

The structure and evolution of stars

Lecture 8: Polytropes and simple models



John Bahcall

Introduction and recap

In previous lecture we saw how a homologous series of models could describe the main-sequence approximately. These models were not full solutions of the equations of stellar structure, but involved simplifications and assumptions

Before we move on to description of the models from full solutions, we will come up with another simplification method that will allow the first two equations of stellar structure to be solved, without considering energy generation and opacity.

This form was historically very important and used widely by Eddington and Chandrasekhar

Learning Outcomes

- What is a polytrope
- Simplifying assumptions to relate pressure and density
- How to derive the Lane-Emden equation
- How to solve the Lane-Emden equation for various polytropes
- How realistic a polytrope is in describing the structure of the Sun

What is a simple stellar model

- We have seen the seven equations required to be solved to determine stellar structure. Highly non-linear, coupled and need to be solved simultaneously with two-point boundary values.
- Simple solutions (i.e. analytic) rely on finding a property that changes moderately from stellar centre to surface such that it can be assumed only weakly dependent on r or m - difficult, as for example T varies by 3 orders of magnitude and P by $>14!$ Chemical composition is a property that can be assumed uniform (e.g. if stars is mixed by convective processes).
- Polytropic models: method of simplifying the equations. Simple relation between pressure and density (for example) is assumed valid throughout the star. Eqns of hydrostatic support and mass conservation can be solved independently of the other 5.
- Before the advent of computing technology, polytropic models played an important role in the development of stellar structure theory.

Recall the equations:

We already have the four eqns of stellar structure in terms of mass (m)

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \quad \frac{dL}{dM} = \varepsilon$$

With boundary conditions:

$$R=0, L=0 \text{ at } M=0$$

$$\rho=0, T=0 \text{ at } M=M_s$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \quad \frac{dT}{dM} = \frac{3\kappa_R L}{64\pi^2 r^4 \sigma T^3}$$

And supplemented with the three additional relations for P, rho, eps (assuming that the stellar material behaves as an ideal gas with negligible radiation pressure, and laws of opacity and energy generation can be approximated by power laws)

$$P = \frac{\mathfrak{R} \rho T}{\mu}$$

Where α, β, η are constants and κ_0 and ε_0 are constants for a given chemical composition.

$$\kappa = \kappa_0 \rho^\alpha T^\beta$$

$$\varepsilon = \varepsilon_0 \rho T^\eta$$

Polytropic models

Take the equation for hydrostatic support (in terms of the radius variable r),

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Multiply by r^2/ρ and differentiating with respect to r , gives

$$\frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -G \frac{dM}{dr}$$

Now substitute the equation of mass-conservation on the right-hand side, and we obtain

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G\rho$$

Adopt an equation of state of the form

$$P(r) = K \rho(r)^\gamma = K \rho(r)^{(n+1)/n} \quad \text{and} \quad \frac{dP(r)}{dr} = K \gamma \rho(r)^{\gamma-1} \frac{d\rho(r)}{dr}$$

where K is a constant and n (not necessarily an integer) is known as the polytropic index.

Substituting for $dP(r)/dr$ in the hydrostatic equilibrium equation combined with the mass conservation equation, and writing ρ for $\rho(r)$ in order to simplify the notation, gives

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2 K}{\rho} \gamma \rho^{\gamma-1} \frac{d\rho}{dr} \right) = -4 \pi G \rho \quad \text{or}$$

$$\frac{1}{\alpha^2} \frac{1}{\xi^2} \frac{d}{d\xi} \left(\frac{\xi^2 K}{\rho} \gamma \rho^{\gamma-1} \frac{d\rho}{d\xi} \right) = -4 \pi G \rho,$$

where the radial variable r has been rescaled by a constant α^{-1} so that $r = \alpha \xi$.

Suppose a radial density dependence

$$\rho = \rho_c \theta(\xi)^n \quad \text{and} \quad \frac{d\rho}{d\xi} = \rho_c n \theta(\xi)^{(n-1)} \frac{d\theta(\xi)}{d\xi},$$

where ρ_c is the central density. Writing θ for $\theta(\xi)$ in order to simplify notation, the above equation then becomes

$$\frac{K (n+1)}{4 \pi G \rho_c^{(1-1/n)} \alpha^2} \frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n.$$

As α is arbitrary, choose

$$\alpha^2 = \frac{K(n+1)}{4\pi G \rho_c^{(1-1/n)}}$$

in which case

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n.$$

The above equation is known as the **Lane-Emden** equation; it defines the rate of change of density within a stellar interior subject to:

- At the centre of the star where $\xi(0) = 0$, $\theta(0) = 1$ so that $\rho = \rho_c$.
- since $dP/dr \rightarrow 0$ as $r \rightarrow 0$, $d\theta/d\xi = 0$ at $\xi = 0$.
- The outer boundary (surface) is the first location where $\rho = 0$ or $\theta(\xi) = 0$; this location is referred to as ξ_1 .

