The structure and evolution of stars

Lecture 7:
The structure of main-sequence stars: homologous stellar models

The structure and evolution of stars

Lecture 7: The structure of main-sequence stars: homologous stellar models

Hans Bethe

Learning Outcomes

Student will learn:

- How to employ approximate forms for the three equations that supplement the stellar structure equations i.e. opacity, equation of state and energy generation
- How to derive a sequence of homologous stellar models
- Why these homologous sequences are useful
- How the approximate homologous sequence compares to observations of stars

Introduction and recap

- We have 4 differential equations of stellar structure
- Completely accurate expressions for pressure, opacity and energy generation are extremely complicated, but we can find simple approximate forms
- Eqns of stellar structure too complicated to find exact analytical solution, hence must be solved with computer
- But we **can verify position of main-sequence and find mass-luminosity relation without solving eqns completely**.

- We will attempt to simply derive relationships between luminosity, temperature and mass for a population of stellar models. This will allow comparison with observations.

Equation of state of an ideal gas

We have seen that stellar gas is ionized plasma, and although density is so high that typical inter-particle spacing is of the order of an atomic radius, the effective particle size is more like a nuclear radius (10^5) times smaller. Hence material behaves like an **ideal gas**.

$$P_{gas} = nkT$$

Where n is number of particles per cubic meter, k is Boltzmann's constant

But we want this equation in the form:

$$P = P(\rho, T, composition)$$

Following the class derivation, this can be written:

$$P = \frac{\Re \rho T}{\mu}$$

R=k/m_H =the gas constant

mu = mean molecular weight = mean mass of particles in terms of H-atom (m_H)

If radiation pressure is important

$$P = \frac{\Re \rho T}{\mu} + \frac{aT^4}{3}$$

Mean molecular weight

We can derive an expression for the mean molecular weight μ. An exact solution is complex, depending on fractional ionisation of all the elements in all parts of the star. We will assume that all of the material in the star is **fully ionised**.

Justified as H and He are most abundant, and they are certainly fully ionised in stellar interiors (assumption will break down near stellar surface).

X=fraction of material by mass of H

Y=fraction of material by mass of He

Z=fraction of material by mass of all heavier elements

$X + Y + Z = 1$

Hence in $1 m^3$ of stellar gas of density ρ there is mass $X\rho$ of H, $Y\rho$ of He, $Z\rho$ of heavier elements. In a fully ionised gas,

H gives 2 particles per m_H

He gives 3/4 particles per m_H (alpha particle, plus two e^-)

Heavier elements give ~1/2 particles per m_H (^{12}C has nucleus plus $6e^- = 7/12$)

(^{12}O has nucleus plus $8e^- = 9/16$)

The total number of particles per cubic metre is then given by the sum:

$$n = \frac{2X\rho}{m_H} + \frac{3Y\rho}{4m_H} + \frac{Z\rho}{2m_H}$$

$$n = \frac{\rho}{4m_H}\left(8X + 3Y + 2Z\right) = \frac{\rho}{4m_H}\left(6X + Y + 2\right)$$

