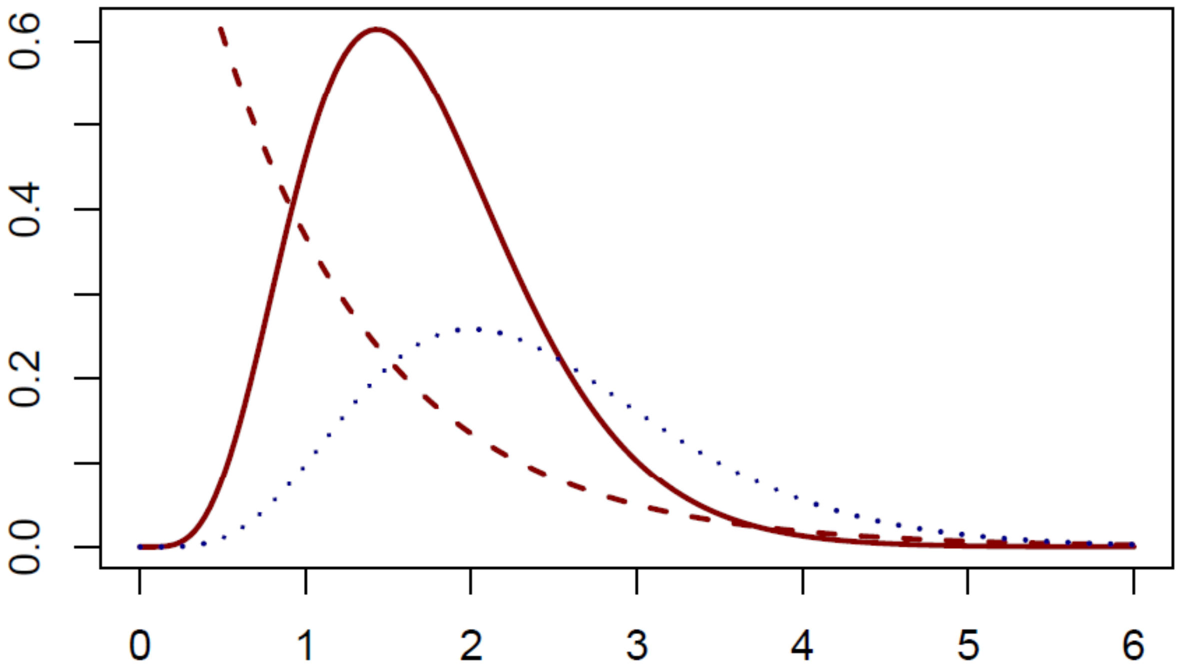


Reparametrization 2 with $\mathrm{Inv}\text{-}\chi^2$: prior

Another convenient parametrization is to write

$$\sigma^2 =_d \frac{\sigma_0^2 \nu_0}{X}, \quad X \sim \chi^2_{\nu_0}$$

following Gelman we call this $\mathrm{Inv}\text{-}\chi^2(\nu_0, \sigma_0^2)$ (**inverse-scaled** χ^2).

This corresponds to $\nu_0 = 2\alpha$ and $\sigma_0^2 = \beta/\alpha$, or, in other terms, implies that $\sigma^2 \sim \mathrm{Inv\text{-}gamma}(\nu_0/2, (\nu_0/2)\sigma_0^2)$, the density is

$$p(\sigma^2) = \frac{((\nu_0\sigma_0^2)/2)^{\nu_0/2}}{\Gamma(\nu_0/2)} (\sigma^2)^{-\nu_0/2-1} \exp\{-(\nu_0\sigma_0^2)/(2\sigma^2)\}$$

Note that

$$E(\sigma^2) = \frac{\nu_0\sigma_0^2}{\nu_0 - 2},\ \nu_0 > 2; \quad \mathrm{Mode}(\sigma^2) = \frac{\nu_0\sigma_0^2}{\nu_0 + 2}$$

$$V(\sigma^2) = \frac{(\nu_0\sigma_0^2)^2}{(\nu_0 - 2)^2(\nu_0 - 1)},\ \nu_0 > 2$$

Reparametrization 2 with $\text{Inv-}\chi^2$: posterior

The posterior is then

$$\sigma^2|y \sim \text{Inv-}\chi^2\left(\nu_0+n, \frac{\nu_0\sigma_0^2+n\hat{\sigma}^2_{MLE}}{\nu_0+n}\right)$$

the scale parameter being a weighted average of the prior variance σ_0^2 and the MLE with weight given by ν_0 and n.

Then

$$E(\sigma^2|y) = \frac{\nu_0\sigma_0^2+n\hat{\sigma}^2_{MLE}}{\nu_0+n-2}$$

$$\text{Mode}(\sigma^2|y) = \frac{\nu_0\sigma_0^2+n\hat{\sigma}^2_{MLE}}{\nu_0+n+2}$$

The priori can then be interpreted as information equivalent to ν_0 observations with variance σ_0^2 (a non-informative prior corresponds to $\nu_0 = 0$).