Solutions of the Lane-Emden equation, known as polytropes, specify $\rho(r)$ although expressed as $\theta(\xi)$. The order of the solution is determined by the index n ; in particular it depends only on n and can be scaled by varying P_c (central pressure) and ρ_c to give solutions for stars over a range of total mass and radius. Analytical solutions exist for $n = 0, 1$ and 5 ; all other solutions need to be obtained numerically.

- For $n = 0$, $\rho(r) = \rho_c$; this is the solution for an incompressible sphere.
- To approximate a fully convective star (such as a M, L or T dwarf) use polytropes having $n = 1 \rightarrow 1.5$.
- The Eddington Approximation discussed below corresponds to $n = 3$; it corresponds to a fully radiative star and is a useful approximation for the Sun.

Solving the Lane-Emden equation

It is possible to solve the equation analytically for only three values of the polytropic index n

$$n = 0, \quad \theta = 1 - \left(\frac{\xi^2}{6} \right)$$

$$n = 1, \quad \theta = \frac{\sin \xi}{\xi}$$

$$n = 5, \quad \theta = \frac{1}{\left(1 + \frac{\xi^2}{3} \right)^{0.5}}$$

Solutions for all other values of n must be solved numerically i.e. we use a computer program to determine θ for values of ξ

Solutions are subject to boundary conditions:

$$\frac{d\theta}{d\xi} = 0, \quad \theta = 1 \quad \text{at} \quad \xi = 0$$

Computational solution of the equation

We start by expressing the Lane-Emden equation in the form:

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi} \frac{d\theta}{d\xi} - \theta^n$$

The numerical integration technique - step outwards in radius from the centre of the star and evaluate density at each radius (i.e. evaluate θ for each of ξ). At each radius, the value of density θ_{i+1} is given by the density at previous radius, θ_i plus the change in density over the step ($\Delta\xi$)

$$\theta_{i+1} = \theta_i + \Delta\xi \frac{d\theta}{d\xi}$$

Now $d\theta/d\xi$ is unknown, but by same technique we can write

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_i + \Delta\xi \frac{d^2\theta}{d\xi^2}$$

Then we can replace the second derivative term in the above by the rearranged form of the Lane-Emden equation:

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_i - \left(\frac{2}{\xi} \frac{d\theta}{d\xi} + \theta^n\right) \Delta\xi$$

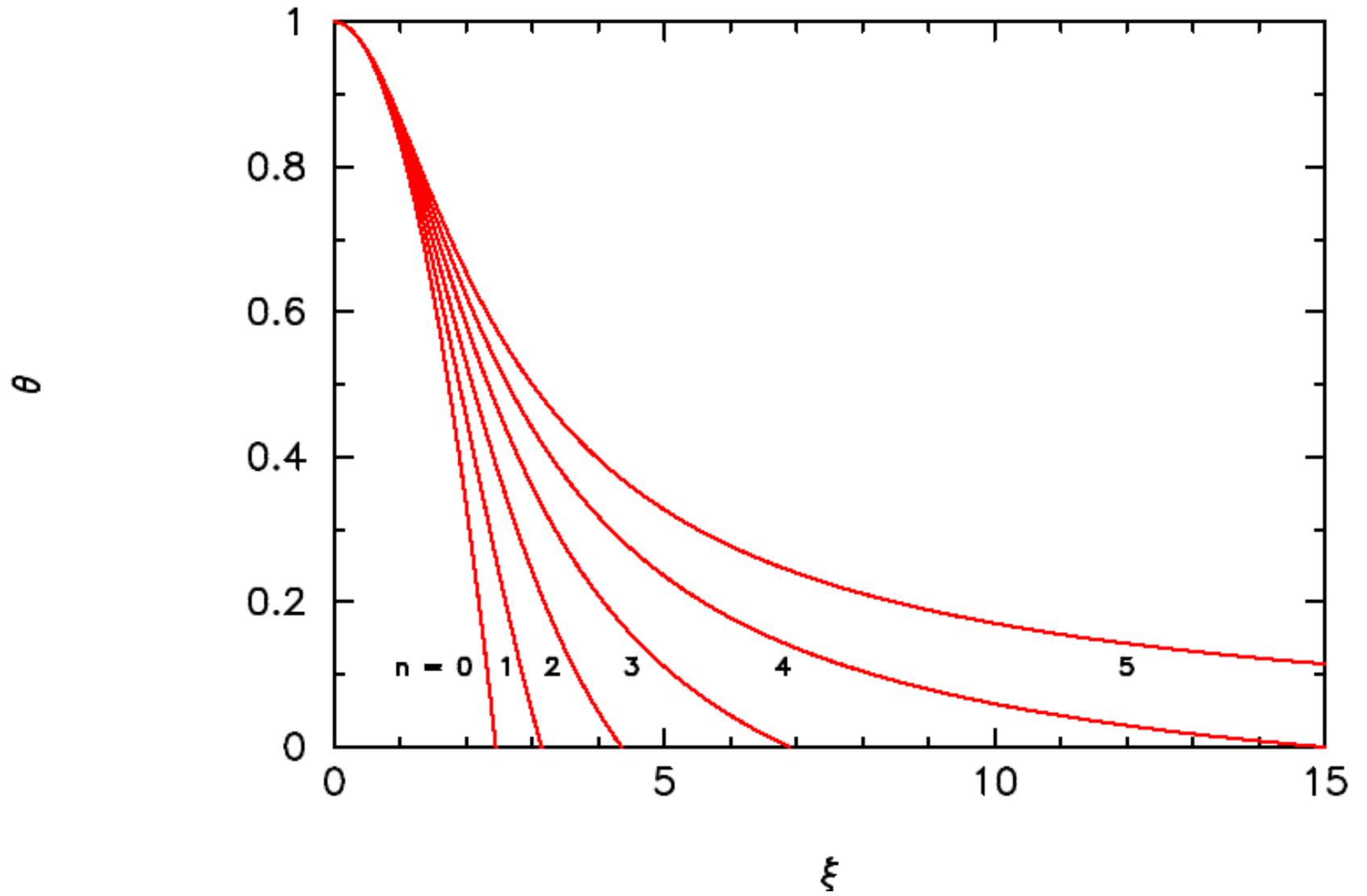
Now we can adopt a value for n and integrate numerically. We have the boundary conditions at the centre.

