Esercizio 1

$$f_\alpha(z) = \frac{z e^{i\alpha z}}{(z^2+1)^2}, \quad \alpha > 0$$

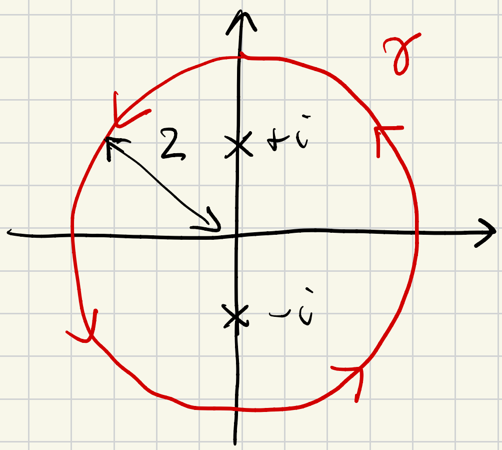
I)

$e^{i\alpha z}$: singolarità essenziale in $z=\infty$

$\frac{1}{(z^2+1)^2}$: polo di ordine 2 in $z=\pm i$

$\Rightarrow$ $f_\alpha(z)$ ha singolarità essenziale in $z=\infty$ e polo doppio in $z=\pm i$. (Notiamo che il numeratore non si annulla in $z=\pm i$).

II)



$$\oint_\gamma f_\alpha(z)\,dz = 2\pi i \left(\mathrm{Res}_{f_\alpha}(z=i) + \mathrm{Res}_{f_\alpha}(z=-i)\right)$$

teorema interno dei residui

$$= 2\pi i \left[\frac{d}{dz}\left((z-i)^2 \frac{z e^{i\alpha z}}{(z^2+1)^2} \right)\Big|_{z=i} + \frac{d}{dz}\left((z+i)^2 \frac{z e^{i\alpha z}}{(z^2+1)^2} \right)\Big|_{z=-i} \right]$$

$$= 2\pi i \left[\frac{d}{dz}\left(\frac{z e^{i\alpha z}}{(z+i)^2} \right)\Big|_{z=i} + \frac{d}{dz}\left(\frac{z e^{i\alpha z}}{(z-i)^2} \right)\Big|_{z=-i} \right]$$

$$= 2\pi i \left[\left(\frac{-2}{(z+i)^3} z e^{i\alpha z} + \frac{e^{i\alpha z}}{(z+i)^2} + \frac{i\alpha z e^{i\alpha z}}{(z+i)^2} \right) \Bigg|_{z=i} \right.$$

$$\left. + \left(\frac{-2}{(z-i)^3} z e^{i\alpha z} + \frac{e^{i\alpha z}}{(z-i)^2} + \frac{i\alpha z\, e^{i\alpha z}}{(z-i)^2} \right) \Bigg|_{z=-i} \right]$$

$$= 2\pi i \left[\frac{-\cancel{2i}\, e^{-\alpha}}{(2i)^{\cancel{3}2}} + \frac{e^{-\alpha}}{(2i)^2} - \frac{\alpha e^{-\alpha}}{(2i)^2} \right.$$

$$\left. + \frac{\overset{-1}{\cancel{2i}}\, e^{\alpha}}{(-2i)^{\cancel{3}2}} + \frac{e^{\alpha}}{(-2i)^2} + \frac{\alpha\, e^{\alpha}}{(-2i)^2} \right]$$

$$= \frac{2\pi i}{-4} \left(\cancel{-e^{-\alpha}} \cancel{+e^{-\alpha}} - \alpha e^{-\alpha} \cancel{-e^{\alpha}} \cancel{+e^{\alpha}} \right.$$

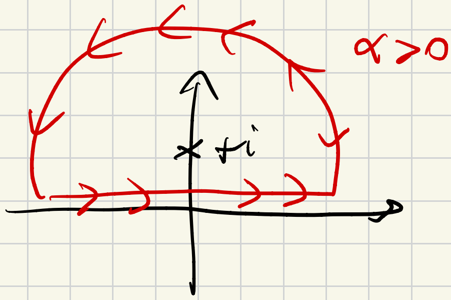
$$\left. + \alpha e^{\alpha} \right)$$

$$= -\frac{\pi i}{2} \alpha \left(e^{\alpha} - e^{-\alpha} \right) = -\pi i \alpha \sinh\alpha .$$

III) $$I(\alpha) = \int_0^{+\infty} dx \, \frac{x \sin(\alpha x)}{(x^2+1)^2}$$

$$\underset{\text{pari fà}}{=} \frac{1}{2} \int_{-\infty}^{+\infty} dx \, \frac{x \sin(\alpha x)}{(x^2+1)^2}$$

$$= \frac{1}{2} \operatorname{Im} \int_{-\infty}^{+\infty} dx \, f_\alpha(x)$$



$$= \frac{1}{2} \operatorname{Im} \left(2\pi i \operatorname{Res}_{f_\alpha}(z=i) \right)$$