Summary of priors for the variance

Distribution	Density	Mean	Mode
$z \sim \text{Gamma}(\alpha, \beta)$	$\frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z}$	$\frac{\alpha}{\beta}$	$\frac{\alpha-1}{\beta},\ \alpha \geq 1$
$z \sim \text{inv-Gamma}(\alpha, \beta)$	$\frac{\beta^\alpha}{\Gamma(\alpha)} z^{-\alpha-1} e^{-\beta/z}$	$\frac{\beta}{\alpha-1},\ \alpha > 1$	$\frac{\beta}{\alpha+1}$
$z \sim \text{inv-}\chi^2(\nu, \sigma)$ $z \sim \text{inv-Gamma}(\nu/2, \nu\sigma/2)$	$\frac{(\nu\sigma/2)^{(\nu/2)}}{\Gamma(\nu/2)} z^{-\nu/2-1} e^{-\nu\sigma/(2z)}$	$\frac{\nu\sigma}{\nu-2},\ \nu > 2$	$\frac{\nu\sigma}{\nu+2}$

Una a priori per la varianza

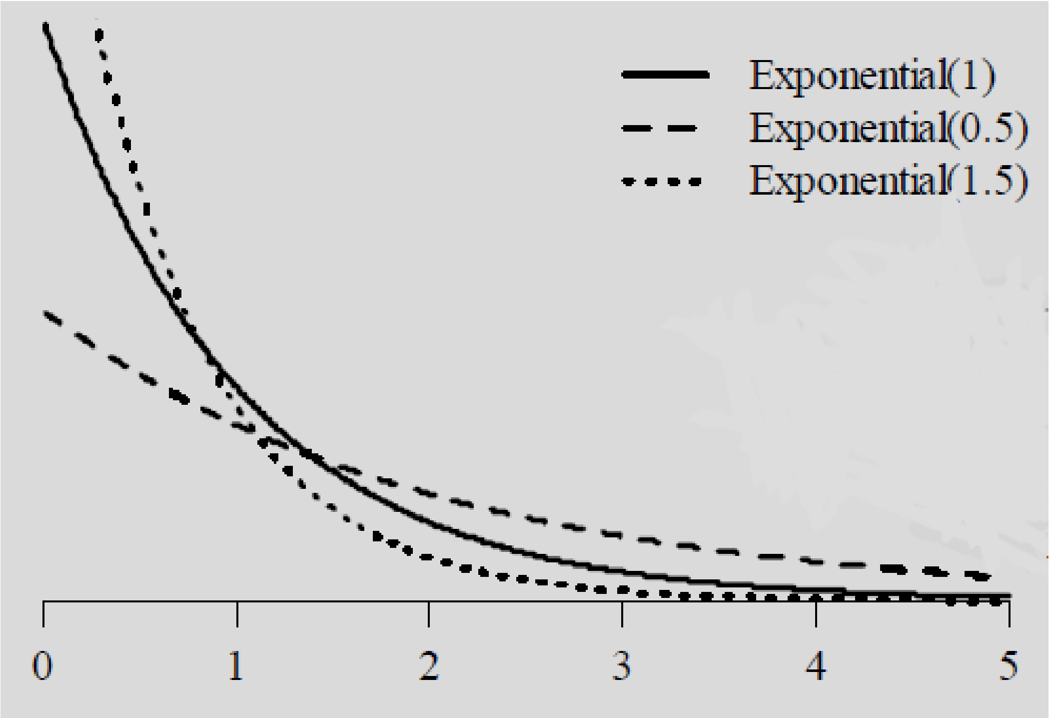
Molti approcci per specificare una a priori per la varianza

- Quale parametro?
  - Varianza σ^2
  - Deviazione standard σ
  - Precisione $\tau = \sigma^{-2}$
- Quale distribuzione?
  - Uniforme, Gamma inversa, Gamma, Esponenziale, semi-Cauchy, half-t, half-normale

Si tende a scriverlo in termini di varianza (σ^2)
Il software Stan utilizza la deviazione standard (σ)
Altri software (eg BUGS) utilizzano la precisione (τ)

A priori esponenziale per la Deviazione Standard

$p(\sigma) = \exp(\lambda)$, $f(\sigma|\lambda) = \lambda e^{-\lambda\sigma}$, con $\sigma \geq 0$, $\lambda > 0$



$E(\sigma) = 1/\lambda$ -> si può pensare che la stima migliore per σ sia $1/\lambda$

E' una Gamma con $\alpha = 1$

A priori diffusa per la variabilità

Campo di ricerca in continua espansione su come rappresentare un'a priori diffusa per la variabilità

- Considera diverse a priori per varianza, precisione, deviazione standard
- $IG(\alpha, \beta)$ per la varianza, equivalentemente $Gamma(\alpha, \beta)$ per la precisione, con $\alpha = \beta \approx 0$ (eg., .001), usate perchè si sosteneva che garantissero un'adeguata diffusione a priori
- La ricerca metodologica ha contestato ciò
- Si suggeriscono altre distribuzioni come esponenziale, half-normal, half-Cauchy, uniforme e altre

Standard distributions

- Binomiale: conteggio di risultati indipendenti ("scambiabili")
- Normale: $X = \sum_{i=1}^{N} Z_i$, N grande, Z_i indipendenti ("scambiabili")
- Poisson: conteggio di occorrenze
- Esponenziale: tempi di attesa per eventi che accadono indipendentemente - dato un tasso costante di occorrenza - ("scambiabilmente") in un dato intervallo di tempo.

Per tutti questi casi, la costante di normalizzazione $p(y)$ è calcolabile e quindi l'a posteriori $p(\theta|y)$ ha forma chiusa.