$$\frac{d\theta}{d\xi} = 0, \quad \theta = 1 \quad \text{at} \quad \xi = 0$$

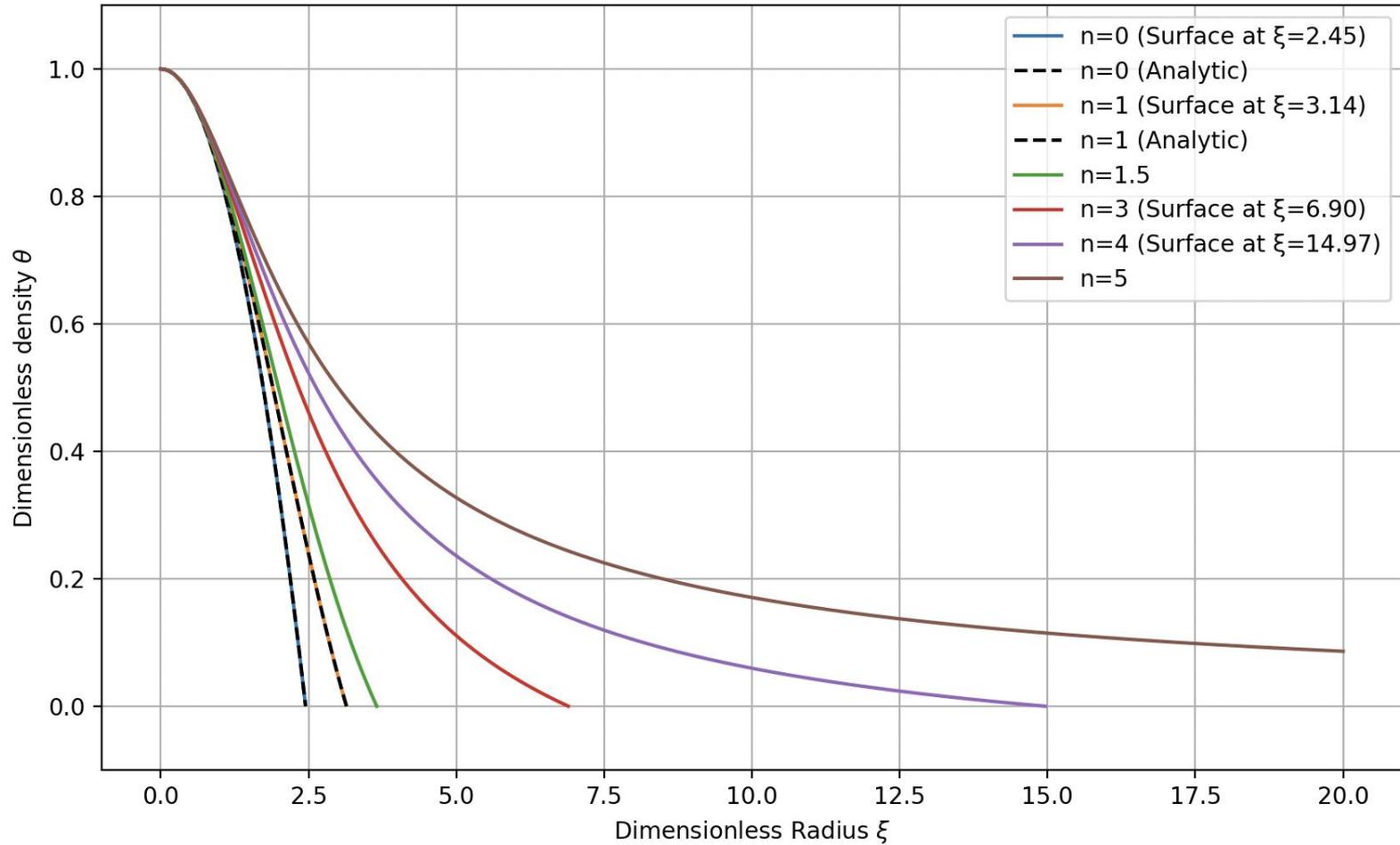
So starting at the centre, we determine $\left(\frac{d\theta}{d\xi}\right)_{i+1}$

