BCAM

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Topological vector space $\quad \mathbb{K}=\mathbb{R}, \mathbb{C}$

<u>Df</u> Consider a vector space X on $\mathbb{K}$.
and let τ be a topological structure.
We say that (X,τ) is a TVL

if it is Hausdorff and if

$$X \times X \longrightarrow X, \quad (x,y) \longmapsto x+y$$

$$\mathbb{K} \times X, \quad (\lambda, x) \longrightarrow \lambda x \in X$$

are continuous.

Def X a TVS on K. Consider a subset $\Omega \subseteq X$. Then

1) Ω is balanced if

$$x \in \Omega \text{ and } |\lambda| \leq 1 \Rightarrow \lambda x \in \Omega$$

2) Ω is absorbing if $\forall x \in X$

$$\exists \lambda \in K \text{ s.t. } x \in \lambda \Omega = \{\lambda \omega : \omega \in \Omega\}$$

Exercise If V is a nbhd of 0 then V is absorbing.

Lemma For any given nbhd U of 0 in a TVS X $\exists$ a balanced nbhd of 0, V, s.t. $V \subseteq U$.

Pf We use the continuity of

$$\begin{aligned} K \times X &\longrightarrow X \\ (\lambda, x) &\longrightarrow \lambda x \end{aligned}$$

Given any nbhd U of $0 \in X$ the exist an $\varepsilon > 0$ and a W nbhd of $0 \in X$ s.t.

$$\lambda W \subseteq U \qquad \text{if } |\lambda| \leq \varepsilon$$

$$U \supseteq \bigcup_{|\lambda| \leq \varepsilon} \lambda W \stackrel{\circ}{=} V$$

It is easy to see that V is balanced.

$x \in V \qquad \exists\ |\lambda| \leq \varepsilon \quad \text{s.t.} \quad x = \lambda w$ and $w \in W$

If I pick $\mu \in K \quad |\mu| \leq 1 \qquad \mu x = \underbrace{\mu \lambda}_{\tilde{\lambda}} w \qquad |\tilde{\lambda}| \leq \varepsilon$

$$|\mu \lambda| = |\mu|\ |\lambda| \leq |\lambda| \leq \varepsilon$$

$$|\mu \lambda| \leq \varepsilon \implies \mu x \in \tilde{\lambda} W \subseteq V$$

__Def__ Given two TVS ~~X~~ and Y on K I will denote by $\mathcal{L}(X,Y)$ the set of linear operators $X \longrightarrow Y$ which are continuous.

When $X = Y$ I will write

$$\mathcal{L}(X) = \mathcal{L}(X,X).$$

$$X' = \mathcal{L}(X,K)$$

is the dual of X.

$L^1(\mathbb{R}^d)$ L^2

$L^p(\mathbb{R}^d)$

$1 < p < +\infty$

$$X \xrightarrow{T} Y$$

T is continuous iff T is continuous
in a single point

$$\begin{tikzcd} & \mathbb{R} \\ K \arrow[ur, dash] \arrow[dr, dash] & \\ & \mathbb{C} \end{tikzcd}$$

If X is a vector space on $\mathbb{R}$
we can complexify by considering

$$\mathbb{C} \otimes_{\mathbb{R}} X \hookleftarrow X$$

$$1 \otimes_{\mathbb{R}} x$$

$$X \xrightarrow{T} Y \qquad \text{TVS on } \mathbb{R}$$

$$\mathbb{C} \otimes_{\mathbb{R}} X \xrightarrow{\tilde{T}} \mathbb{C} \otimes_{\mathbb{R}} Y$$

$$X = Y$$

$$X \xrightarrow{T} X$$

Def Given a TVS X a subset $B \subseteq X$ is bounded in X if for any nhbd V of 0 $\exists \lambda \in K$ st. $\lambda V \supseteq B$.

Def A linear map $T: X \to Y$ between two TVS's is bounded if for any $B \subseteq X$ bounded also TB is bounded in Y.

Lemma Let X be a TVS on K.

$T: X \to K$ linear and suppose $\exists\, x \in X$

s.t. $Tx \neq 0$. The following are equivalent

a) $T \in X'$

b) $\ker T$ is closed in X

c) $\ker T$ is not dense in X

d) T is bounded in some nbhd of 0 in K.

Pf $a \Rightarrow b \Rightarrow c$

$x \cap \ker T = \phi$ $\quad$ $\cdot 0$ $\quad K$

$c \Rightarrow d$

$\exists\, x \in X$ $\quad$ $\exists$ nbhd of x, it is of the form $x+V$ with V a nbhd of 0 in X

s.t. $(x+V) \cap \ker T = \phi$

Here I can assume V is balanced

$TV \subseteq K$ $\quad$ TV is balanced in K

$\lambda TV \subseteq TV \quad \forall\, |\lambda| \leq 1$

$\lambda TV = T\lambda V \subseteq TV \subseteq K$

either TV is bounded

or TV is unbounded

Let us assume TV is unbounded

Then $\exists\, \{Tx_n\}_{n \in \mathbb{N}}$ with $x_n \in V$

s.t. $|Tx_n| \xrightarrow{n \to +\infty} +\infty$

However then, since TV is absorbing

it contains all the points of the form

$\{\lambda Tx_n : \text{for } |\lambda| \leq 1\}$

$= \overline{D_K(0, |Tx_n|)} \subseteq TV \quad \forall n$

Hence $TV = K$

consider $-Tx \in K$ $\quad \exists\, y \in V$

s.t. $Ty = -Tx$

$\Rightarrow T(x+y) = 0 \Rightarrow (x+V) \cap \ker T \neq \phi$

Conclusion $T|_V$ is bounde

$d \Rightarrow a$

T bounded in some nbhd of 0 in X

$\Rightarrow$ T is continuous.

d) guarantees $\exists\, V$ nbhd of $0 \in X$

and $M > 0$ s.t.

$|Ty| < M \quad \forall\, y \in V.$

Let $\varepsilon > 0$ and let $x \in \frac{\varepsilon}{M} V$ $\quad$ $x = \frac{\varepsilon}{M} y$, $y \in V$

$|Tx| = |T \frac{\varepsilon}{M} y| =$

$= |\frac{\varepsilon}{M} Ty| = \frac{\varepsilon}{M} |Ty| < \frac{\varepsilon}{\cancel{M}} \cancel{M} = \varepsilon$

We proved that T is continuous in $0 \in X$.

Def A metric d on a vector space X is translation invariant if

$$d(x,y) = d(x+z, y+z) \qquad \forall\, x, y, z$$

Def A TVS (X, τ) is metrizable if $\exists$ a metric d on X compatible with τ.

Def Given a topological space X and $x \in X$ a family $\mathcal{B}$ of nbhd's of x is a subbasis of nbhds of x if for any nbhd V of x there are finitely many $B_1, \dots, B_n \in \mathcal{B}$

s.t. $B_1 \cap \dots \cap B_n \subset V$.

Theorem ~~If for~~ FV