Thm
Let X be a TVS. Then X is metrizable if there exists a subbasis of nbhds of $0 \in X$ which is countable. Furthermore, the metric can be taken translation invariant.

for proof
See Rudin Functional Analysis

X a vector space on K

$\|\cdot\| : X \longrightarrow [0, +\infty)$

1) $\|x\| = 0 \iff x = 0$

2) $\|x + y\| \leq \|x\| + \|y\|$

3) $\|\lambda x\| = |\lambda| \, \|x\|$

$d(x,y) = \|x - y\|$

d is a metric on X and the metric induces on X a topology τ which makes X a TVS.

$(X, \|\cdot\|)$ is a normed space and whenever it is complete we say it is a Banach space

B-space.

non complete

$$\begin{tikzcd}
(X, d_X) \arrow[r, "f"] \arrow[d, hook] & (Y, d_Y) \\
(\overline{X}, d_{\overline{X}}) \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

complete

complete

$$\begin{tikzcd}
(X, \|\ \|_X) \arrow[r, "T"] \arrow[d, hook] & (Y, \|\ \|_Y) \\
(\overline{X}, \|\ \|_{\overline{X}}) \arrow[ur, "\overline{T}"'] &
\end{tikzcd}$$

Examples of B-spaces.

$\emptyset \neq \Omega \subseteq \mathbb{R}^d$ open

$L^p(\Omega)$ $1 \leq p \leq +\infty$ are complete

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$\|f\|_{L^\infty(\Omega)} = \sup\{c \geq 0 : |\{x \in \Omega : |f(x)| \geq c\}| > 0\}$$

$$f \in BC^0(\Omega) = L^\infty(\Omega) \cap C^0(\Omega)$$

$$\|f\|_{L^\infty(\Omega)} = \sup\{|f(x)| ; x \in \Omega\}$$

$$C^0_0(\mathbb{R}^d) = \{f \in C^0(\mathbb{R}^d) : \lim_{x \to \infty} f(x) = 0\}$$

$$\cap$$

$$BC^0(\mathbb{R}^d)$$

$$C^0_0(\mathbb{R}^d) = \overline{C^0_c(\mathbb{R}^d)} \quad \text{in } BC^0(\Omega)$$

$$C^0_c(\mathbb{R}^d) = \{f \in C^0(\mathbb{R}^d) : \operatorname{supp} f \text{ is compact in } \mathbb{R}^d\}$$

$\vartheta \in (0,1)$ $\quad \Omega \subseteq \mathbb{R}^d)$

$$C^{0,\vartheta}(\Omega) = \{ f \in BC^0(\Omega) : [f]_{C^{0,\vartheta}(\Omega)} < +\infty \}$$

where

$$[f]_{C^{0,\vartheta}(\Omega)} = \sup_{\substack{x \neq y \\ x,y \in \Omega}} \frac{|f(x) - f(y)|}{|x-y|^\vartheta} \quad (< \infty)$$

$C^{0,\vartheta}(\Omega)$

$$\|f\|_{C^{0,\vartheta}} = \|f\|_{L^\infty} + [f]_{C^{0,\vartheta}}$$ it is a norm.

$C^{0,\vartheta}$ is a B-space

Let us consider a Cauchy sequence $\{f_n\}$ in $C^{0,\vartheta}(\Omega)$

$$\|f\|_{C^{0,\vartheta}(\Omega)} = |f|_{L^\infty(\Omega)} + [f]_{C^{0,\vartheta}}$$

$\{f_n\}$ in $C^{0,\vartheta}(\Omega) \subseteq BC^0(\Omega) \subseteq L^\infty(\Omega)$

Cauchy

f_n is Cauchy in $BC^0(\Omega)$. So $\exists\, f \in BC^0(\Omega)$

s.t. $f_n \longrightarrow f$ in $BC^0(\Omega)$.