Which can be used to determine θ_{i+1} . The radius is then incremented by adding $\Delta\xi$ to ξ and the process is repeated until the surface of the star is reached (when θ becomes negative).

In your own time – try to create a python program to do these calculations - useful experience!



Numerical Solutions of the Lane-Emden Equation



Predictions of the $n = 3$ polytropic model for the Sun of mass, density, pressure and temperature variations with radius are needed for comparison with the Standard Solar Model of Bahcall *et al.* (1998 Physics Letters B, 433, 1).

At the surface of the $n = 3$ polytrope where $\theta = 0$

$$\alpha = \frac{R_{\odot}}{\xi_1} = \frac{7 \times 10^8}{6.9} \text{ m} = 1.01 \times 10^8 \text{ m}.$$

The rate of change of mass with radius is given by the Equation of Mass Conservation

$$\frac{dM(r)}{dr} = 4 \pi r^2 \rho(r).$$

Integrating and substituting $r = \alpha \xi$ and $\rho = \rho_c \theta^n$ gives

$$M_{\odot} = \int_0^{R_{\odot}} 4 \pi r^2 \rho dr = 4 \pi \alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n d\xi.$$

The Lane-Emden equation may expressed in the form

$$\xi_1^2 \left. \frac{d\theta}{d\xi} \right|_{\xi=\xi_1} = - \int_0^{\xi_1} \xi^2 \theta^n d\xi$$

and substituting in the above expression for M_{\odot} gives

$$M_{\odot} = -4 \pi \alpha^3 \rho_c \xi_1^2 \left. \frac{d\theta}{d\xi} \right|_{\xi=\xi_1}.$$

The Lane-Emden Equation for $n = 3$ has a solution ($\theta = 0$) relevant to stellar structure at

$$\xi_1 = 6.90 \quad \text{and} \quad \left. \frac{d\theta}{d\xi} \right|_{\xi=\xi_1} = -4.236 \times 10^{-2}.$$

Taking $M_\odot = 2 \times 10^{30}$ kg and the Lane-Emden Equation solution for $n = 3$, the expression for M_\odot above gives an estimate for the central density of the Sun of

$$\rho_c = 7.66 \times 10^4 \text{ kg m}^{-3}$$

and the dependence of density on radial distance from the solar centre immediately follows from

$$\rho = \rho_c \theta^n$$

since θ varies from $\theta = 1$ at the centre to $\theta = 0$ at the surface.

By definition

$$\alpha^2 = \frac{K(n+1)}{4\pi G \rho_c^{(1-1/n)}}$$

and as ρ_c and α are known, $K = 3.85 \times 10^{10} \text{ Nm kg}^{-1}$. It then follows since $P = K \rho^\gamma$ that an estimate of the pressure at the centre of the Sun (where $\rho = \rho_c$) is

$$P_c = 1.25 \times 10^{16} \text{ Nm}^{-2},$$

and the dependence of gas pressure on radial distance follows directly by substituting the appropriate ρ .

