

Esercizio 1

$$f(z) = \frac{(\log z)^2}{z^2 + 2z + 2}$$

I) Zeri del denominatore:

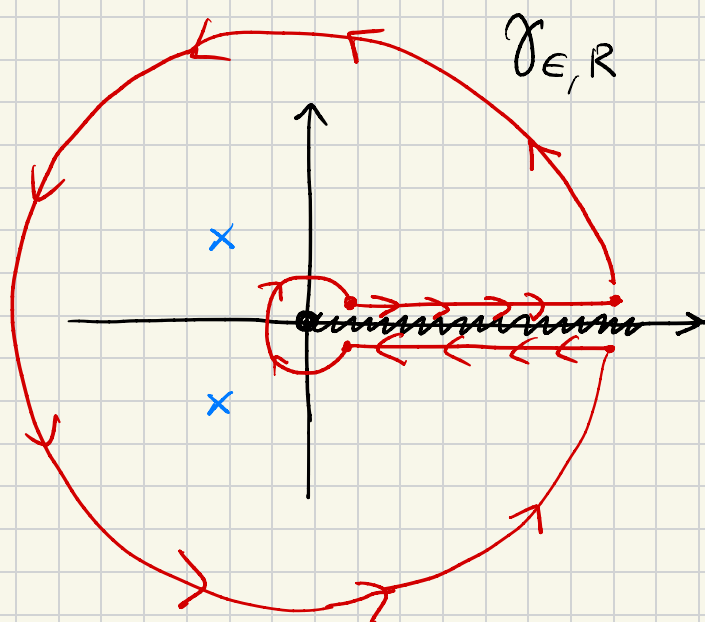
$$z_{\pm} = -1 \pm \sqrt{1-2} = -1 \pm i$$

$$\Rightarrow z^2 + 2z + 2 = (z + 1 - i)(z + 1 + i)$$

Il $\log$ non si annulla in questi punti, quindi sono poli di ordine 1 per $f(z)$.

Inoltre per via del $(\log(z))^2$, $f(z)$ ha punti di diramazione in $z=0$ e $z=\infty$ (che quindi non sono singolarità isolate).

II)



$$\oint_{\gamma_{E,R}} f(z) dz = \int_E^R dx (f(x))^{\text{sopra}} - \int_E^R dx (f(x))^{\text{sotto}} \\ + \int_{\gamma_R} dz f(z) - \int_{\gamma_E} dz f(z)$$

γ_R : cerchio di raggio R in senso antiorario

γ_E : cerchio di raggio E in senso antiorario

Ricordiamo che con questo ramo del logaritmo, per $x > 0$ si ha:

$$(\log(x))^{\text{sopra}} = \log^R(x), \quad (\log(x))^{\text{sotto}} = \log^R(x) + 2\pi i \\ \Rightarrow (f(x))^{\text{sopra}} = \frac{(\log^R(x))^2}{x^2 + 2x + 2},$$

$$(f(x))^{\text{sotto}} = \frac{(\log^R(x))^2 + 4\pi i \log^R(x) - 4\pi^2}{x^2 + 2x + 2}$$

$$\Rightarrow \int_E^R dx (f(x))^{\text{sopra}} - (f(x))^{\text{sotto}} =$$

$$= \int_E^R dx \frac{4\pi^2 - 4\pi i \log^R(x)}{x^2 + 2x + 2}$$

---> sottintendendo che nell'integrale su $x > 0$ $\log(x)$ indica $\log^R(x)$.

$$\left| \int_{\gamma_R} f(z) dz \right| \leq 2\pi R \max_{z \in \gamma_R} \left| \frac{(\log(z))^2}{z^2 + 2z + 2} \right|$$

$$\stackrel{R \rightarrow +\infty}{=} 2\pi R \frac{(\log R)^2 (1 + \mathcal{O}(\frac{1}{\log R}))}{R^2 (1 + \mathcal{O}(\frac{1}{R}))}$$

$$\stackrel{R \rightarrow +\infty}{=} \frac{2\pi (\log R)^2}{R} (1 + \mathcal{O}(\frac{1}{\log R})) \xrightarrow{R \rightarrow +\infty} 0$$

$$\Rightarrow \lim_{R \rightarrow +\infty} \int_{\gamma_R} f(z) dz = 0.$$

$$\left| \int_{\gamma_\epsilon} f(z) dz \right| \leq 2\pi \epsilon \max_{z \in \gamma_\epsilon} \left| \frac{(\log(z))^2}{z^2 + 2z + 2} \right|$$

$$\stackrel{\epsilon \rightarrow 0^+}{=} 2\pi \epsilon \frac{|\log \epsilon|^2 (1 + \mathcal{O}(\frac{1}{|\log \epsilon|}))}{2(1 + \mathcal{O}(\epsilon))}$$

$$\stackrel{\epsilon \rightarrow 0^+}{=} \pi \epsilon |\log \epsilon|^2 (1 + \mathcal{O}(\frac{1}{|\log \epsilon|})) \xrightarrow{\epsilon \rightarrow 0^+} 0.$$