Now as before we define $\rho = n\, m_H\, \mu$

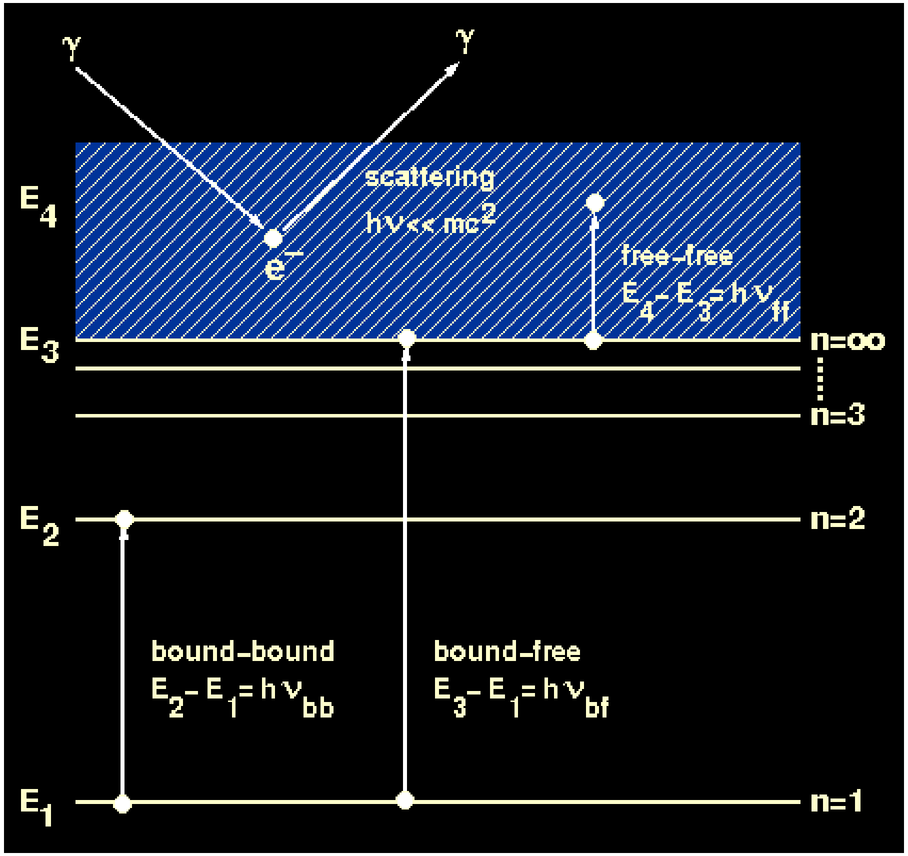
$$\mu = \frac{4}{6X + Y + 2}$$

Which is a good approximation to μ except in the cool outer regions of stars. For solar composition, $X_\odot$=0.747, $Y_\odot$=0.236, $Z_\odot$=0.017, resulting in μ~0.6,

i.e. the mean mass of particles in a star of solar composition is a little over half the mass of the proton

Opacity

Concept of opacity introduced when deriving the equation of radiation transport, and will be discussed extensively in the lectures of **Radiative Processes**. Opacity is the resistance of a material to the flow of radiation through it. In most stellar interiors, it is determined by all the processes which scatter and absorb photons



Four processes:

- Bound-bound absorption
- Bound-free absorption
- Free-free absorption
- scattering

Approximate form for opacity

We need an expression for opacity to solve the eqns of stellar structure. For stars in thermodynamic equilibrium with only a slow outward flow of energy, the opacity should have the form