By a similar argument, the equation of state for a perfect gas

$$P_{\text{gas}} = \frac{k}{m_{\text{H}} \bar{\mu}} \rho T$$

gives the dependence of T on radial distance (r) on substituting the $P_{\text{gas}}(r)$ and adopting $\bar{\mu} \simeq 0.6$ as previously derived. In particular, setting $P_{\text{gas}}(r) = P_{\text{c}}$ gives a temperature at the solar centre of

$$T_{\text{c}} = 1.19 \times 10^7 \text{ K.}$$

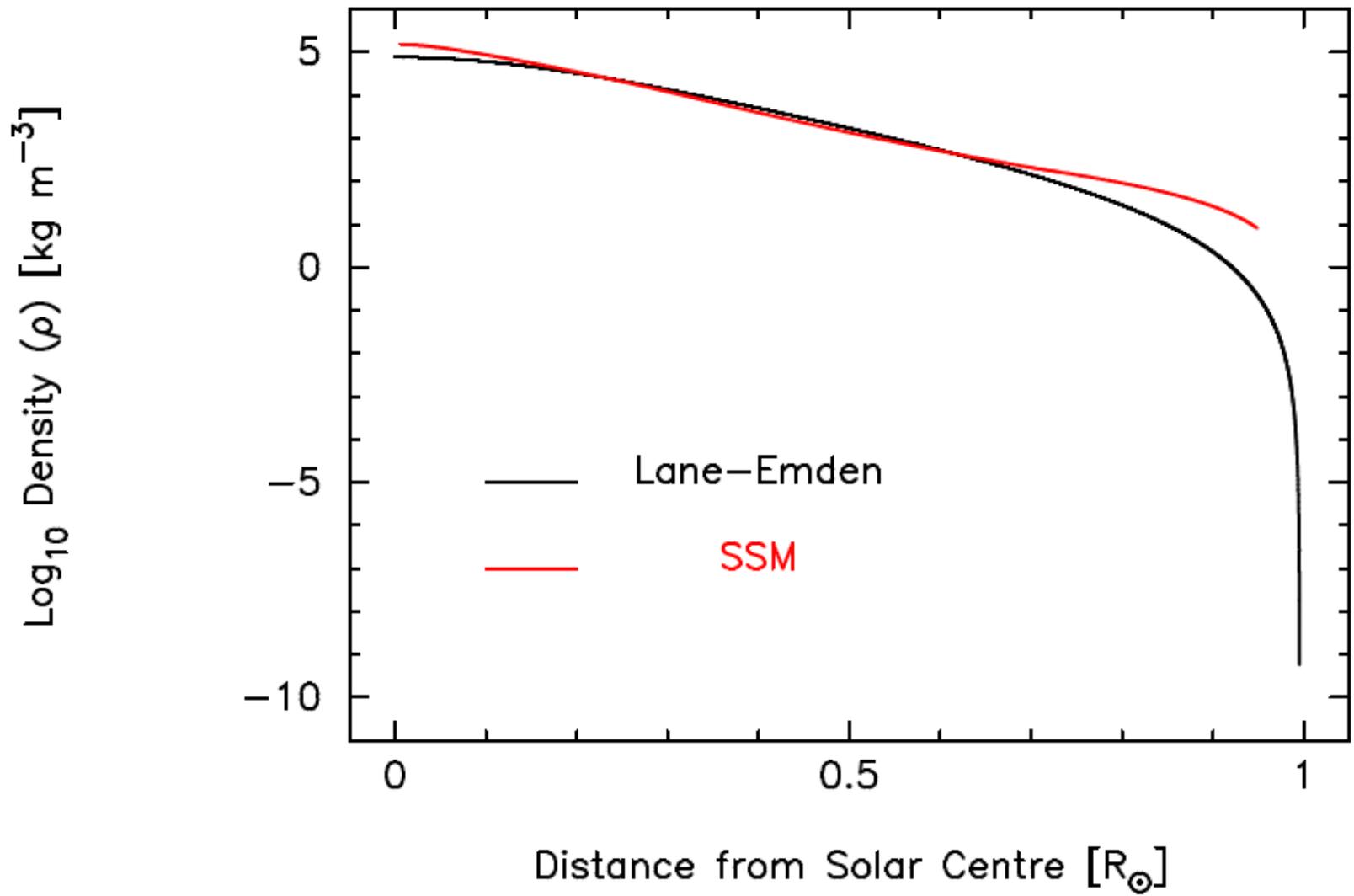
As previously discussed, the mass ($M(r)$) interior to some distance r from a stellar centre is given by the mass conservation equation, to which the Lane-Emden equation may be applied, to give