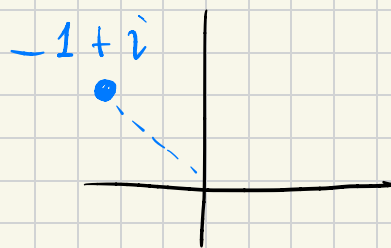
$$\Rightarrow \lim_{\epsilon \rightarrow 0^+} \int_{\gamma_\epsilon} f(z) dz = 0$$

$$\Rightarrow \lim_{\epsilon \rightarrow 0^+} \lim_{R \rightarrow +\infty} \oint_{\gamma_{R,\epsilon}} f(z) dz = \int_0^{+\infty} dx \frac{4\pi^2 - 4\pi i \log x}{x^2 + 2x + 2}$$

III) $\oint_{\gamma_{R,\varepsilon}} f(z) dz = 2\pi i (\text{Res}_f(-1+i) + \text{Res}_f(-1-i))$
 $\downarrow$
 somma dei residui

$$\text{Res}_f(-1+i) = \lim_{z \rightarrow -1+i} (z+1-i) \frac{(\log z)^2}{\cancel{z^2} + \cancel{2z} + 2} \frac{1}{z+1+i}$$

$$= \frac{(\log(-1+i))^2}{2i}$$



$$-1+i = \sqrt{2} e^{i \frac{3\pi}{4}}$$

$$\Rightarrow \log(-1+i) = \log \sqrt{2} + i \frac{3\pi}{4} \rightarrow \text{corretto}$$

perché $\text{Im} \in (0, 2\pi)$.

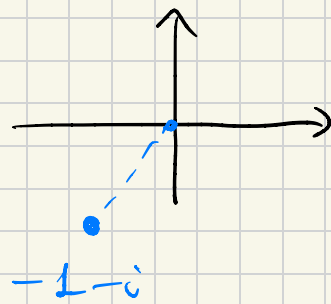
$$\Rightarrow (\log(-1+i))^2 = (\log \sqrt{2})^2 + \frac{3\pi i}{2} \log \sqrt{2} - \frac{9\pi^2}{16}$$

$$= \left((\log \sqrt{2})^2 - \frac{9\pi^2}{16} \right) + i \frac{3\pi}{2} \log \sqrt{2}$$

$$\Rightarrow \text{Res}_f(-1+i) = \frac{(\log \sqrt{2})^2 - \frac{9\pi^2}{16} + i \frac{3\pi}{2} \log \sqrt{2}}{2i}$$

$$\text{Res}_f(-1-i) = \lim_{z \rightarrow -1-i} (z+1+i) \frac{(\log z)^2}{\cancel{z^2} + \cancel{2z} + 2} \frac{1}{z+1-i}$$

$$= \frac{(\log(-1-i))^2}{-2i}$$



$$-1-i = \sqrt{2} e^{i \frac{5\pi}{4}}$$

$$\Rightarrow \log(-1-i) = \log \sqrt{2} + i \frac{5\pi}{4} \rightarrow \text{ok perché } \text{Im} \in (0, 2\pi).$$

$$(\log(-1-i))^2 = (\log \sqrt{2})^2 + i \frac{5\pi}{2} \log \sqrt{2} - \frac{25\pi^2}{16}$$

$$= \left((\log \sqrt{2})^2 - \frac{25\pi^2}{16} \right) + i \frac{5\pi}{2} \log \sqrt{2}$$

$$\Rightarrow \text{Res}_f(-1-i) = \frac{(\log \sqrt{2})^2 - \frac{25\pi^2}{16} + i \frac{5\pi}{2} \log \sqrt{2}}{-2i}$$

$$\Rightarrow \oint_{\gamma_{\varepsilon, R}} f(z) dz = \pi \left(\frac{25\pi^2}{16} - \frac{3\pi^2}{16} + i \left(-\frac{5\pi}{2} + \frac{3\pi}{2} \right) \log \sqrt{2} \right) = \pi \left(\pi^2 - i\pi \log \sqrt{2} \right)$$

$$\text{Im} \oint_{\gamma_{\varepsilon, R}} f(z) dz = -4\pi I \quad (\text{dal punto precedente})$$

$$\text{Per tanto: } -4\pi I = -\pi^2 \log \sqrt{2}$$

$$\Rightarrow I = \frac{\pi}{4} \log \sqrt{2}$$

ESERCIZIO 2

$$G_a(t) = \frac{\sin(at)}{\pi t}$$

I) Notiamo che $|G_a(t)|$ va come $1/|t|$ per $|t| \rightarrow +\infty$, inoltre G_a è
e limitata $\Rightarrow G_a(t) \in L^2(\mathbb{R})$
 $\Rightarrow \hat{G}_a(\omega) \in L^2(\mathbb{R})$.