Need to show

a) $f \in C^{0,\vartheta}(\Omega)$

b) $f_n \to f$ in $C^{0,\vartheta}(\Omega)$

$x \neq y \qquad [f]_{C^{0,\vartheta}(\Omega)} < +\infty$

$$\frac{|f(x)-f(y)|}{|x-y|^{\vartheta}} = \frac{|f(x)-f_n(x)+f_n(x)-f_n(y)+f_n(y)-f(y)|}{|x-y|^{\vartheta}}$$

$$\leq \frac{|f_n(x)-f_n(y)|}{|x-y|^{\vartheta}} + \frac{|f(x)-f_n(x)|+|f(y)-f_n(y)|}{|x-y|^{\vartheta}}$$

$$\leq [f_n]_{C^{0,\vartheta}(\Omega)} + \frac{2}{|x-y|^{\vartheta}} |f-f_n|_{L^\infty(\Omega)}$$

$\forall\, x \neq y$ in Ω

$$\frac{|f(x)-f(y)|}{|x-y|^{\vartheta}} \leq [f_n]_{C^{0,\vartheta}(\Omega)} + \frac{2}{|x-y|^{\vartheta}} |f-f_n|_{L^\infty(\Omega)}$$

$$\leq C + \frac{2}{|x-y|^{\vartheta}} |f-f_n|_{L^\infty(\Omega)}$$

Recall that $\{f_n\}$ is Cauchy in $C^{0,\vartheta}(\Omega)$

So $\exists\, C>0$ st. $|f_n|_{C^{0,\vartheta}(\Omega)} \leq C$

$\forall\, x,y \quad x \neq y$

$$\frac{|f(x)-f(y)|}{|x-y|^{\vartheta}} \leq C + \frac{2}{|x-y|^{\vartheta}} |f-f_n|_{L^\infty(\Omega)}$$

for $n \to +\infty$

$$\frac{|f(x)-f(y)|}{|x-y|^{\vartheta}} \leq C \qquad \forall\, x \neq y \text{ in } \Omega$$

$$\Rightarrow [f]_{C^{0,\vartheta}(\Omega)} < +\infty$$

We need to show

$$\lim_{n\to+\infty} \|f-f_n\|_{C^{0,\vartheta}(\Omega)} = 0$$

~~It~~ We know already

$$\lim_{n\to+\infty} \|f-f_n\|_{L^\infty(\Omega)} \quad ;$$

what is left is

$$\lim_{n\to+\infty} [f-f_n]_{C^{0,\vartheta}(\Omega)} = 0$$

$x \neq y$

$$\frac{|(f(x)-f_n(x))-(f(y)-f_n(y))|}{|x-y|^\vartheta} = \lim_{m\to+\infty} \frac{|f_m(x)-f_n(x)-(f_m(y)-f_n(y))|}{|x-y|^\vartheta}$$

$\leq \varepsilon$

$$f(x) = \lim_{m\to+\infty} f_m(x)$$
$y \qquad\qquad y$

$\forall \varepsilon>0 \quad \exists\ N_\varepsilon$ s.t. if $m, n \geq N_\varepsilon$

$\Longrightarrow \quad [f_m-f_n]_{C^{0,\vartheta}(\Omega)} < \varepsilon$

$$\frac{|f_m(x)-f_n(x)-(f_m(y)-f_n(y))|}{|x-y|^\vartheta} \leq [f_m-f_n]_{C^{0,\vartheta}} < \varepsilon$$

$\Longrightarrow$ for $n \geq N_\varepsilon$

$$[f-f_n]_{C^{0,\vartheta}(\Omega)} \leq \varepsilon$$

$1 \leq p < \infty \qquad L^p(\Omega)$

$\Omega \subseteq \mathbb{R}^d$ open $\qquad 0 < \vartheta < 1$

$$W^{\vartheta,p}(\Omega) = \{ f \in L^p(\Omega) : [f]_{W^{\vartheta,p}(\Omega)} < +\infty \}$$

$$[f]^p_{W^{\vartheta,p}(\Omega)} = \int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{\vartheta p + d}} \, dx \, dy$$

$$\|f\|_{W^{\vartheta,p}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{W^{\vartheta,p}(\Omega)}$$

It is a B-space

Let $\{f_n\}$ be a Cauchy sequence

Then $\{f_n\}$ is Cauchy in $L^p(\Omega)$

Then $f_n \longrightarrow f$ in $L^p(\Omega)$. Two tasks now:

a) $f \in W^{\vartheta,p}(\Omega)$

b) $f_n \to f$ in $W^{\vartheta,p}(\Omega)$

the is in $L^p(\Omega\times\Omega)$

$$\frac{|f(x)-f(y)|}{|x-y|^{\vartheta+\frac{d}{p}}} \le \frac{|f_n(x)-f_n(y)|}{|x-y|^{\vartheta+\frac{d}{p}}} + \frac{|f(x)-f_n(x)|}{|x-y|^{\vartheta+\frac{d}{p}}} + \frac{|f(y)-f_n(y)|}{|x-y|^{\vartheta+\frac{d}{p}}}$$