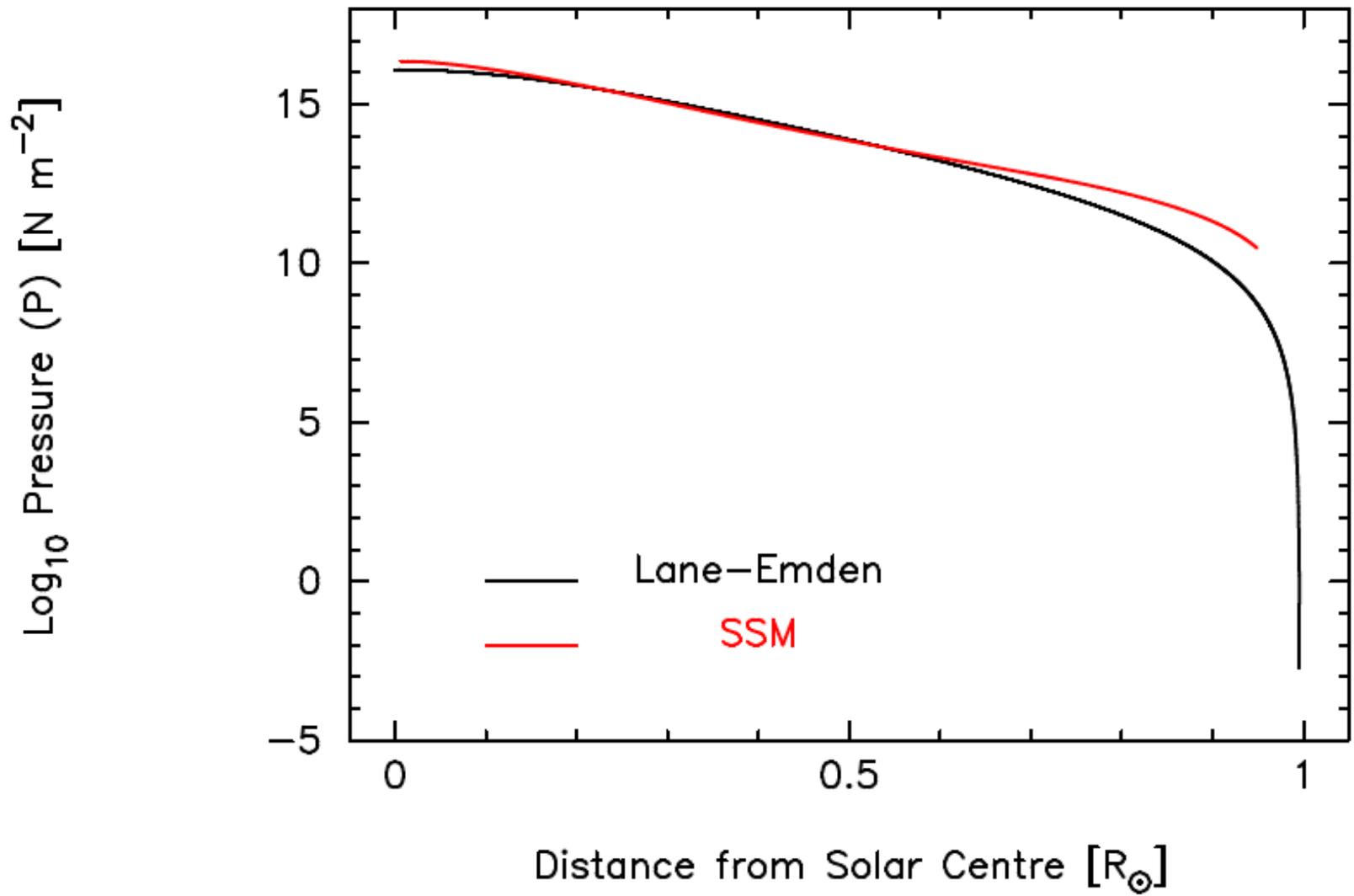
$$M(r) = -4 \pi \alpha^3 \rho_{\text{c}} \xi_r^2 \left. \frac{d\theta}{d\xi} \right|_{\xi=\xi_r}$$