D'altra parte l'andamento $\frac{1}{|t|}$ implica
che $G_a(t) \notin L^1(\mathbb{R})$. Pertanto non possiamo
concludere che $\hat{G}_a(\omega)$ sia continua e limitata.

Calcoliamo $\hat{G}_a(\omega)$:

$$\hat{G}_a(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} G_a(t) \quad \rightarrow \text{scriviamo: } \sin(at) = \frac{e^{iat} - e^{-iat}}{2i}$$

$$= P \int_{-\infty}^{+\infty} dt \frac{e^{i(\omega+a)t}}{2\pi i t} - P \int_{-\infty}^{+\infty} dt \frac{e^{i(\omega-a)t}}{2\pi i t}$$

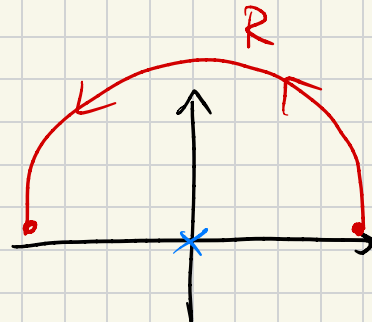
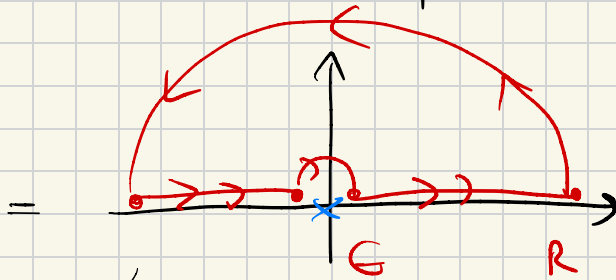
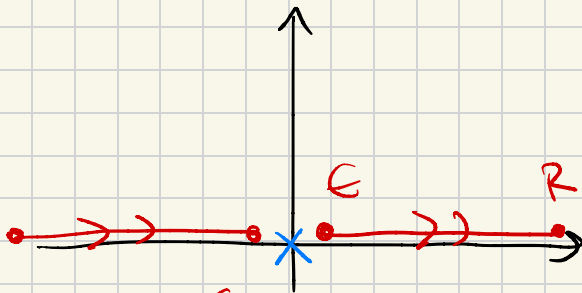
$\rightarrow$ usiamo la parte principale per rendere
l'integrale ben definito

$$\text{Calcoliamo: } P \int_{-\infty}^{+\infty} dt \frac{e^{i\gamma t}}{2\pi i t}$$

(oppure usiamo
il suggerimento)

$$= \lim_{\substack{\epsilon \rightarrow 0^+ \\ R \rightarrow +\infty}} \left(\int_{-R}^{-\epsilon} dt + \int_{+\epsilon}^{+R} dt \right) \frac{e^{i\gamma t}}{2\pi i t}$$

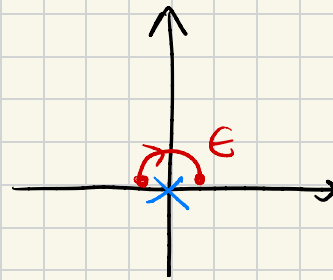
$\gamma > 0$:



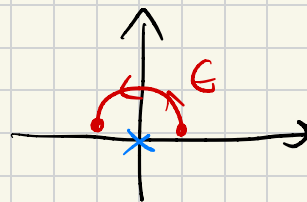
tende a
0 per $R \rightarrow +\infty$
per Jordan

$= 0$ per teorema
dei residui

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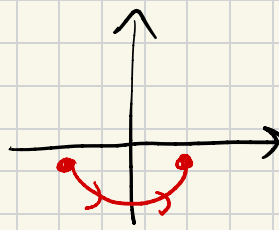
$$\Rightarrow P \int_{-\infty}^{+\infty} dt \frac{e^{i\gamma t}}{2\pi i t} = \lim_{\epsilon \rightarrow 0^+}$$



$$= \pi i \operatorname{Res}_{\substack{e^{i\gamma z} \\ 2\pi i z}}(z=0) = \frac{\pi i}{2\pi i} = \frac{1}{2}$$