Poisson Model

Count data: Poisson likelihhod

Assume that y_i is a count (e.g., a disease incidence) modelled as a Poisson variate and that $y = (y_1, \ldots, y_n)$ are observed and

$$y_1, \ldots, y_n | \theta \sim \mathrm{Poisson}(y_i | \theta) \text{ iid}$$

then, if we let $t(y) = \sum_{i=1}^{n} y_i$, the likelihood is

$$\begin{aligned} p(y|\theta) &= \prod_{i=1}^{n} \frac{\theta^{y_i} e^{-\theta}}{y_i!} \\ &\propto \theta^{t(y)} e^{-n\theta} \end{aligned}$$

where $t(y)$ is the sufficient statistics.

The likelihood

$$p(y|\theta) \propto (e^{-\theta})^n e^{t(y) \log \theta}$$

belongs to an exponential family with natural parameter $\phi(\theta) = \log \theta$.

Conjugate prior to Poisson likelihood

The conjugate prior has to be

$$p(\theta) \propto \theta^{\alpha-1} e^{-\beta\theta} \propto \mathrm{Gamma}(\alpha, \beta)$$

which is equivalent to a total count of $\alpha - 1$ in β prior observations.

The posterior is then

$$p(\theta|y) = \mathrm{Gamma}(\alpha + \sum y_i, \beta + n)$$

The posterior mean being

$$E(\theta|y) = \frac{\alpha + n\bar{y}}{\beta + n}$$

Distribution of y is Negative Binomial

Prior predictive distribution of a Poisson model has a negative binomial distribution.

Note that $p(y) = \frac{p(y|\theta)p(\theta)}{p(\theta|y)}$, then, for a single poisson observation y,

$$p(y) = \frac{\text{Poisson}(y|\theta)\text{Gamma}(\theta|\alpha, \beta)}{\text{Gamma}(\theta|\alpha + y, \beta + 1)}$$

which reduces to

$$p(y) = \binom{\alpha + y - 1}{y} \left(\frac{\beta}{\beta + 1}\right)^{\alpha} \left(\frac{1}{\beta + 1}\right)^{y}$$

known as negative binomial

$$y \sim \text{Neg-Bin}(\alpha, \beta)$$

The negative binomial distribution describes the number of failures that occur until a predefined number of successes α occurs in a Bernoulli process with parameter $\beta/(\beta + 1)$.

Negative Binomial is a mixture of Poisson

The above derivation shows how the negative binomial distribution is a mixture of Poisson distributions with θ rates following a Gamma distribution.

$$\text{Neg-Bin}(\alpha, \beta) = \int \text{Poisson}(y|\theta)\text{Gamma}(\theta|\alpha, \beta) d\theta$$

Poisson model with exposure

Assume we know for each observation y_i the value of an explanatory variable x_i (> 0), e.g.

- y_i is the incidence of a desease in unit i
- x_i is the exposure in unit i
- θ is the incidence *rate*

$$y_i|\theta \sim \mathrm{Poisson}(\theta x_i)$$

$$\theta \sim \mathrm{Gamma}(\alpha, \beta)$$

$$p(y|\theta) \propto \theta^{\sum_i y_i} e^{-\theta \sum_i x_i}$$

$$p(\theta|y) \sim \mathrm{Gamma}(\alpha + \sum_i y_i, \beta + \sum_i x_i)$$

Note: model non exchangeable in y_i's but exchangeable in couples $(x, y)_i$.

Example: estimate of incidence rate from count data

- In a given year, in a US city of 200,000 inhabitants, we observed $y = 3$ deaths for asthma
- Let θ be the underlying long-term asthma mortality rate (measured in cases per 100,000 persons per year)
- The exposure is then $x = 2$
- Then, $y|\theta \sim Pois(2\theta)$
- (prior elicitation) If we assume $\theta \sim \text{Gamma}(3, 5)$ (assuming that the city and year are exchangeable with other years and cities from which we get information about θ)
- Then, $\theta|y \sim \text{Gamma}(6, 7)$
- *Do you notice a* shrinkage *toward the prior compared to the "raw" mortality rate?*
- Imagine that we have more data: $y = 30$ in the following 10 years (in the same city, with a constant population of 200,000). How does the posterior change?

Prior elicitation

We want to choose a prior within the conjugate family, so

$$p(\theta) = \text{Gamma}(\alpha, \beta)$$

In order to assess appropriate hyperparameters α and β we use the fact that according to epidemiological literature

- (1) rates above 1.5 per $100\,000$ are rare
- (2) typical mortality rate is around 0.6 per $100\,000$