$$\left(\int_{\substack{\Omega\times\Omega \\ |x-y|\ge\varepsilon}} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy\right)^{\frac{1}{p}} \le$$

$$\le \left(\int_{\substack{\Omega\times\Omega \\ \cancel{|x-y|\ge\varepsilon}}} \frac{|f_n(x)-f_n(y)|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy\right)^{\frac{1}{p}}$$

$$+\,2\left(\int_{\substack{\Omega\times\Omega \\ |x-y|\ge\varepsilon}} \frac{|f(x)-f_n(x)|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy\right)^{\frac{1}{p}}$$

$$= [f_n]_{W^{\vartheta,p}(\Omega)} + 2\left(\int_\Omega dx\,|f(x)-f_n(x)|^p \int_{\substack{|y-x|\ge\varepsilon \\ \cancel{y\in\Omega}}} \frac{dy}{|x-y|^{\vartheta p+d}}\right)^{\frac{1}{p}}$$

$$= [f_n]_{W^{\vartheta,p}(\Omega)} + 2\left(\int_\Omega dx\,|f(x)-f_n(x)|^p\, \varepsilon^{-\vartheta p}\right)^{\frac{1}{p}}$$

$$= [f_n]_{W^{\vartheta,p}(\Omega)} + 2\,\varepsilon^{-\vartheta}\,|f-f_n|_{L^p(\Omega)}$$

$$\left(\int_{\substack{\Omega\times\Omega \\ |x-y|\ge\varepsilon}} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy\right)^{\frac{1}{p}} \le \underbrace{[f_n]_{W^{\vartheta,p}(\Omega)}}_{\le C} + 2\,\varepsilon^{-\vartheta}\underbrace{|f-f_n|_{L^p(\Omega)}}_{\downarrow\, n\to\infty \atop 0}$$

$$\left(\int_{\substack{\Omega\times\Omega \\ |x-y|\geq\varepsilon}} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy\right)^{\frac{1}{p}} \leq \underbrace{[f_n]_{W^{\vartheta,p}(\Omega)}}_{\leq C}$$

$\varepsilon_n \to 0^+$

$$\underbrace{1_{|x-y|\geq\varepsilon_n} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}}$$

$$\Big\downarrow \varepsilon\to 0^+$$

$$\frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}$$

Fatou lemma

$$\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy \leq \liminf_{n\to+\infty} \int_{\Omega\times\Omega} 1_{|x-y|\geq\varepsilon_n} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy$$