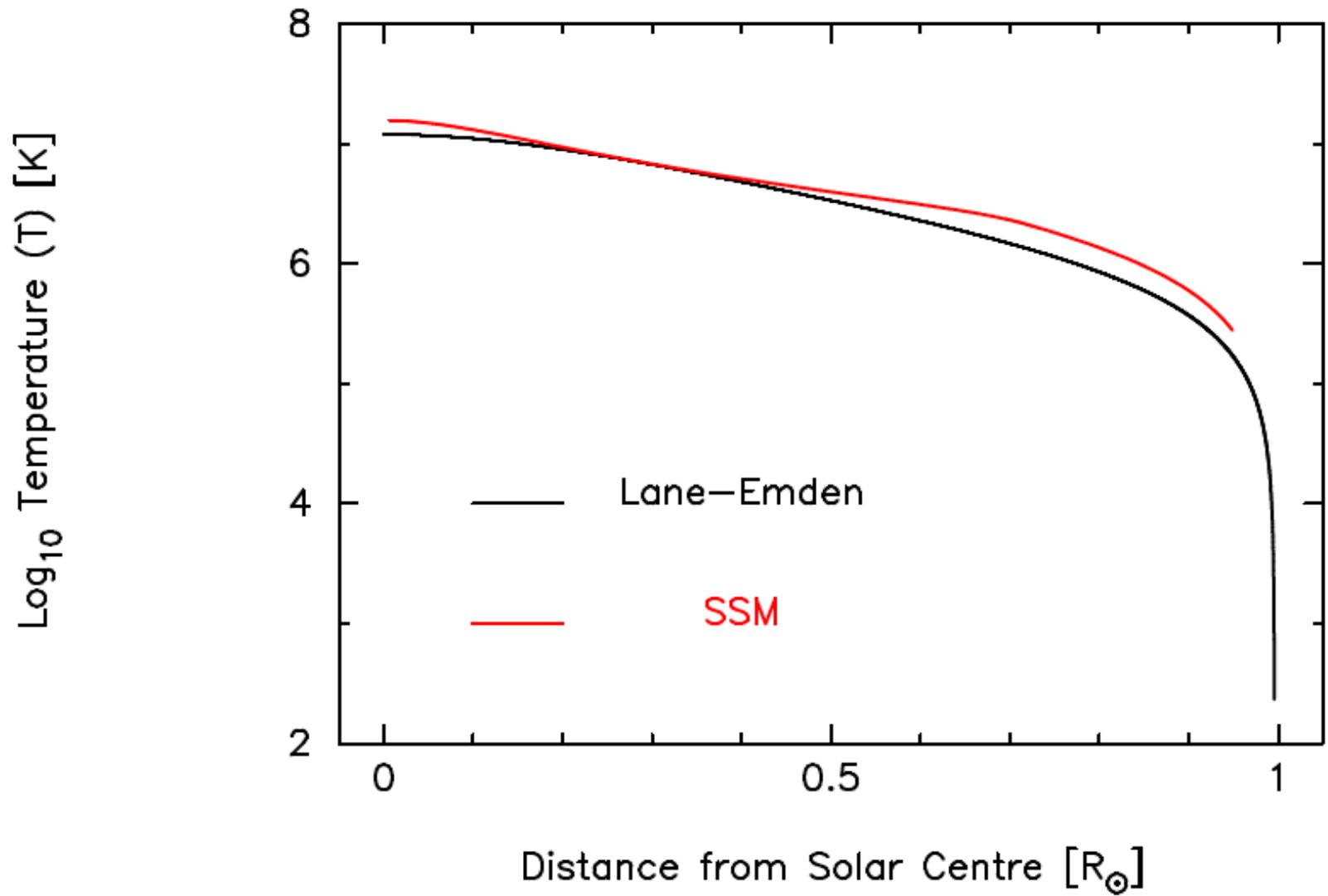
where ξ_r is the scaled radial distance r/α at distance r from the centre of the Sun. Evaluating the right-hand side for successive values of ξ_r gives the mass interior to those points. Comparisons with the Standard Solar Model (SSM) are shown in the plots which follow.