Fact (2) suggests

$$E(\theta) = \frac{\alpha}{\beta} = 0.6$$

Fact (1) suggests that we should keep $P(\theta < 1.5)$, for example

$$\alpha = 3; \quad \beta = 5$$

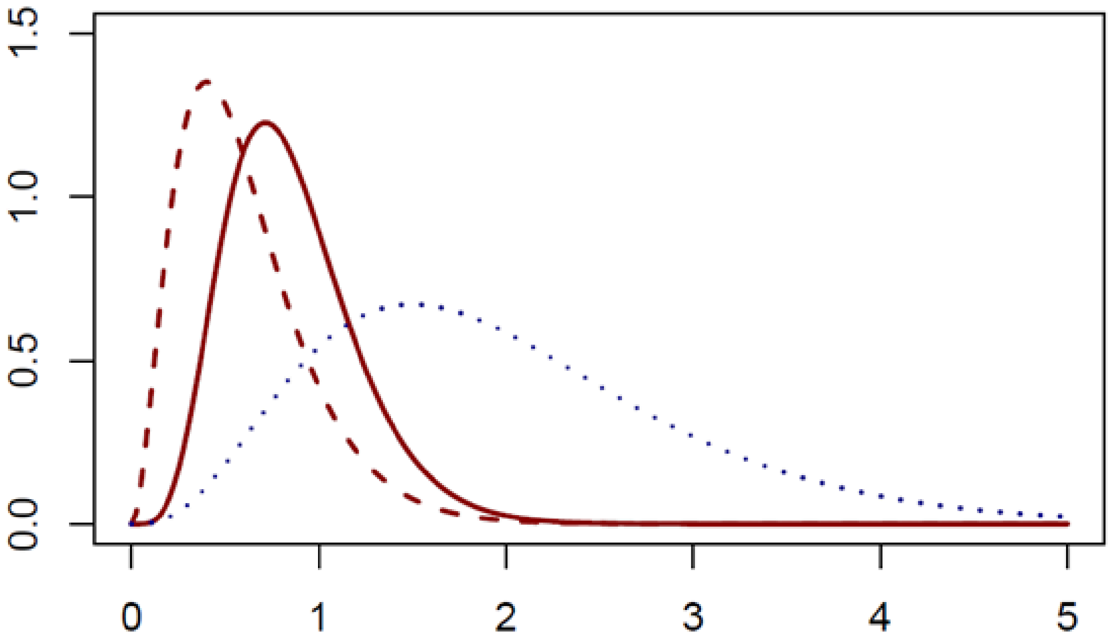
leads to $P(\theta < 1.44) = 0.975$.

Shrinkage to the prior

Starting from a prior $p(\theta) = \text{Gamma}(3, 5)$ the observation $y = 3$ with exposure $x = 2$ leads to the posterior

$$\theta | y \sim \text{Gamma}(3 + 3, 5 + 2)$$

whose mean is 0.86



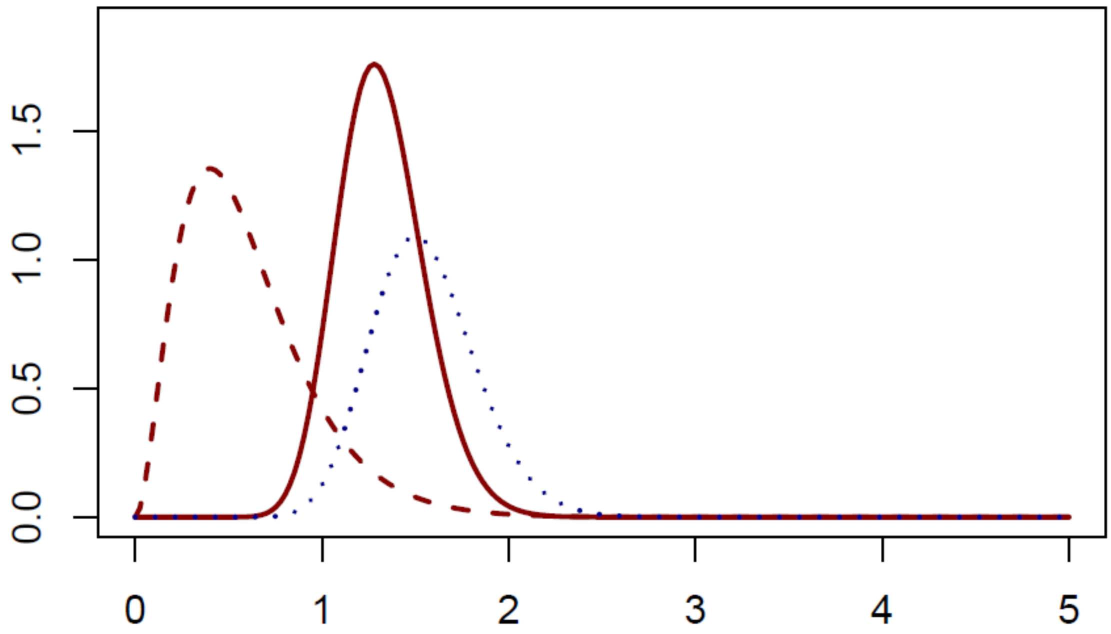
The posterior mean is shrunken towards the prior mean 0.6 away from the observed mortality rate 1.5.

More (exchangeable) observations

With a constant population of 200,000 and assuming that the ten-year results are independent with a constant long-run rate θ, the posterior distribution of θ is then

$$\theta | y \sim \text{Gamma}(33, 25)$$

whose mean is 1.32.



Exponential Model

Exponential Model

- y continuous variable, $y \geq 0$, real-valued, e.g. a waiting time
- property: *memoryless*

$$P(y > t + s | y > s,\, \theta) = P(t > t | \theta)\ \forall\, t,\, s$$

- Sampling model $\mathrm{Exp}(y|\theta)$

$$p(y|\theta) = \theta \exp\left(-\theta y\right) \text{ for } y \geq 0,\ \theta > 0$$

$E(y) = 1/\theta, \theta = 1/E(y|\theta)$ is called the *rate*

- $\mathrm{Exp}(\theta) = \mathrm{Gamma}(\alpha = 1, \beta = \theta)$
- If n observations, the likelihood is $p(y|\theta) = \theta^n \exp\left(-\theta \sum_i y_i\right)$
- Prior $p(\theta) = \mathrm{Gamma}(\alpha, \beta) \propto \theta^{\alpha - 1} e^{-\beta\theta}$
- From the likelihood prior parameters can be interpreted as $\alpha - 1$ exponential observations with total waiting time β
- Posterior $p(\theta|y) = \mathrm{Gamma}(\alpha + 1, \beta + y)$

Prior distribution

Prior distribution

The prior distribution is a novelty in Bayesian statistics with respect to classical statistics.