$$\kappa = \kappa(\rho, T, \text{chemical composition})$$

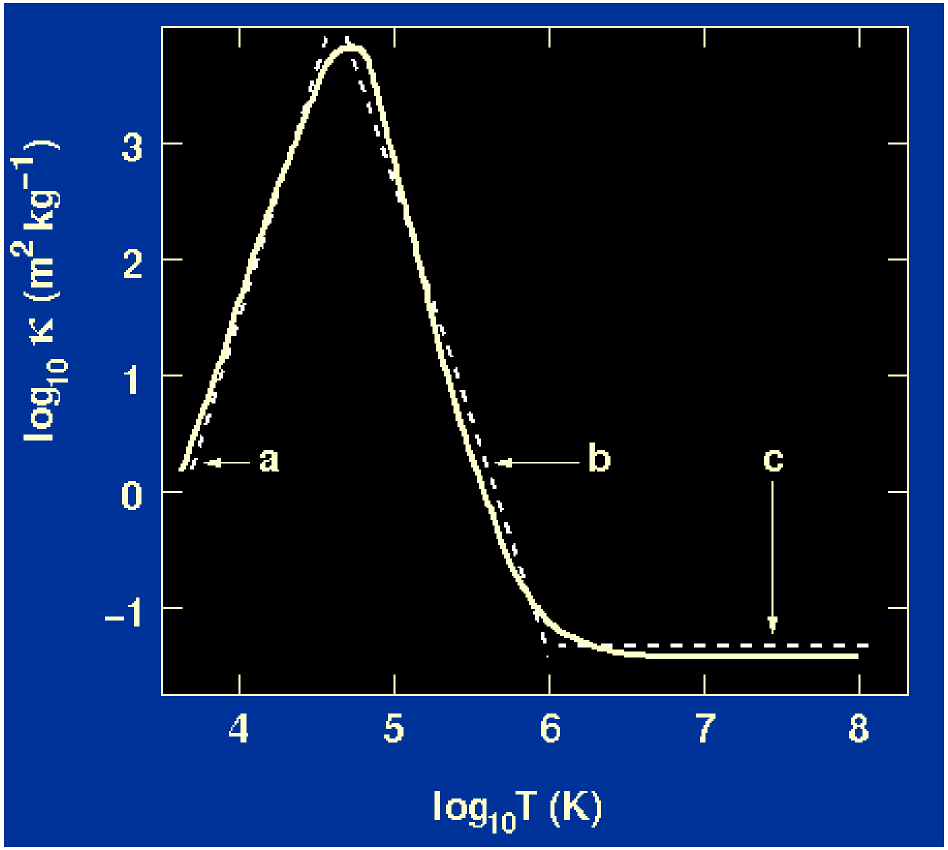
Opacity coefficients may be calculated, taking into account all possible interactions **between the elements and photons of different frequencies**.

This requires an enormous amount of calculation and is beyond the scope of this course. When it has been done, the results are usually approximated by the relatively simple formula :

$$\kappa = \kappa_0 \rho^{\alpha} T^{\beta}$$

Where alpha, beta are slowly varying functions of density and temperature and k_0 is a constant for a given chemical composition

Figure shows opacity as a function of temperature for a star of given ρ (10^{-1} kgm^{-3}). Solid curve is from detailed opacity calculations. Dotted lines are approximate power-law forms.



At high T: k is low and remains constant. Most atoms fully ionised, high photon energy, hence free-free absorption unlikely, Dominant mechanism is electron scattering, independent of T, alpha=beta=0

$$\kappa = \kappa_0 \text{ (curve c)}$$

Opacity is low a low T, and decreases with T. Most atoms not ionised, few electrons available to scatter photons or for free-free absorption. Approx analytical form is alpha=1/2 , beta=4

$$\kappa = \kappa_0 \rho^{1/2} T^4 \text{ (curve a)}$$

At intermediate T, k peaks, when bound-free and free-free absorption are very important, then decreases with T (Kramers opacity law, see Böhm-Vitense Ch. 4)

$$\kappa = \kappa_0 \rho T^{-3.5} \text{ (curve b)}$$

Homologous stellar models

We already have the four eqns of stellar structure in terms of mass (m)

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \qquad \frac{dL}{dM} = \varepsilon$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \qquad \frac{dT}{dM} = \frac{3\kappa_R L}{64\pi^2 r^4 \sigma T^3}$$

With boundary conditions:

R=0, L=0 at M=0

ρ=0, T=0 at M=M_s

And supplemented with the three additional relations for P, k, ε (assuming that the stellar material behaves as an ideal gas with negligible radiation pressure, and laws of opacity and energy generation can be approximated by power laws)