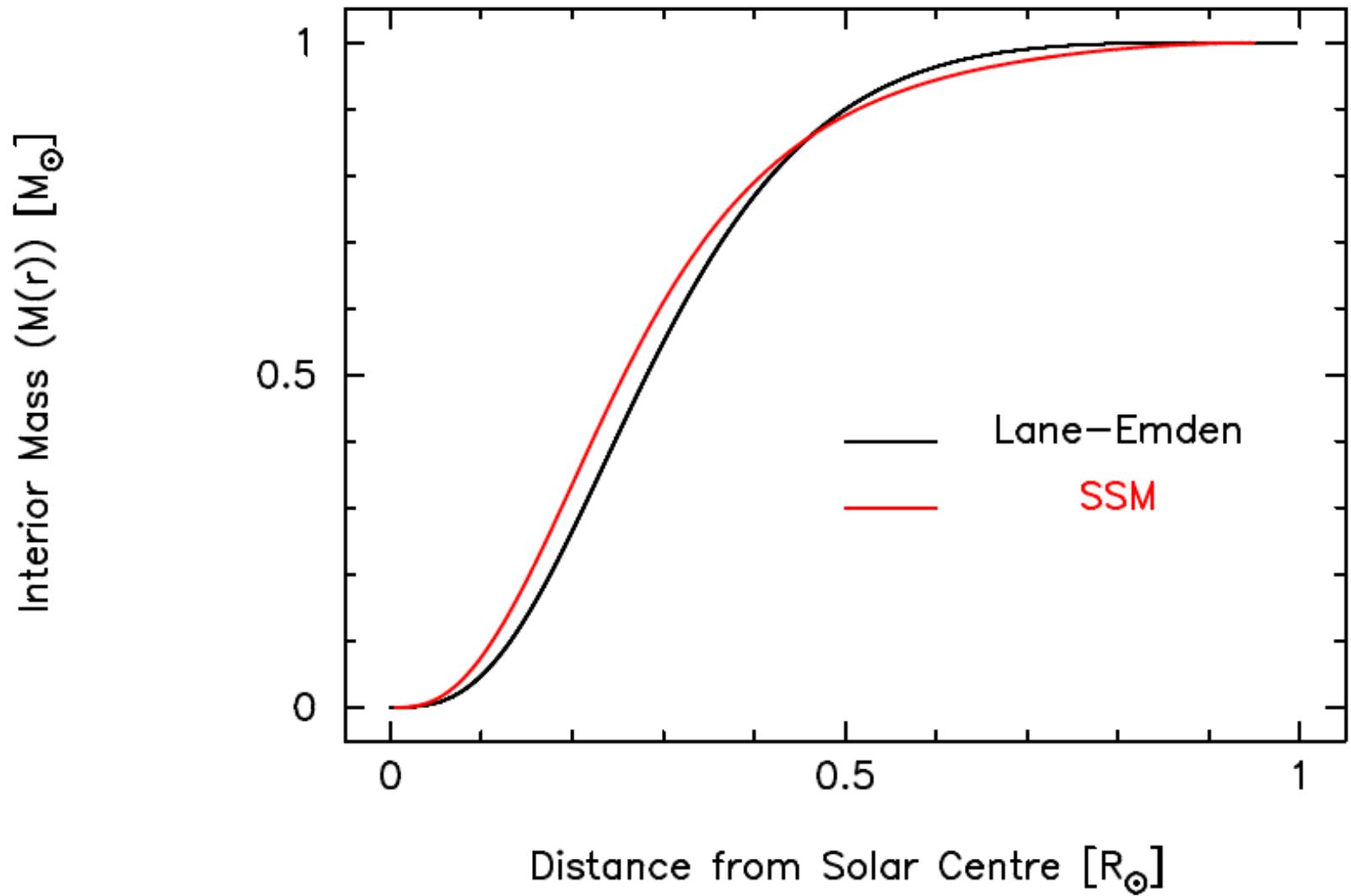
To summarise this is the comparison between the polytrope model $n=3$ and a proper solar model

| Property | $n=3$ polytrope | SSM |
|----------|---|---|
| ρ_c | 7.65×10^4 kgm^{-3} | 1.52×10^5 kgm^{-3} |
| P_c | 1.25×10^{16} Nm^{-2} | 2.34×10^{16} Nm^{-2} |
| T_c | 1.18×10^7 K | 1.57×10^7 K |









Summary

- We have defined a method to relate the internal pressure and density as a function of radius - the polytropic equation of state
- We derived the Lane-Emden equation
- We saw how this equation could be numerically integrated in general
- We compared the $n=3$ polytrope with the Standard Solar model, finding quite good agreement considering how simple the input physics was
- Now we are ready to discuss modern computational solutions of the full structure equations

Now, the temperature inside the Sun is sufficiently high that radiation pressure cannot be completely neglected with respect to conventional gas pressure. In other words, we must write the solar equation of state in the form:

$$p = p_g + p_r,$$

where:

$$p_g = \frac{\rho k T}{\mu m_p}$$

is the gas pressure (modeling the plasma within the Sun as an ideal gas of free electrons and ions), and (Chandrasekhar 1967)

$$p_r = \frac{1}{3} \alpha T^4$$

the radiation pressure (assuming that the radiation within the Sun is everywhere in local thermodynamic equilibrium with the plasma). Here, $T(r)$ is the Sun's internal temperature, k the Boltzmann constant, m_p the mass of a proton, and μ the relative molecular mass (i.e., the ratio of the mean mass of the free particles making up the solar plasma to that of a proton). Note that the electron mass has been neglected with respect to that of a proton.

Furthermore $\alpha = 4\sigma/c$, where σ is the Stefan-Boltzmann constant, and c the velocity of light in a vacuum. Incidentally, in writing Equation (13.26), we have expressed

M/R in the equivalent form $\mu m_p/k$

$$p_r = (1 - \beta) p, \quad p_g = \beta p,$$

where the parameter β is assumed to be uniform. In other words, the ratio of the radiation pressure to the gas pressure is assumed to be the same everywhere inside the Sun.

This fairly drastic assumption turns out to lead to approximately the correct internal pressure-density relation for the Sun. In fact, Equations (13.26)-(13.29) can be combined to give

$$p = K \rho^{4/3}, \quad \text{where} \quad K = \left[\left(\frac{k}{m_p} \right)^4 \frac{3}{\alpha} \frac{\beta}{\mu^4 (1 - \beta)^4} \right]^{1/3}.$$