It is

- an opportunity, since we **can** formally include information other than observations in inference
- a problem, since we **must** include in inference informations which do not come from the experiment (observations).

From what we already discussed we know that

- Attitude toward subjective priors are the most various, from essential to unacceptable.
- (Reasonably specified) prior information vanishes as the sample size tends to infinity. This helps but is not a panacea, we have finite samples, so in practice our inference will be affected by the prior.

Sensitivity of results to prior's choice

The following example, due to Berger, shows that the same experimental result may lead to different conclusions depending on the prior distribution.

$$Y_1, \ldots, Y_n \ \ \mathrm{iid}(\mathcal{N}(\theta, 1)) \text{ hence } L(\theta) \propto \exp(-\frac{n}{2}(\bar{y} - \theta)^2)$$

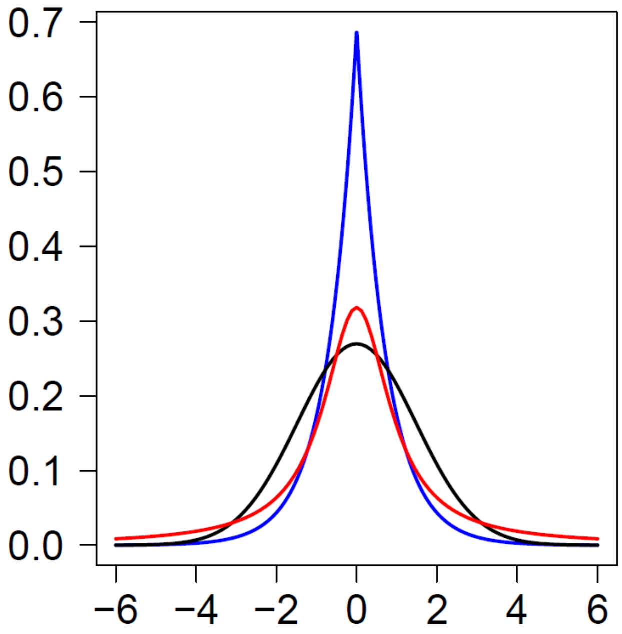
Let us fix the quartiles of the prior distribution:

$$IQ = -1 \ \ ; \quad Me = 0 \ \ ; \quad IIIQ = 1$$

There are infinite probability distribution coherent with the above values, let us consider

- Gaussian: $\theta \sim \mathcal{N}(0, 2.19)$
- Laplace: $\theta \sim La(1.384)$
- Cauchy: $\theta \sim Ca(0, 1)$

Sensitivity of results to prior's choice (cont.)



Laplace
$p(\theta) = \frac{\lambda}{2}\exp(-\lambda|\theta|)$

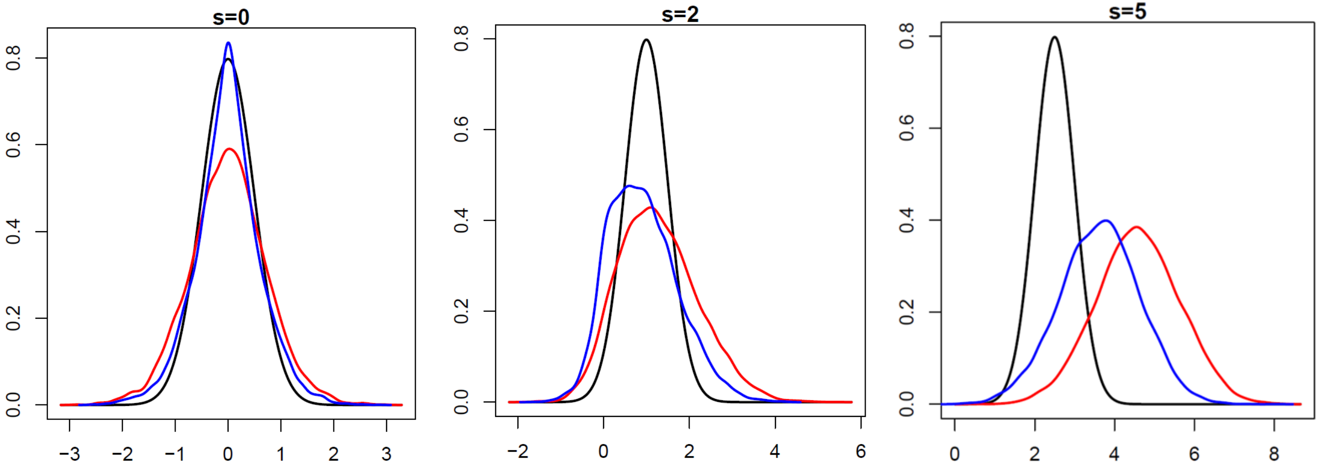
Cauchy
$p(\theta) = \frac{1}{\pi(1+\theta^2)}$

Gaussian
$p(\theta) = \frac{1}{\sqrt{2\pi}\tau}\exp(-\frac{1}{2\tau^2}\theta^2)$

Sensitivity of results to prior's choice (cont.)