$$P = \frac{\Re \rho T}{\mu}$$

$$\kappa = \kappa_0 \rho^\alpha T^\beta$$

$$\varepsilon = \varepsilon_0 \rho T^\eta$$

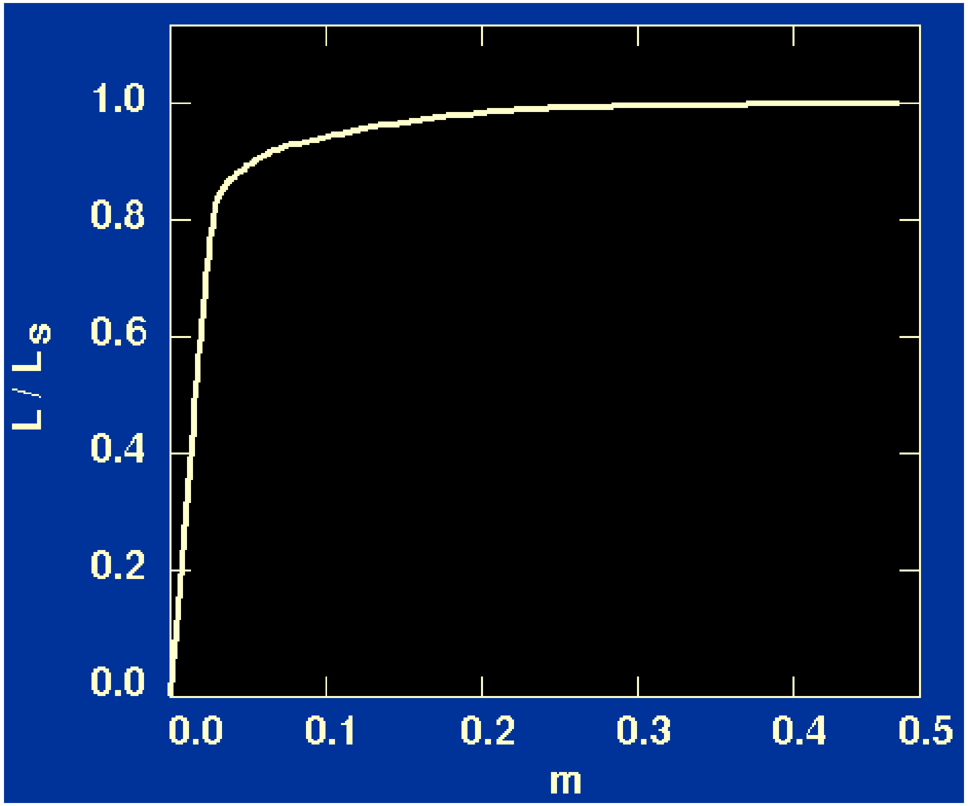
Where alpha, beta, eta are constants and k_0 and ε_0 are constants for a given chemical composition.

Homologous models

We aim to formulate the eqns of stellar structure so that they are independent of mass M_S. Hence, we will **assume** that the way in which a physical quantity (e.g. L) varies from centre of star to surface is the same for all stars of all masses (only absolute L varies).

Schematic illustration: ratio of luminosity to surface luminosity is plotted against fractional mass (m) , which is defined as the ratio of mass to total mass

$m=M/M_s$



We then assume this curve is the same for ALL stars with the same laws of opacity and energy generation. But that L_S is proportional to some power of M_S, which depend on the values of alpha, beta, eta

The same will also be true for r_s and T_e (effective temperature)

$$
\begin{aligned}
r &= M^{a_1}\, r_{\mathrm{s}}(x) & dr &= M^{a_1}\, dr_{\mathrm{s}}(x) \\
\rho(r) &= M^{a_2}\, \rho_{\mathrm{s}}(x) & d\rho(r) &= M^{a_2}\, d\rho_{\mathrm{s}}(x) \\
T(r) &= M^{a_3}\, T_{\mathrm{s}}(x) & dT(r) &= M^{a_3}\, dT_{\mathrm{s}}(x) \\
P(r) &= M^{a_4}\, P_{\mathrm{s}}(x) & dP(r) &= M^{a_4}\, dP_{\mathrm{s}}(x) \\
L(r) &= M^{a_5}\, L_{\mathrm{s}}(x) & dL(r) &= M^{a_5}\, dL_{\mathrm{s}}(x)
\end{aligned}
$$

where the a_i exponents are constants to be determined and the variables $r_{\mathrm{s}}(x)$, $\rho_{\mathrm{s}}(x)$ etc. depend only on the fractional mass x. Also $M(r) = Mx$, $dM(r) = M\,dx$,