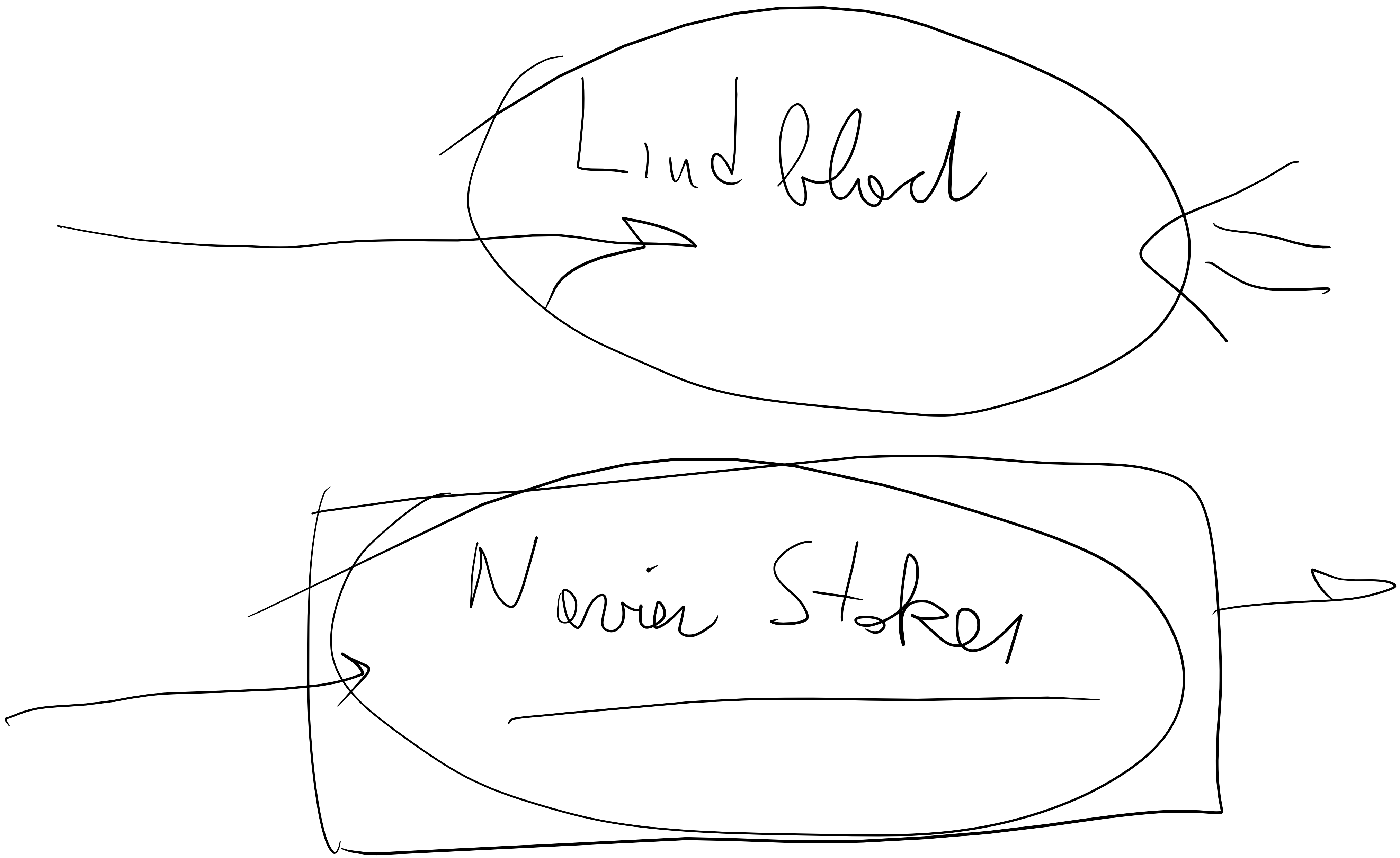
$x \leq x_n$

$x \leq \liminf_{n\to\infty} x_n$

$L^1(X, d\mu)$

$f_n \qquad f_n(x) \to f(x)$ for a.e $x \in X$

$\Longrightarrow \quad f \in L^1(X, d\mu) \qquad |f|_{L^1} \leq \liminf_{n\to+\infty} |f_n|_{L^1}$



Lindblad
Navier Stokes

$f \in W^{\vartheta,p}(\Omega)$

$f_n \to f$ in $W^{\vartheta,p}(\Omega)$

$[f_n - f]^p \xrightarrow{n\to\infty} 0$

$$\frac{|(f(x)-f_n(x))-(f(y)-f_n(y))|^p}{|x-y|^{\vartheta p+d}}$$

$$\int_{\Omega\times\Omega} \frac{|f(x)-f_n(\dots)\dots|}{|x-y|^{\vartheta p+d}}\,dx\,dy \xrightarrow{n\to+\infty} 0$$

$f_n \to f$ in $L^p(\Omega) \not\Longrightarrow f_n(x) \to f(x)$ a.e.

But there exists a subsequence f_{n_k}

s.t. $f_{n_k}(x) \to f(x)$ a.e.

$$\frac{|(f(x)-f_n(x))-(f(y)-f_n(y))|^p}{|x-y|^{\vartheta p+d}} = \lim_{k\to+\infty} \frac{|(f_{n_k}(x)-f_n(x))-(f_{n_k}(y)-f_n(y))|^p}{|x-y|^{\vartheta p+d}}$$

for a.e. $(x,y) \in \Omega\times\Omega$

$$\int_{\Omega\times\Omega} \frac{|(f(x)-f_n(x))-(f(y)-f_n(y))|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy$$

$$\le \liminf_{k\to+\infty} \int_{\Omega\times\Omega} dx\,dy \frac{|(f_{n_k}(x)-f_n(x))-(f_{n_k}(y)-f_n(y))|^p}{|x-y|^{\vartheta p+d}} \le \varepsilon^p$$

$\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $n,m \ge N_\varepsilon \Rightarrow [f_n - f_m]_{W^{\vartheta,p}} \le \varepsilon$

$n \ge N_\varepsilon \Rightarrow [f - f_n]_{W^{\vartheta,p}} \le \varepsilon$