Assume $n = 1$, consider three different samples

$y = 0$ $\qquad$ $y = 2$ $\qquad$ $y = 5$



Caution: relatively similar prior could lead to different posterior.

How do we choose a prior?

Any probability distribution (and not only) can be a prior for

$$\theta \in \Theta$$

A reasonable requirement is that $\mathrm{supp}(p(\theta)) = \Theta$
(note that the support of the posterior distribution is, whatever the likelihood, a subset of the support of the prior distribution).

Typical choices are

- conjugate distributions;
- noninformative (reference) priors
  - uniform prior
  - Jeffreys prior
  - improper prior
- weakly informative distributions

Conjugate priors: pros and cons

A family of distributions $f(\theta; \nu)$ is a natural conjugate for the likelihood $L(\theta)$ if, assuming $p(\theta) = f(\theta; \nu)$ the posterior distribution is in the same family, that is $p(\theta|y) = f(\theta; \nu_1)$ for some ν_1.

$+$ the main advantage is that solutions are available in closed form and are easily obtained;

$-$ restricting the choice to the conjugate family may be too restrictive;

$\times$ conjugate families are less relevant today due to the use of MCMC and similar method to explore posterior distribution (closed forms are not needed anymore).

Conjugate priors examples

- Beta + Binomial;
- Gaussian + Gaussian (for the mean, variance known);
- Gamma + Poisson.

Noninformative (reference) priors

Abandon the idea that the prior distribution is meant to reflect the opinion of the researcher prior to observing any data.

Rather, we want to model the absence of any opinion (whether this is realistic is disputable).

This is a relevant issue also as a possible answer to the objection which are put forward by those who do not like the results of inference to depend on subjective opinions: the rationale is to **let the data speak for themselves**.

These kind of priors have been called *noninformative* or reference priors and are sometimes associated to adjectives such as *vague*, *flat* or *diffuse*.

> Reference analysis produces objective Bayesian inference, in the sense that inferential statements depend only on the assumed model and the available data, and the prior distribution used to make an inference is least informative in a certain information-theoretic sense. (Berger et al, 2009)

The problem is that it is not so obvious what "noninformative" means.

Non informative priors: uniform

An intuitive solution is to assume

$$p(\theta) \propto k$$

so that no values of θ are privileged (*principle of insufficient reason*).

There are two difficulties

- What if the parameter space is not limited?
    - If the parameter space is not limited a constant has an infinite integral and so is not a probability distribution.
- Is it really non informative?

Improper priors

If we apply (the algebra of) Bayes theorem

$$p(\theta|y) \propto p(y|\theta)p(\theta)$$

with a function $p(\theta)$ which is not a valid probability distribution, then

- $p(\theta|y)$ is not necessarily a valid distribution (and if it is not then it is not useful)
- if $p(\theta|y)$ is a valid distribution then it is reasonable to interpret it as a posterior distribution

  See examples of uniform prior for the mean of a normal and of improper prior for the variance of a normal

In practice: the uniform prior may work even if the parameter space is not limited (on a case by case basis).

($\nu_0 = 0$) An improper prior for the variance

Esempio con una Normale di media nota e varianza non nota

$$\begin{aligned} y|\sigma^2 &\sim N(\theta, \sigma^2) \\ \sigma^2 &\sim IG(\alpha, \beta) \end{aligned}$$

Se $\alpha = \beta = 0$, ovvero $\nu_0 = 0$ per la chi-quadro inversa scalata equivalente, allora

$$\begin{aligned} p(\sigma^2) &\propto \frac{1}{\sigma^2} \\ \Rightarrow \sigma^2|y &\approx \text{Inv}{-}\chi^2(n, v = \sum_i (y_i - \mu)^2/n) \end{aligned}$$

si noti che $p(\sigma^2) \propto \frac{1}{\sigma^2}$ è impropria avendo integrale infinito nel range $(0, \infty)$

A priori improprie possono portare ad a posteriori proprie, da valutare caso per caso.

improper priors

"Informativeness" of the uniform distribution

The non informative nature of the uniform distribution in general is disputable.

Let

$$p(\theta) \propto k$$

consider the reparametrization $\psi = \psi(\theta)$, then

$$p(\psi) = p(\theta(\psi)) \left| \frac{d\theta}{d\psi} \right|$$

which is not uniform in general.

That is, assuming that uniform means non informative, by specifying a uniform distribution for the parameter θ, we are specifying an informative prior on its transform $\psi = \psi(\theta)$.

Principio di invarianza di Jeffreys

Il problema appena evidenziato può essere risolto dalle distribuzioni a priori (noninformative) introdotte da Jeffreys il cui principio base è quello di invarianza rispetto a trasformazioni uno-a-uno del parametro: $\phi = h(\theta)$.

Il principio generale di Jeffreys è che qualsiasi regola per determinare la densità a priori $p(\theta)$ dovrebbe comportare un risultato *equivalente* se applicato al parametro trasformato ϕ; cioè, $p(\phi)$ calcolato determinando prima $p(\theta)$ e applicando poi il **teorema del cambio di variabile**, i.e.,

$$p_\phi(\phi) = p_\theta(h^{-1}(\phi))\,|J| = p_\theta(\theta)\left|\frac{d\theta}{d\phi}\right|$$

dovrebbe corrispondere alla distribuzione $p(\phi)$ ottenuta utilizzando la "regola" direttamente sul modello trasformato, $p(y, \phi) = p(\phi)p(y|\phi)$.