$$\kappa(r) = \kappa_0\, \rho^{\alpha}(r)\, T^{\beta}(r) = \kappa_0\, {\rho_{\mathrm{s}}}^{\alpha}(x)\, {T_{\mathrm{s}}}^{\beta}(x)\, M^{\alpha a_2}\, M^{\beta a_3}$$

and

$$\epsilon(r) = \epsilon_0\, \rho(r)\, T^{\eta}(r) = \epsilon_0\, \rho_{\mathrm{s}}(x)\, {T_{\mathrm{s}}}^{\eta}(x)\, M^{a_2}\, M^{\eta a_3}.$$

Substitution into the stellar structure equations now allows these to be expressed in terms of the dimensionless mass x:

- Mass Conservation:

$$\frac{dr}{dM(r)} = \frac{1}{4\,\pi\,r^2\,\rho(r)} \quad \text{becomes}$$

$$M^{(a_1-1)}\frac{dr_s(x)}{dx} = \frac{1}{4\,\pi\,{r_s}^2(x)\,\rho_s(x)}M^{-(2a_1+a_2)}.$$

  A requirement of the homology condition is that scaling is independent of actual mass; the M exponents on either side of the above equation must then be equal giving

$$3a_1 + a_2 = 1.$$

- Hydrostatic Equilibrium:

$$\frac{dP(r)}{dM(r)} = -\frac{G\,M(r)}{4\,\pi\,r^4} \quad \text{becomes}$$

$$M^{(a_4-1)}\frac{dP_s(x)}{dx} = -\frac{G\,x}{4\,\pi\,{r_s}^4}M^{(1-4a_1)} \quad \text{and by the homology condition}$$

$$4a_1 + a_4 = 2.$$

- Energy Production:

$$\frac{dL(r)}{dM(r)} = \epsilon(r)$$
$$= \epsilon_0\, \rho(r)\, T^{\eta}(r) \quad \text{becomes}$$
$$M^{a_5-1}\frac{dL_s(x)}{dx} = \epsilon_0 \rho_s(x) T^{\eta}(x) M^{a_2+\eta\, a_3} \quad \text{and by the homology condition}$$
$$a_2 + \eta\, a_3 + 1 = a_5.$$

- Radiative Transport:

$$\frac{dT(r)}{dM(r)} = -\frac{3\,\bar{\kappa}_{\rm Ross}\, L(r)}{16\,\pi^2\, r^4\, a\, c\, T(r)^3} \quad \text{becomes}$$
$$M^{(a_3-1)}\frac{dT_s(x)}{dx} = -\frac{3(\kappa_0\, \rho_s(x)^{\alpha} T_s(x)^{\beta}) L_s(x)}{16\,\pi^2\, r_s(x)^4\, a\, c\, T_s(x)^3} M^{(a_5+(\beta-3)a_3+\alpha a_2-4a_1)} \quad \text{and}$$
$$4a_1 + (4-\beta)a_3 = \alpha a_2 + a_5 + 1 \quad \text{by the homology condition.}$$

- Equation of State:
  If gas pressure dominates and we neglect any radial dependence of the mean molecular weight ($\bar{\mu}$):

$$P_{\rm gas}(r) = \frac{\rho(r)\, k\, T(r)}{\bar{\mu}\, m_{\rm H}} \quad \text{becomes}$$

$$M^{a_4}\, P_{\rm s}(x) = \frac{\rho_{\rm s}(x)\, k\, T_{\rm s}(x)}{\bar{\mu}\, m_{\rm H}}\, M^{a_2+a_3} \quad \text{and by the homology condition}$$

$$a_2 + a_3 = a_4.$$

  Alternatively, if radiation pressure dominates:

$$P_{\rm rad} = \frac{1}{3}\, a\, T^4 \quad \text{becomes}$$

$$M^{a_4}\, P_{\rm s}(x) = \frac{1}{3}\, a\, (M^{a_3} T_{\rm s}(x))^4 \quad \text{and by the homology condition}$$

$$a_4 = 4a_3.$$

In matrix form, the five equations (assuming $P_{\rm gas} \gg P_{\rm rad}$) for the exponents a_i are

$$\begin{bmatrix} +3 & +1 & & & \\ +4 & & & +1 & \\ & +1 & +\eta & & -1 \\ +4 & -\alpha & +(4-\beta) & & -1 \\ & +1 & +1 & -1 & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

and which can be solved for given values of α, β and η.