Analogamente, per $\gamma < 0$:

$$P \int_{-\infty}^{+\infty} dt \frac{e^{i\gamma t}}{2\pi i t} = - \lim_{\epsilon \rightarrow 0^+}$$



$$= -\pi i \operatorname{Res}_{\substack{z=0 \\ \frac{e^{iyz}}{2\pi i z}}} = -\frac{\pi i}{2\pi i} = -\frac{1}{2}$$

$$\Rightarrow P \int_{-\infty}^{+\infty} dt \frac{e^{iyt}}{2\pi i t} = \frac{1}{2} \operatorname{sign}(y)$$

$$\Rightarrow \hat{G}_a(\omega) = \frac{1}{2} \operatorname{sign}(\omega + a) - \frac{1}{2} \operatorname{sign}(\omega - a)$$

$$= \begin{cases} \omega > a: & \frac{1}{2} - \frac{1}{2} = 0 \\ -a < \omega < a: & \frac{1}{2} + \frac{1}{2} = 1 \\ \omega < -a: & -\frac{1}{2} + \frac{1}{2} = 0 \end{cases}$$

$= \Theta(a - |\omega|) \in L^1(\mathbb{R}) \text{ e } L^2(\mathbb{R})$ perché
limitate e $\neq 0$ solo in un intervallo. Non è continue.

$$\text{II)} \quad \widehat{G_a * G_b}(\omega) = \Theta(a - |\omega|) \Theta(b - |\omega|)$$

$$= \begin{cases} \Theta(a - |\omega|), & \text{se } a < b \\ \Theta(b - |\omega|), & \text{se } b < a \end{cases}$$

$$\Rightarrow (G_a * G_b)(t) = \begin{cases} \frac{\sin(at)}{\pi t}, & \text{se } a < b \\ \frac{\sin(bt)}{\pi t}, & \text{se } b < a \end{cases}$$

$$\int_{-\infty}^{+\infty} dt' \frac{\sin(at') \sin(b(t-t'))}{\pi^2 t' (t-t')}$$

$$\Rightarrow (G_a * G_b)(0) = \int_{-\infty}^{+\infty} dt' \frac{\sin(at') \sin(bt')}{\pi^2 (t')^2}$$

Dunque l'integrale richiesto da $\pi^2 (G_a * G_b)(0)$, ovvero dell'espressione sopra:

$$\int_{-\infty}^{+\infty} dt \frac{\sin(at) \sin(bt)}{t^2} = \begin{cases} \pi a, & \text{se } a < b \\ \pi b, & \text{se } b < a. \end{cases}$$

ESERCIZIO 3 $L_q(v) = \sum_{n=1}^{+\infty} q^{n-1} (e^{(n)}, v)$

$$v \in H, \quad q \in \mathbb{C}, \quad |q| < 1.$$

I) La linearità segue dalle linearità del prodotto scalare. Limitatezza:

$$\begin{aligned} \|L_q(v)\| &\leq \sum_{n=1}^{+\infty} |q|^{n-1} |(e^{(n)}, v)| \\ &\leq \left(\sum_{n=1}^{+\infty} (|q|^{n-1})^2 \right)^{1/2} \left(\sum_{n=1}^{+\infty} |(e^{(n)}, v)|^2 \right)^{1/2} \\ \text{C.S.} \end{aligned}$$

$$= \sqrt{\frac{1}{1-|q|^2}} \|v\| \Rightarrow \|L_q\| \leq \sqrt{\frac{1}{1-|q|^2}}$$

(Si può vedere che questo stima è anche

ottimale scegliendo: $v_0 = \sum_{n=1}^{+\infty} (\bar{q})^{n-1} e^{(n)}$,

infatti: $L_q(v_0) = \sum_{n=1}^{+\infty} |q|^2 = \frac{1}{1-|q|^2}$

e $\|v_0\| = \left(\sum_{n=1}^{+\infty} |q|^2 \right)^{1/2} = \sqrt{\frac{1}{1-|q|^2}}$

$\Rightarrow \|L_q(v_0)\| / \|v_0\| = \sqrt{\frac{1}{1-|q|^2}} \Rightarrow \|L_q\| \geq \sqrt{\frac{1}{1-|q|^2}}$

$\Rightarrow \|L_q\| = \sqrt{\frac{1}{1-|q|^2}}).$

II) $F_v(q)$: funzione analitica su $\{|q| < 1\}$

Se una funzione analitica $\bar{e}$ 0 in un aperto del dominio, allora $\bar{e}$ 0 ovunque nel dominio. Pertanto: $F_v(q) \equiv 0$.

In particolare la serie di Taylor in $q=0$ ha tutti i coefficienti nulli.

Quindi: $(e^{(n)}, v) = 0, \forall n \geq 1$

Ma $\{e^{(n)}\}$ $\bar{e}$ un s.o.c. $\Rightarrow v=0$.