> equivalent, in terms of expressing the same beliefs

N.B. Nel caso multivariato $|J|$ è il determinante dello Jacobiano J della trasformazione inversa $\theta = h^{-1}(\phi)$, dove J è la matrice (quadrata) delle derivate parziali con elemento (i, j) uguale a $\partial\theta_i/\partial\phi_j$.

Jeffreys' prior

Jeffreys' principle leads to defining the noninformative prior density as

$$p(\theta) \propto \sqrt{\mathcal{I}(\theta)}$$

where $\mathcal{I}$ is the Fisher information, that is

$$[\mathcal{I}(\theta)] = E\left(\left(\frac{\partial \log\, p(y|\theta)}{\partial\theta}\right)^2 |\theta\right) = -E\left(\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2}|\theta\right)$$

with this, for any parametrization $\phi = h(\theta)$, it is easy to derive that, evaluating $I(\phi)$ in $\theta = h^{-1}(\phi)$,

$$\mathcal{I}(\phi) = \mathcal{I}(\theta)\left|\frac{d\theta}{d\phi}\right|^2$$

whence $\mathcal{I}(\phi)^{1/2} = \mathcal{I}(\theta)^{1/2}\left|\frac{d\theta}{d\phi}\right|$, that is what we wanted to prove:

$$p(\phi) \propto \sqrt{\mathcal{I}(\phi)} = \sqrt{\mathcal{I}(\theta)}\left|\frac{d\theta}{d\phi}\right|$$

Jeffreys' prior: example

Consider a Binomial experiment, so the log-likelihood is

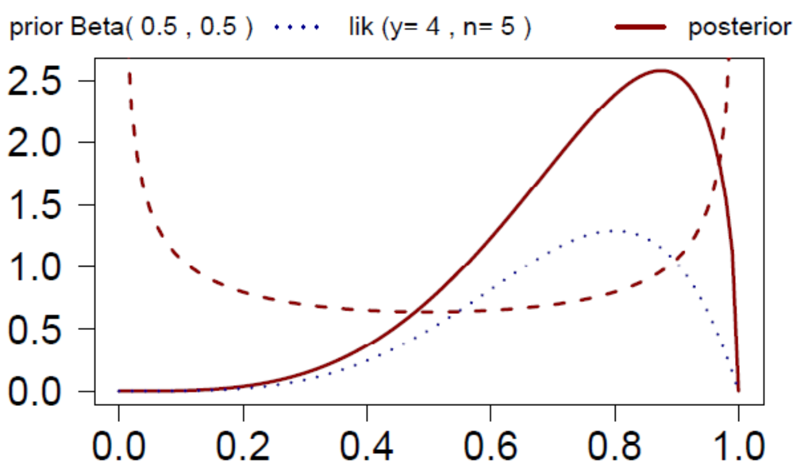
$$\log p(y|\theta) = y \log \theta + (n-y)\log(1-\theta)$$

then

$$[\mathcal{I}(\theta)] = -E\left(\frac{d^2}{d\theta^2}\log p(y|\theta)\right) = \frac{n}{\theta(1-\theta)}$$

the prior is then a $\mathrm{Beta}(1/2, 1/2)$

$$p(\theta) \propto \theta^{-1/2}(1-\theta)^{-1/2}$$



Binomial, reparametrization

Consider the reparametrization

$$\psi = \log\left(\frac{\theta}{1-\theta}\right) \in \mathbb{R}$$

If we assumed a uniform prior on θ then

$$p(\psi) = p(\theta^{-1}(\psi))\left|\frac{d\theta}{d\psi}\right| = \frac{e^{\psi}}{(1+e^{\psi})^2}$$

If the Jeffrey's prior is chosen then it implies

$$p(\psi) = \frac{e^{\psi/2}}{1+e^{\psi}}$$

which is also equal to

$$\sqrt{\mathcal{I}(\psi)}$$

Binomial, reparametrization (cont.)

In fact

$$\begin{aligned}
\sqrt{\mathcal{I}(\psi)} &= \sqrt{-E\left(\frac{d^2 \log p(y|\psi)}{d\psi^2}\right)} \\
&= \sqrt{-E\left(\frac{d^2}{d\psi^2}\left(y\psi - \log(1+e^{\psi})\right)\right)} \\
&= \sqrt{E\left(\frac{e^{\psi}}{(1+e^{\psi})^2}\right)} \\
&= \sqrt{\frac{e^{\psi}}{(1+e^{\psi})^2}} \\
&= \frac{e^{\psi/2}}{1+e^{\psi}}
\end{aligned}$$