Consider two cases:

- Low-mass stars ($\sim 0.7 \lesssim M \lesssim 2\,M_\odot$) corresponding to spectral types F and later: Adopt Kramers opacity $\kappa \propto \rho T^{-3.5}$ and a nuclear generation rate for PP-Chain $\epsilon \propto T^4$ so that $\alpha = 1$, $\beta = -3.5$ and $\eta = 4$.

- Higher mass stars ($M \gtrsim 2\,M_\odot$) corresponding to spectral types A and earlier: Assume opacity to be dominated by electron scattering and a nuclear generation rate dominated by the CNO cycle with a stronger temperature dependence $\epsilon \propto T^{16}$ so that $\alpha = 0$, $\beta = 0$ and $\eta = 16$.

Regime	α	β	η	a_1	a_2	a_3	a_4	a_5
Low-Mass	1	-3.5	4	$1/13$	$10/13$	$12/13$	$22/13$	$71/13$
Higher-Mass	0	0	16	$15/19$	$-26/19$	$4/19$	$-22/19$	3

The system of stellar structure equations in the scaled mass (x) representation may be solved numerically as previously described, subject to the same boundary conditions. But the beauty of the homology approximation is that useful conclusions may be derived analytically from the a_i obtained from the above matrix equation and summarised in the table:

- From the definition of homologous models:

$$L = M^{a_5}\, L_{\rm s}(1) \quad \text{and} \quad R = M^{a_1}\, r_{\rm s}(1)$$

  it follows immediately that

  - Low-mass stars $\quad L \propto M^{71/13} \sim M^{5.5} \quad R \propto M^{1/13} \sim M^{0.1}$
  - Higher-mass stars $\quad L \propto M^{3} \quad R \propto M^{15/19} \sim M^{0.8}$

which is not too bad when compared with the actual Main Sequence mass-luminosity relationship.

- Secondly, since $L \propto R^2 {T_{\rm eff}}^4$ it also follows that

$$M^{a_5} L_s(1) \propto M^{2a_1} r_s(1)^2 {T_{\rm eff}}^4 \quad \text{or}$$
$$T_{\rm eff} \propto M^{(a_5-2a_1)/4}.$$

  And so

  - In the low-mass case $T_{\rm eff} \sim M^{1.2}$.
  - In the higher-mass case $T_{\rm eff} \sim M^{0.35}$.

- Combining with the mass-luminosity relationship gives

$$L \propto {T_{\rm eff}}^{4a_5/(a_5-2a_1)}$$

  - which for low-mass stars implies $L \sim {T_{\rm eff}}^{4.5}$,
  - and for higher-mass stars $L \sim {T_{\rm eff}}^{8.5}$.

The qualitative result that there **is** a luminosity-temperature relationship is, in effect, a prediction that a main sequence exists in the HR Diagram.

Convection becomes increasingly important at $M \lesssim 0.7_{\odot}$ and radiation pressure becomes more important as the stellar mass increases; in these regimes the above homology approximation begins to breakdown.

Summary and conclusions

Revisit the learning outcomes

- How to employ approximate forms for the three equations that supplement the stellar structure equations i.e. opacity, equation of state and energy generation
- How to derive a sequence of homologous stellar models
- Why these homologous sequences are useful
- How the approximate homologous sequence compares to observations of stars

Next lecture: Another method of simplifying the solution of the stellar structure equations. After that we will move on to discussing the output of full numerical solutions of the equations and realistic predictions of modern theory