↳ Lemma di Jordan: $\alpha > 0$ richiede semipiano superiore

$$= \frac{1}{2} \operatorname{Im}\left(2\pi i \frac{\alpha}{4} e^{-\alpha}\right) = \frac{\pi}{4} \alpha e^{-\alpha}.$$

ESERCIZIO 2 $\quad H_a(t) = e^{-at}\theta(t)$

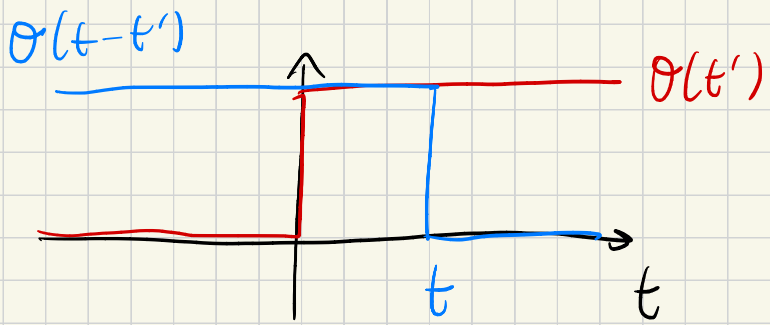
$H_b(t) = e^{-bt}\theta(t)$, $\quad F_{a,b} = H_a * H_b$

I) $$\hat{H}_a(\omega) = \int_0^{+\infty} dt\, e^{-at} e^{i\omega t} = \frac{1}{a - i\omega}$$

$$\hat{H}_b(\omega) = \frac{1}{b - i\omega}$$

$$\hat{F}_{a,b}(\omega) = \hat{H}_a(\omega)\hat{H}_b(\omega) = \frac{1}{a - i\omega}\frac{1}{b - i\omega}$$

$$F_{a,b}(t) = \int_{-\infty}^{+\infty} dt'\, H_a(t') H_b(t-t')$$



$$= \theta(t) \int_0^t dt'\, e^{-at'} e^{-b(t-t')}$$

$$= \theta(t)\, e^{-bt} \int_0^t dt'\, e^{(b-a)t'}$$

$$= \theta(t)\, e^{-bt} \frac{1}{b-a}\left(e^{(b-a)t} - 1\right)$$

$$= \theta(t)\ \frac{e^{-at} - e^{-bt}}{b-a}$$

II) $$\int_{-\infty}^{+\infty} d\omega\ \frac{1}{(a^2+\omega^2)(b^2+\omega^2)}$$

$$= \int_{-\infty}^{+\infty} d\omega\ |\hat{F}_{a,b}(\omega)|^2 = 2\pi \int_{-\infty}^{+\infty} dt\ |F_{a,b}(t)|^2$$

$$= 2\pi \int_0^{+\infty} \frac{e^{-2at} + e^{-2bt} - 2e^{-(a+b)t}}{(b-a)^2}\, dt$$

$$= 2\pi \frac{1}{(b-a)^2}\left[\frac{1}{2a} + \frac{1}{2b} - \frac{2}{a+b}\right]$$

$$= \cancel{2}\pi \frac{1}{(b-a)^2}\ \frac{\cancel{ab} + b^2 + \cancel{ab} + a^2 - \cancel{4ab}^{\,-2ab}}{\cancel{2}ab(a+b)}$$

$$= \frac{\pi}{ab(a+b)}.$$

Esercizio 3

$$A(v) = (u_1, v)\, u_2$$

I) $$\|A(v)\| = \|u_2\|\,|(u_1,v)| \underset{\text{Cauchy Schwartz}}{\le} \|u_2\|\,\|u_1\|\,\|v\|$$

$\Rightarrow \|A\| \le \|u_1\|\,\|u_2\| \Rightarrow$ operatore limitato.

D'altre parte: $A(u_1) = \|u_1\|^2 u_2$

$$\Rightarrow \quad \frac{\|A(u_1)\|}{\|u_1\|} = \|u_1\| \, \|u_2\|$$

$$\Rightarrow \quad \|A\| \geq \|u_1\| \, \|u_2\| \quad \Rightarrow \quad \|A\| = \|u_1\| \, \|u_2\| .$$

II) $\forall \, v, w \in \mathcal{H}$

$$(w, A(v)) = (w, (u_1, v) u_2)$$

$$= (u_1, v)(w, u_2)$$

$$= ((w, u_2)^* u_1, v)$$

$$= ((u_2, w) u_1, v) \equiv (A^+(w), v)$$

$$\Rightarrow \quad A^+(w) = (u_2, w) u_1$$