Riassunto: Varie a priori non-informative per il parametro della Binomiale

- $y|\theta \sim Bin(y|\theta, n)$
- $I(\theta) = \frac{n}{\theta(1-\theta)}$ da cui la a priori di Jeffrey

$$p(\theta) \propto \theta^{-1/2}(1-\theta)^{-1/2} \propto Beta(1/2, 1/2)$$

- A priori Bayes-Laplace è l'uniforme

$$p(\theta) \sim Beta(1, 1)$$

- A priori uniforme per il $logit(\theta)$

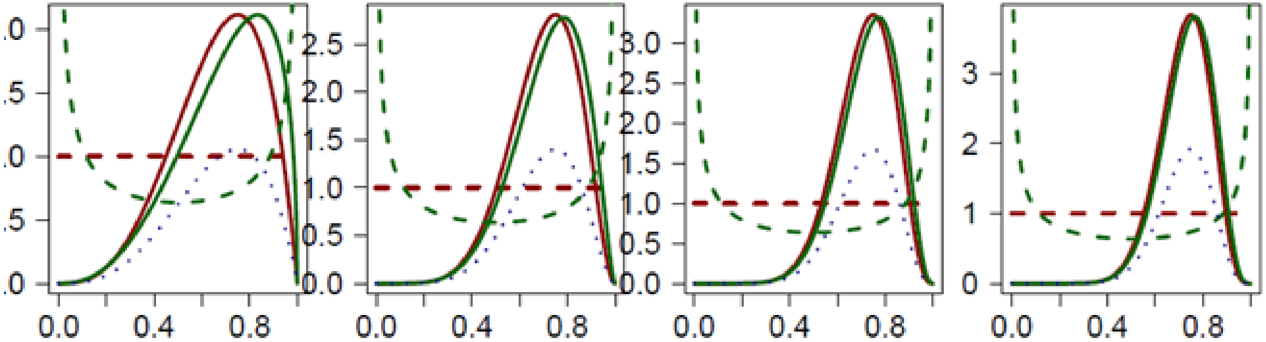
$$p(logit(\theta)) \propto \text{constant}$$

corrisponde ad una $Beta(0, 0)$ per θ, ed è impropria

ATTENZIONE: se l'a priori è $Beta(0, 0)$ e $y = 0$ o $y = n$ allora l'a posteriori è impropria!

Sensitivity to the prior choice: Jeffrey's v. uniform

Samples imply $\hat{\theta} = 0.75, n = 4, 8, 12, 16.$



Sensitivity to the prior choice

Consider a Beta-Binomial model where 4 successes are observed on $n = 10$ trials, so that the ML estimate is 0.4, consider as a prior a $Beta(\alpha, \beta)$ with $\alpha = \beta$ (so $E(\theta) = 0.5$), compare below the effect of different choices on the posterior means and variances

	$\alpha + \beta$	$V(\theta)$	$E(\theta \vert y)$	$V(\theta \vert y)$
Jeffrey	1	0.1250	0.409	0.0201
Uniform	2	0.0833	0.417	0.0187
	5	0.0417	0.433	0.0153
	10	0.0227	0.450	0.0118
	20	0.0119	0.467	0.0080
	50	0.0049	0.483	0.0041
	100	0.0025	0.491	0.0023

Note su a priori non-informative

- Se la verosimiglianza è veramente dominante, allora la scelta nella gamma di a priori relativamente piatte è sostanzialmente indifferente
- Una densità piatta per una data parametrizzazione può non essere piatta in un'altra, e.g., $p(\log \sigma^2) = 1$ ma $p(\sigma^2) = 1/\sigma^2$
- Vantaggio: non sembra valga la pena esprimere la reale conoscenza a priori come una distr di prob propria se si controlla che la distr a post sia propria e si effettui l'analisi di sensitività

Weakly informative prior

The rationale is that we usually do not really need to start from complete ignorance (which is what reference priors try to describe).

On the contrary there usually is some information

- for the probability of a female birth we are pretty sure it is not 0.1 or 0.9,

The idea is than to use a prior conveying less information than what we actually have

- for the probability of a female birth we may use $p(\theta) \sim N(0.5, 0.1^2)$, or $p(\theta) \sim \mathrm{Beta}(20, 20)$
- for the inference on the mean $p(\theta) \sim N(0, A^2)$ with A large (where what large means depends on the problem)

Prior distribution, in brief

There are some situations in which it is sensible to put relevant information into the prior distribution (especially with few data).

In general, even if we had information, it may be deemed inconvenient to include it wholly in the model (prior), possible reasons include

- difficulties to elicit the prior
- mathematical simplicity

In this case we have a number of options

- uniform / improper priors
- non informative priors (Jeffrey's priors)
- weakly informative priors (possibly conjugate)

These are all valid options none of which is clearly superior, in fact, if we have enough data to rely exclusively on them, then the choice among relatively flat priors should not matter.

On the contrary it is advisable to avoid automatic use of a particular specification and do some sensitivity analysis.

Appendix

Theorem: Mixture of Normals

Theorem

If $Y|\theta \sim \mathrm{N}(\theta,\, \sigma^2)$ and $\theta \sim \mathrm{N}(\mu,\, \tau^2)$ then

$$Y \sim \mathrm{N}(\mu,\, \sigma^2 + \tau^2).$$

This is easily seen, let

$$Z = Y - \theta;\ \text{then} \quad Z \sim \mathrm{N}(0,\, \sigma^2)\ \ \forall\theta$$

then $Y = Z + \theta$

- Y is a sum of normal r.v. so it is normal,
- $E(Y) = E(Z) + E(\theta) = 0 + \mu = \mu$
- $V(Y) = V(Z) + V(\theta) + 2\mathrm{Cov}(Z, \theta) = \sigma^2 + \tau^2 + 2\mathrm{Cov}(Z, \theta) = \sigma^2 + \tau^2$ since $\mathrm{Cov}(Z, \theta) = E(Z\theta) = E(E(Z\theta|\theta)) = E(E((Y - \theta)\theta|\theta)) = 0$

Back to normal predictive